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A TREATISE

ON

ANALYTICAL MECHANICS.

BY

BARTHOLOMEW PRICE, M.A., F.R.S., F.R.A.S.,

SEDLEIAN PROFESSOR OF NATURAL PHILOSOPHY, OXFORD.

VOL. I.

STATICS, ATTRACTIONS,
DYNAMICS OF A MATERIAL PARTICLE.

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PREFACE TO THE SECOND EDITION.

THIS volume is the third of a Treatise on Infinitesimal Calculus and its capital applications. It is also the first of a Treatise on Mechanics, and may be considered and studied independently of the two preceding volumes. In it are contained Statics ordinarily so called, Attractions, and the Dynamics of a Material Particle.

The investigations are for the most part confined to subjects which are within the range of the general principles of Mechanics, and are not extended to particular sciences wherein these principles are specifically applied. Thus, the principles are discussed on which the equilibrium and stability of bridges, arches, and roofs depend; yet the practical rules of the engineer's and the builder's arts are not considered. Also as physical astronomy, the theories of light, heat, and electricity require the explanation and discussion of certain experimental laws which rule their subject-matter, so the inquiry into these special subjects is beyond the scope of this work at its present stage.

Chapter I is introductory to the whole of this part of the Treatise on Infinitesimal Calculus. It seemed desirable to explain as accurately as possible the relation between "applied Mathematics," as some parts of the subject are called, and the sciences of number and geometrical space; and so I have entered on a discussion of one or two salient points of the subject with the object of shewing that an exact knowledge of Mathematics is necessary for the complete inquiry into such sciences. I have also ventured to submit to the common judgment of Mathematicians the statement, that Mechanics, enlarged in its idea and principles, as I have attempted to enlarge it, is nothing else than the science of motion, and ought, as such, to be called by that name. Thus there are three principal mathematical sciences, those viz. of number, space, and motion: the last of which it has been my purpose to develope in the following pages.

A course of inquiry somewhat irregular has been followed, because it has been found most convenient for a didactic treatise; and Chapters II—V contain Statics, wherein the laws of pressure as they produce equilibrium, or neutralize each other's effects, are considered. In Chapter VI I have considered the theory of Attractions at some length, and have also employed the indirect mode of investigation which the potential-function supplies. In Chapter VII the principles, incidents, laws, and conditions of the science of motion are formally drawn out. The Chapter is thus introductory to Dynamics. The mode of investigation and the forms of statement of some of the pregnant principles are different from those which are commonly given. Only two of the three ordinary laws of motion (*axiomata motûs*, as they are called by Newton) are admitted. The truth of these is made to depend on and to flow from an intelligible conception of the idea of motion and its incidents; and on an inductive verification only so far as the science is applied. This distinction is important, and appears to solve some questions which are in dispute between the two schools of writers on Mechanics. The method which I have taken is indeed counter to that of most English authorities on the subject: it is rather in accordance with that of foreign, and chiefly French, writers. If any one after reflection should hesitate or refuse to admit my principles, and the mode of arriving at and of stating them, I must ask him to consider the subject from the point of view which the Infinitesimal Calculus and a reasonable conception of Infinitesimals present to him; and which, with great respect for the great names and the sober judgment of those who take the opposite course, I venture to think to be the most natural and the most rational.

The first principles of the science are drawn from an intelligible conception of motion itself. For the mathematical expression of these, the language and the symbols of Infinitesimals are peculiarly appropriate: effects are produced by causes which act according to continuous laws: thus the effects become continuously developed, and a peculiar system of symbols is required to express them. New ideas necessitate a new language, and new language requires new characters; and these are supplied by the Infinitesimal Calculus.

A license has been taken, for which I must crave some indulgence; certain words are used which are either new or are used in a new relation. In the absence of generally recognised rules for the formation of scientific language, I have used compounded words; and have thereby obtained expressive, though somewhat long, words. This course I found myself obliged to take. For ideas which are in themselves clear and distinct have been so much obscured by ambiguity and indistinctness of language, that there is no source of error more fertile. Let me cite an instance. In former books no word occurs more frequently than the word "force." Indeed Mechanics has been called the science of *forces*. But what does "force" mean? Will any one give an accurate definition of it? a definition, that is, which will be correct, when the word is applied to "the cause of motion," to "accelerating forces," to "effective forces," to "forces lost and forces gained," to "living force," to "labouring force?" In some of these various meanings it indicates effect, in others it indicates cause. Surely herein is confusion; and herein too, as it seems to me, is the reason why the principles of mechanical science, or the science of motion, are so imperfectly understood. Similar is the ambiguity of the word "motion:" it is frequently used synonymously with the word *velocity:* thus "*momentum*" has been called "quantity of motion:" it is *quantity of velocity;* and it is at all events perplexing to most minds to have a thing called by a name which means what it is not. Thus I have endeavoured in those parts of the treatise where first principles are expounded, and where clearness of language no less than clearness of conception is required, to call things by names which are expressive *vi significationis*; although in the more popular parts I have used words in their ordinary and less exact meaning. The subject is not in itself difficult, but it has been made difficult by the maze of indistinct nomenclature by which its fundamental notions have been obscured.

As in the previous volumes, I am under obligation to many friends, and to many writers on these subjects. It is almost superfluous to mention Euler, Lagrange, Laplace, Poisson, Poinsot, Jacobi, M. Bertrand, Sir W. R. Hamilton of Dublin, and now, Sir William Thomson and Professor P. G. Tait, the authors of the treatise on Natural Philosophy, the first volume of which has lately been published at the Clarendon Press; because no one has a right to form a judgment, and much less to compose a didactic treatise, on the subject of Mechanics, without a previous and preparatory study of the works of these eminent men. From the works of Dr. Whewell, lately the Master of Trinity College, Cambridge, I have derived much aid: I know not how much: for in the Appendices to the second volume of his Philosophy of the Inductive Sciences so much suggestive matter on Mechanical Philosophy is contained, that opinions which appear to be one's own may perhaps owe their origin to those essays. The Journals of Crelle and Liouville have given much assistance. To the editors of those Journals and to their contributors, whose names are too many to be mentioned here, I tender my acknowledgments.

References are made to the second editions of the Differential and Integral Calculus, which are the two preceding volumes of this treatise; and also to the numbers of the Articles and of the equation as in these volumes. The colloquial style has been retained.

11, St. Giles', Oxford,
Nov. 3, 1868.

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ANALYTICAL MECHANICS.

CHAPTER I.

INTRODUCTORY; THE METHOD OF THE TREATISE.

ARTICLE 1.] Of all parts of Infinitesimal Calculus, Analytical Mechanics, or (as I shall hereafter have reason to call it) the Science of Motion, is in its results and its applications the most important; the principles and processes of all mathematical physics are derived from it; and as, for reasons which shall be assigned hereafter, it is in itself the most perfect of physical sciences, so do the others approach more or less to completeness according as the laws and methods of mechanics are more or less satisfied by them; and the object to be attained in all is, to make them parts of this principal and normal science. Now in the process of our application of the science of number to that of motion, new subject-matter, or new kinds of quantity measurable by number, will be introduced; and also as the results of our investigations will be applicable to the phenomena of the external world, and to the unravelling of complex effects, it is necessary to premise some few observations on the method of our inquiry; and especially to shew how, and how far, the pure sciences of number, space, and motion may aid us in the discovery of the proximate causes of such effects; proximate, I say, in order that the objects of our search may be definite and intelligible, and that we may not be lost in the subtleties of metaphysics.

2.] There are generally two processes, by one or other of which our knowledge of natural phenomena is obtained, and with both of which it is in many cases absolutely necessary, and in all cases desirable, that an inquirer into nature's laws should be acquainted; and although in their use one of these processes frequently runs into the other, and they are alternately applied for the purposes of discovery and verification, yet they are in themselves distinct, and for philosophical reasons it is requisite to keep them so. In one of these processes we take the facts of nature as they are presented to us in their simple and concrete forms; and animated by a conviction deep-seated in our nature that they are not isolated, but instances of a grand and comprehensive law, which has been impressed on them, and by virtue of which they are, we seek for that law: with this object in view we study them, analyse them; and in the analysis we subject them to trials of various kinds, if they admit of experiment, or observe them in such varying relations as they exhibit to us, if they do not; we separate what is extraneous and thus accidental from that by virtue of which they seem to us to be, and without which they would not be: and by this process detect the general law which lies latent in the fact; or, in perhaps more precise terms, the cause of which the fact is the effect. Thus we ascend from the fact to the cause; and when many facts have been subjected to a similar process, and the same law has been detected in all, we collect them under a general formula which expresses this law, and thereby a cause of which all the examined facts are the effects; and the human mind, endowed with a love of continuity, extends this to other facts similar in kind, and beside those which have been examined. In this process therefore we interrogate nature as she offers herself to us in her simple forms and particular developments; and so long as any branch of knowledge consists only of such isolated facts it is little else than mere empiricism; but when a bond of union has been imported from some other source, and these facts have been collected into general propositions; when on these phenomena has been induced a distinct idea, and the information obtained from them in their isolated forms has been studied, arranged, and reasoned upon, then, and not until then, has it a right to bear the name of Science; it is then no longer accumulated experience, but it is experience systematized, digested, assimilated, organized into a whole; it has harmony, regularity, and law; and the physical sciences thus formed will be found to satisfy another most exact, and perhaps the most searching, test of their truth; they predict similar effects from similar causes.

3.] All the physical sciences are progressive, and pass through the experimental or observational stage which I have described;

in their infancy the subjects of them have been in this disconnected state. Experience in the way of experiment and observation has been the chief instrument by which their boundaries have been advanced, distinct ideas for the colligation of facts been obtained, and inquirers been led to the discovery and enuntiation of their peculiar laws: the discovery of the laws of motion by Galileo, of the laws of planetary orbits by Kepler, of the law of refraction by Snell, are, amongst many, early and salient instances. Invariably, so long as any science is in this imperfect condition, its phenomena must be examined for the purpose of discovering such normal laws, and it is in the prosecution of this work that the most eminent philosophers of the present age have earned their glory: in short, the analysis of such facts has been in a great measure the characteristic of the science of the XIXth century: and no mean work is it: it demands the highest intellectual and moral qualities that can adorn human nature; an eager and honest desire after truth; patience and endurance of labour; a courage that never fails under non-success; the keenest intellectual acuteness in detecting resemblances; a mind gifted with a plastic power of framing an idea distinct and pregnant, which shall collect all into one general formula; an inventiveness and a never-failing command of resources: and in our days these qualities have not been wanting, and have not been unrewarded. It is however unnecessary for me to do more than to indicate the methods of experimental philosophy, in order that I may contrast with them, and thus bring into greater prominence, the process of investigating truth which will be developed in the following volume; and the reader desirous of further information on the methods of inductive philosophy must have recourse to works wherein such subjects are specially treated of. Let me refer him to Sir John Herschel's Treatise of Natural Philosophy, a work which contains in a short compass a masterly exposition of the methods, and to Dr. Whewell's Philosophy of the Inductive Sciences, wherein he will find the subject treated by an eloquent author, whose knowledge of physical science seems to be limited only by the limits of science itself[*].

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[*] See also an article on 'Whewell on Inductive Sciences' in the Quarterly Review, Vol. LXVIII: and subsequently republished in the Collection of Essays contributed to the Edinburgh and Quarterly Reviews by Sir John F. W. Herschel.

4.] It will be seen then that the first step in experimental philosophy is to colligate facts by means of a distinct and appropriate idea; afterwards a consilience of inductions takes place; and hereby we arrive at the last step in the construction of a science, which is the enuntiation of a *theory;* the determination, that is, of a law which rules all the subject facts, and the discovery of a general cause, of which the facts of the science are the single and (as they seem at first) isolated or independent effects; and when such perfection is attained the aggregate of the knowledge receives the name of a science, having all the characteristics of arrangement, order, system, completeness, which are necessary for such perfection.

And now comes in the second process to which allusion has been made. If the theory is true, not only is it an explanation of all the facts which it comprises in its formula, but it has also a prophetic power: when the cause is active, results similar to the former ones must be produced; the theory requires verification; and the verification consists in the prediction of the future: and it is only when such future facts have been shewn to accord with a theory, that it satisfies those stringent rules of induction which have been constructed in a jealous care of truth. The theory may also be pregnant with results different from those out of which it has grown; these must also be traced and examined: the theory must be tested in all ways and in all directions; and when such tests have been satisfied, it has a claim on our acceptance, and for this purpose a process, the reverse of the former, is necessary: facts were in that analysed, so that their latent cause might be detected; in this causes are to be developed into their effects; the former is the *historical* process through which the science has grown from an imperfect state to perhaps full maturity; the latter takes the science in its perfect state, and explores the riches which it contains; the former is the process by which the science has been constructed, and is somewhat analogous to the manner in which we individually *learn* it; the latter is the form wherein the man of science *knows* it. Now this distinction is important: for as it is under the latter and more perfect aspect that I shall have to consider the science of motion, so the method is dogmatic; and the fundamental and axiomatic laws will be enuntiated, and no formal proof of them will be given; it may sometimes be desirable to indicate the steps by which historically they have been arrived at, but such an explanation will be only incidental and that the learner may have adequate knowledge of them; and I shall not lose sight of the chief object, which is to trace into their farthest results those general laws which an inductive philosophy has supplied.

5.] Mathematics is the most powerful instrument, which we possess, for this purpose: in many sciences a profound knowledge of mathematics is indispensable for a successful investigation. In the most delicate researches into the theories of light, heat, and sound it is the only instrument; they have properties which no other language can express; and their argumentative processes are beyond the reach of other symbols. For other sciences, for Mechanics, and Astronomy, and for Mechanism they are almost as necessary; and I am sure that to any one who has taken the pains to compare the general explanation of planetary disturbances given in Sir John Herschel's Outlines of Astronomy with that of the same phenomena as discussed with the aid of mathematical appliances, there cannot be a doubt that, however successful Sir John Herschel may have been, even beyond his expectation, yet for an accurate comprehension of the circumstances the other method is absolutely necessary. The following extract from that work[*] is unimpeachable testimony: 'Admission to its sanctuary' (that is, of astronomy) 'and to the privileges and feelings of a votary is only to be gained by one means—*sound and sufficient knowledge of mathematics, the great instrument of all exact inquiry, without which no man can ever make such advances in this or in any other of the higher departments of science as can entitle him to form an independent opinion on any subject of discussion within their range.*' I can truly use the same language as to the necessity of mathematics for the successful study of the other higher branches of the science of motion.

6.] Here it may be asked, What are mathematics? Define them. Do they require and apply reasoning processes different from those of the ordinary discourse of men? have they a different logic? and a different language? What distinction exists between pure and mixed mathematics, since they are commonly divided into these two classes? and what does the term include? Many of these questions may be matter of words only; it is not necessary for me to define mathematics in a way which would satisfy a metaphysician, or to inquire how far 'science of quantity,' or 'science of measuring quantity' may be a sufficient definition, and whether there is not a large class of propositions of geometrical *position* which such definitions will not include; it is enough for me to be able to give you such an account of the means which mathematics afford for pursuing our present inquiry that I may excite in you good hope of final success. I would however observe, that the reasoning process is not different from that of any other branch of knowledge; their logic is the same as that of chemistry, of political economy, or moral philosophy; it is addressed to the same faculties of man, and does not require any peculiar formation or deformity of human nature, as some seem to think; but there is required, and in a great degree, that attention of mind which is in some part necessary for the acquisition of all knowledge, and in this branch is indispensably necessary. This must be given in its fullest intensity; this is the excellency which Sir Isaac Newton claimed for himself, and thus placed his superiority on moral rather than on intellectual grounds: the other elements especially characteristic of a mathematical mind are quickness in perceiving logical sequence, love of order, methodical arrangement and harmony, distinctness of conception. The language of mathematics is to a certain extent peculiarly its own; its symbols are certainly its own; but these may generally, if it is desirable, be translated into ordinary language; and its language is peculiar, because the subjects of which it treats are peculiar. Now mathematics include three normal sciences; (1) science of number, (2) science of space, (3) science of motion; and under one or other of these all sciences which are treated mathematically may be ranged; or the several parts of any one may come under different normal sciences: thus, formal or geometrical optics is an application of geometry; physical optics of the science of motion; plane astronomy is geometrical, physical astronomy is mechanical. The division of mathematics into pure and mixed is arbitrary and useless, because it leads to no practical result; and therefore I do not care to retain it. I may however observe that the first two sciences, those viz. of number and space, are commonly included under the term pure mathematics, and that the last one and its subordinates are called mixed; the reason being that the subject-matter of the last has been thought to be terrestrial, or, at all events, cosmical matter; and that therefore the science involves considerations of the properties of this matter, and which must be discovered by examination and analysis, and that these processes are extraneous to pure motion: whereas the other sciences consider subjects only which are proper to them, and therefore they are called *pure*.

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[*] See *Outlines of Astronomy*. 4th edition. p. 5. Longman and Co., London, 1851.

7.] The science of number, or, as the French call it, *le calcul*, has for its subject-matter number in its pure and abstract form; number, that is, as an abstract quantuplicity; not this or that *thing* taken so many times, but the *times* which it is taken; it does not treat therefore of concrete things; and it is important to observe this property of the science, because the truths of number are for this reason so generally, almost universally, applicable; time, space, pressure, weight, velocity, quantity of light, of heat, of electrical action, may be all measured by it; and so long as the conditions imposed by the numerical science are observed, the truths of number have their counterpart in the applied science. The science also includes number in its twofold division of discontinuous and continuous number; the former of which is the subject of arithmetic and algebra, and the latter of infinitesimal calculus; these being distinguished by a difference of species of subject-matter, and not of process. It is most important to observe that the numerical symbols represent abstract quantuplicities, and that the results are true, because they are correct developments of the idea of number, and are independent of the concrete matter to which they are applied. Yet they may be applied, and by the following process: the numerical proposition is operated on by the concrete unit of the matter of the particular science; whether it be linear length, or area, or cubical content, or weight, or velocity; that is, each term of the numerical equation has the concrete unit affixed to it, and thereby itself becomes concrete, and expresses the concrete thing taken a certain number of times; thus suppose we have a numerical equation

$$4+3=7,$$

and suppose that the operating concrete unit is an inch: then we have

$$4 \text{ times} \times \text{one inch} + 3 \text{ times} \times \text{one inch} = 7 \text{ times} \times \text{one inch};$$

an inch being matter of such a kind as to be consistent with the fundamental operations of arithmetic; that is, if one inch is added to one inch, no part of either one is absorbed into the other, but the matter is continuously additive. Similarly might the operating unit be a pound, or an unit of velocity, and in both cases the result would be true because the arithmetical equality is correct.

8.] Now this process of introducing a concrete factor into an arithmetical equation is of the greatest importance, and deserves careful consideration. The effects of it will frequently be discussed hereafter; but one above all others requires explanation at the outset of our work. Although the equations are made concrete by the process, yet they are still subject to the laws of algebra. In being made concrete they become also homogeneous as to the concrete unit; consequently they are intelligible and interpretable: indeed no meaning can be attached to an equation which is not homogeneous. Also if an equation is once homogeneous, it continues homogeneous, whatever are the algebraical processes to which it is subjected. Hence homogeneity supplies a test of the correctness of the operations; if this character of an equation is lost, error has been introduced. The principle of expressing homogeneity in reference to various concrete units will be explained hereafter.

9.] The second mathematical science is that of space, or, as it is usually called, geometry; the subject-matter is in general tridimensional space; whatever is the origin of our conception of it, whether it is experience, or whether space is a phenomenal condition of our knowing things at all, or whether it is an intuitive notion, yet at all events the subject-matter of geometry is space, abstracted from all consideration of the space which we occupy, and in which we are: and the science consists in the development of this idea of space. The axioms contain enuntiations of constituent parts and properties of it; the definitions are explanations of terms arising out of, and necessary to, the division of space which flows from the fundamental idea; thus, for instance, space is such that the whole is greater than its part; that if equal spaces are added to equal spaces, the wholes are equal; spaces are equal which occupy equal parts of space, the comparison being made on the principle of superposition. The truths of geometry may be directly deduced from the axioms and definitions by means of postulates and more complex constructions, and the science of space thus treated of is called pure geometry; as such it neither requires nor involves the properties of number; its additions and subtractions and equalities are made on the principle of superposition; thus, if an angle is added to an angle, no reference is made to any unit angle, but one concrete angle is superposed on the other; and the symbols in pure geometry are symbols of the concrete quantities and are not the subjects of arithmetical laws and operations. The old geometricians employed this process only. But Descartes, perceiving that geometrical space accords with the fundamental requirements of number, treated of its properties by means of arithmetic and algebra: in this view we may operate on any numerical equation with a concrete geometrical unit whereby it becomes concrete and homogeneous, and becomes a geometrical proposition; and whatever numerical truths are contained in, and deducible from, the numerical equation, analogous geometrical propositions are also deducible; and therefore if the equation is transformed or operated on according to arithmetical laws, so will the transformation carry with it the correctness of the corresponding geometrical changes; the geometrical process is parallel with, and proved by, the numerical process. Thus suppose the following equation to be true for certain *numerical* values,

$$y^2 = 2ax - x^2:$$

then by operating on each term with the *linear unit,* and interpreting x and y according to the conventional signification of rectangular axes, we have the geometrical property of the curve of which it is the equation, viz. (y^2) times the linear unit $= (2ax)$ times the linear unit $-(x^2)$ times the linear unit; y, x, and a being numbers. Or otherwise suppose that we operate on the same equation with the (linear unit)^2, then the equation becoming arithmetically

$$y \times y = (2a - x)x;$$

and we have the square of the ordinate = the rectangle contained by the segments of the base.

By this process algebraical geometry has been constructed: the equations in their original forms are numerical; but as geometrical space satisfies the conditions as to quantity which the science of number requires, we operate on these numerical equations with a geometrical unit, and hereby transform them into geometrical propositions; and we can further employ all the processes of algebra for deducing and proving geometrical truths which are contained in other given geometrical propositions.

In both these sciences it will be observed that the process of inference is the same: the deduction from the fundamental ideas of number and space of the truths with which they are pregnant.

10.] The third and last of the mathematical sciences is that of motion; into the foundation, laws, and processes of which I shall enter at length in the following pages; but as my method is that of a positive deductive science, intended for didactic use, and therefore to a certain extent dogmatical, it is not necessary formally to discuss the history of the laws of motion, or the growth of the fundamental idea, and the successive steps through which it has reached that perfect state in which parts of it can be expressed in definite axioms, and thus be made the major premisses of the first syllogisms from which all the other truths of the science are to be inferred. I shall not relate the logomachy of mechanics in the days of Aristotle, and the disputations of the Schoolmen who taught that rest was natural and motion was unnatural, and that some bodies fall faster than others because they are heavier; nor shall I indicate the several steps by which Galileo first obtained a clear insight into the laws of motion, and how Stevinus first proved the laws of oblique pressure by means of a continuous chain resting on two inclined planes: neither shall I generally detail or explain experiments by which evidence is given to the truth of the axioms. My work, on the contrary, is to take the *idea* of motion as recognized, and its laws as acknowledged, and to deduce from them their results. To this end mathematics, and especially the science of continuous number, will be found most useful instruments of inquiry: a word or two will shew this. Matter of motion, space, time, velocity, and combinations of these, such as momentum, work, vis viva, pressure, weight, will come under consideration. All these quantities are continuously additive and subtractive, and satisfy the requirements of the science of number: and they admit of infinite divisibility; nay, more than this, some of these are within the grasp of our minds only when they are resolved into infinitesimal elements: as, for instance, it is necessary to know the law of change of velocity of a particle moving with a varying velocity, before we can determine the actual change of velocity which takes place in a given finite time; that is, the infinitesimal increment must be known, and this is determined by the law, before we can find the finite change, the latter being determined from the former by means of integration; in these respects then the subject-matter of our science will be found to harmonize with the laws of the science of number; and these latter may be applied.

11.] Suppose now that the axiomatic laws of mechanics are deduced from the fundamental idea of motion, and that we know them: let them be translated into mathematical language and symbols, and so stated that the propositions take the form of equations; if the concrete mechanical unit be removed, the equation will stand as a numerical equation: so in this state all the rules of the science of number may be applied, and whatever are the results which can be inferred by means of them, they may be translated by an operating factor into their mechanical equivalents, and these again into ordinary language. As therefore the resources which the science of number supplies become more numerous, the more fruitful is the deductive process; and hence it is that the progress of the sciences is simultaneous; whatever retards the one is also an obstacle to the progress of the other.

Consequently the following will be the course of our enquiry. The idea of motion will be first described together with space and time which are two incidentals of it. This is the fundamental idea of the science; and pregnant properties of it will be considered: as matter is the subject of motion, so will certain properties of matter have to be explained, and especially the property which is called *inertia*, as we are hereby led to the formation of equations of motion, in which the equality of momentum impressed and momentum expressed will be stated. These pregnant properties of motion and of matter are called *Laws of Motion*, and will be found to be only two: we shall translate them into mathematical language and symbols; and by the process of infinitesimal calculus deduce from them their results, which we shall in many cases trace in the applications of mechanics, and especially in the phenomena of gravitation, whether in the case of bodies being near to the earth and falling towards it, or in the case of the approximate motion of the planetary bodies, herein laying the dynamical foundations of physical astronomy. By this method the foundations of mechanics will be laid in breadth sufficient to include all kinds of matter; whether cosmical or of that of light, if there is an ethereal medium; and all kinds of motion, whether direct or orbital or oscillatory; the basis therefore will be wide enough to comprehend the mathematical theories of hydromechanics, light, heat, electricity, magnetism; these several sciences, as they advance towards perfection, satisfy more and more the notes of the science of motion, but the perfect state will be reached only when they wholly do so.

12.] Such is the philosophical form of the perfect and exact science of motion; and such is the philosophical course of learning it; but there are reasons why a different method is more suitable to a didactic treatise. It is better to begin with what is apparently more simple and more concrete, than with an abstract verity; we are not accustomed to analyse cases of motion, but we are familiar with an effect of the same cause as that which produces motion, but which in mechanics is actually more complex; we have all of us a notion more or less exact of pressure or of weight; the tension of a string caused by a weight suspended at the end of it, or a pressure caused by a weight resting on the hand, gives us a notion more distinct than that of a body falling under the action of the earth's attraction. Now let me analyse such a pressure from a dynamical point of view: take the case of a weight resting on a table; the same force which produces the pressure on the table would cause the body to fall towards the earth, if the table were removed; the falling effort is the same, although the table is there: the earth attracts the body, impresses velocity on it, and causes it to penetrate the table; but the material of the table is elastic, and therefore so often as the body penetrates the table and causes the particles of the table which are in contact with or are near the body to approach each other, an elastic force of recoil is called into action and causes the body to retire: thus an oscillatory motion of the body is established, which is however so slight that the motion of the body is to the senses imperceptible. It may perhaps be thought that this is an indirect mode of considering such a simple case as that of a body resting on a table: perhaps it is; but it is the mode of applying the principles of the science of pure motion to the case of a body resting on a table.

Thus although in the order of the pure science other and more simple cases of motion would be discussed before this, yet as this case of pressure is so simple, as it seems, and so common, it is for didactic purposes desirable, even if it does cost a loss of order scientifically correct, to consider first those forms of problems with which a learner is most familiar; we shall hereby take advantage of his previous knowledge, and lead him from that which is to him more simple to that which is more complex. I propose therefore to defer the pure science of motion to the second part of the treatise; and to consider at present *pressures* only, and these apart from the properties, real or virtual, of motion. The science of pressures is called *statics*; and in establishing the principles from which I shall begin, I shall be obliged to appeal to *experience*, to what we see and observe: and whatever assumptions or hypotheses I may make, I shall refer for proof to our observation of such pressures and to the common sense of mankind. Let me make one other observation on the difference which exists in the views of the same effect as presented to us in a statical and a dynamical light. Suppose that a pound weight rests on the hand, which is at rest; a pressure is experienced which the hand bears; and if another pound be added a pressure twice as great is experienced; but are you conscious of or do you think about the cause of that pressure? are you aware that it is due to the earth's attraction, and to a motion which the body would have if your hand were removed? I think that you consider it as a pressure only, and not in reference to velocity: this is, I say, the common judgment about such pressures: it does not refer them to motion; and it is to such common judgment that I shall appeal in laying the foundation of statics: it may be that I shall now and then use language appropriate to the conception of a real or virtual motion, and that I thereby elucidate difficulties; but it must be remembered that such conceptions are extraneous to statics thus considered, and are such as the subject does not of itself require.

ANALYTICAL MECHANICS.

PART I.

STATICS.

CHAPTER II.

STATICAL PRESSURES ACTING AT THE SAME POINT.

SECTION 1.—*Explanation of matter, force, mechanics.*

13.] A formal definition of *matter* such as would satisfy a metaphysician or a physicist is not required for this work. It is sufficient for us to conceive of it, as the subject of pressure: capable of receiving and of, as we shall hereafter see, transmitting pressure: and as such, having volume and form; because it is in this aspect only that it is of importance to us in the present treatise*. Matter is rigid or stiff, when its component particles are kept in a state of relative rest by the action of cohesion or attraction, or of similar molecular forces; and of these we require at present only to know that the external pressures acting on matter are in magnitude, in comparison of these internal forces, infinitesimal. The consideration of other properties of matter, as the subject of force, will be undertaken in the sequel.

Matter is assumed to be infinitely divisible; an infinitesimal portion of it is called a particle: and the space occupied by a particle is so small that it is a geometrical point. A finite portion of matter is called a body. The quantity of matter contained in a body is called the *mass* of the body.

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* M. Poisson says, 'La matière est tout ce qui peut affecter nos sens d'une manière quelconque.' Dr. Whewell, 'Body or matter is anything extended and possessing the power of resisting the action of force.' Mechanics, 5th edition, Cambridge, 1836.

Force is a cause which changes or tends to change matter's state as to motion or rest. A particle is at rest when it constantly occupies the same place in space. A particle moves when the place occupied by it changes its position.

Mechanics is the science which treats of the action and effects of forces in this respect.

Statics is that part of Mechanics in which the relations of forces are considered as they produce pressure or a tendency to motion.

Dynamics, or as they are sometimes termed Kinetics, is that part of Mechanics in which the relations of forces are considered as they produce motion. In the first part of this work I consider Statics, and only so far, for the most part, as the bodies on which the forces act are rigid. Dynamics and other subjects will be considered in subsequent parts.

14.] When force acts definitely on matter, it is subject to the four following incidents: it acts (1) at a certain point; (2) along a definite line; (3) in a given direction along that line; (4) with a certain magnitude or intensity. And a force is not said to be given unless all these four incidents of it are given.

As Statics is that part of Mechanics which considers the relations of forces as they produce pressure or a tendency to motion, so are statical forces pressures. Weight is one of the most common forms of pressure. Whenever in this first part I speak of forces, the term signifies pressures; but I employ the word force in accordance with common usage.

The point at which a force acts is called its *point of application*. The straight line passing through the point of application of a force, along which the force tends to make the particle at the point of application of the force move, is called the *line of action* or the *action-line of the force;* the direction of the line *towards* which the force tends to make the particle move is called the direction of the force. Thus we take the direction to be that in which the force pulls or attracts the particle at its point of application. The magnitudes of forces are measured by comparing them with some other force, the magnitude of this latter force being taken to be an unit-force. The following is the mode of measuring force.

Two forces are equal, which acting at the same point, along the same line of action, and in opposite directions, neutralize each other.

Statical forces are continuously additive, and, as such, satisfy the requirements of the science of number: thus, if one pound is added to one pound, the sum is two pounds; no part of either of the weights is absorbed into the other; the weight of a basket of stones is the same, whatever is the arrangement of the stones. Statical forces also admit of continuous increase and decrease, and of infinite divisibility: they thus satisfy the requirements of the science of continuous number.

If two statical forces, thus proved to be equal, act on a particle at a point along the same line and in the same direction, the acting force is twice each of the original forces: if three forces act similarly, the resulting force is thrice each of the original forces: and so on. Thus it is that forces admit of measurement: an *unit* of force is chosen, and other forces are compared with it; and are expressed as being so many times the unit-force. Thus forces are expressed by numbers, being referred to a concrete unit-force. The unit-force is arbitrary, and may be a finite or an infinitesimal force. If forces are expressed by numbers which are commonly called incommensurable, they possess the properties of commensurables, if they are referred to an infinitesimal unit-force. If the unit-force is changed, the numbers expressing the forces which are referred to it are also changed in an inverse ratio. Thus a weight of six pounds is expressed by 6, if a pound is the unit-force; by 12, if one-half of a pound is the unit-force; by 3, if two pounds is the unit-force. It is manifest that general laws connecting the point of application, action-line, direction, and magnitude of a force, must be independent of the conventional unit-force.

Statical forces will hereafter be expressed by symbols, such as P, Q, R, ... These are numbers expressing the number of times which the concrete unit-force is contained in the given force; hence also when we meet with such symbols as P^2, Q^2, ... these are also numbers. It is plain that if P represents a concrete force, P^2 is uninterpretable and unintelligible.

Forces may be represented by geometrical straight lines. As a force has a definite point of application, a definite action-line, a definite direction, and is of a definite magnitude, so does a line starting from the point of application of the force and coincident with the action-line in its direction, and in length containing the same number of linear units that the force contains units of force, adequately and completely represent the force in all its circumstances. This mode has the advantage not only of simplifying the enuntiation of many theorems, but also of enabling us to infer mechanical propositions from their geometrical analogues; and vice versâ. Of this process we shall hereafter have many instances.

15.] When a material particle is acted on by many forces simultaneously, there is generally a definite line and a definite direction along which it experiences a definite pressure, or, in other words, along which it has a tendency to move. Now the one force which would produce on this particle a pressure equal, along the same action-line and in the same direction, is called the *resultant* of the acting or impressed forces: and its action-line is called the action-line of the resultant: and the several impressed forces are called *components* in reference to it. The resultant is evidently *unique,* definite as to its point of application, action-line, direction of action, and magnitude.

If the forces acting on a particle are so related as to produce a resultant whose magnitude is zero, then the forces are said to be in *equilibrium,* and the system of forces is called an equilibrium-system.

Hence we infer that when many forces act on a particle, if a new force is introduced equal in magnitude to their resultant, and acting along the same line and in an opposite direction, it neutralizes the effects of all the others, the system of forces is in equilibrium, and the particle is at rest.

The process of combining the effects of many forces, and of thereby determining one force which would produce an equal effect, is called *the composition of forces.* And as the process evidently admits of inversion, and the effect of one force may be decomposed into the effects of many forces acting simultaneously at the same point, so this latter process is called *the resolution* of a force. These processes will be very extensively employed in the sequel.

SECTION 2.—*The composition and resolution of many forces acting on a material particle, the lines of action of which are in one plane.*

16.] Let us first take the case of many forces acting on a particle along the same action-line, and in the same direction.

Let O, fig. 1, be the particle, and let OA be the line of action of all the forces, and let them act from O towards A. Let them be represented by the symbols $P_1, P_2, \ldots P_n$; then, since statical forces acting at a point along the same line and in the same direction are continuously additive, the resultant is equal to the sum of all. So that if R represents the resultant,

$$R = P_1 + P_2 + \ldots + P_n \tag{1}$$

$$= \Sigma . P, \tag{2}$$

where P is the type-symbol of a force, and Σ is the summation-symbol.

Again, suppose O to be acted on by two forces, along the same line, and in opposite directions: let them be P and Q, of which P is the greater: let P be resolved into two parts, Q and $P-Q$; then at the point O three forces act, viz. $P-Q$, Q, and Q, of which the last two act in opposite directions; therefore they neutralize each other; and, if R is the resultant, we have

$$R = P - Q. \tag{3}$$

And as a similar result is true for any number of forces acting in either direction, and along the same action-line, the equation (2) may be extended so as to include the *algebraical* sum of the forces acting on a point and along the same line.

Hence we infer that a particle is in equilibrium under the action of many forces acting along the same line, if the sum of those acting in one direction is equal to the sum of those acting in the opposite direction; and the condition of equilibrium is

$$\Sigma . P = 0. \tag{4}$$

Let us also take another simple case: that of three equal forces P, Q, R, see fig. 2, acting at O, all of which are in the same plane, and the lines of action of which are inclined to each other at 120°. Let the forces be represented, both in direction and in intensity, by the equal definite lines OP, OQ, OR: then the particle at O is in equilibrium: for by the principle of sufficient reason it cannot move out of the plane of the forces, neither can there be any resultant pressure in the plane; the particle therefore is in equilibrium; and either of the forces may be considered to be equal in magnitude to the resultant of the other two, and to act in the same line, but in an opposite direction. Hence we have the following geometrical construction of the resultant. Let P and Q be the components; then R neutralizes the effects of P and Q on O; produce RO to R′ so that OR′ is equal to OR; then the force of which OR′ is the geometrical representative neutralizes R; but the resultant of P and Q also neutralizes R: therefore the force R′ is the resultant of P and Q; and by the geometry OR′ is the diagonal of the parallelogram of which OP and OQ are the adjacent containing sides.

17.] The more general problem however is the determination as to action-line, direction, and magnitude, of the resultant of two forces acting on a particle. This proposition is commonly called *the parallelogram of forces* by reason of the geometrical form of it.

Let the meaning of the problem be clearly understood; it is required to determine the line of action, the direction, and the magnitude of a force which acting at a given point shall produce the same effect in all respects as two forces acting simultaneously at the same point.

It is evident by the principle of sufficient reason that the line of action of the resultant is in the same plane with the lines of action of the components.

[*] Let us first take the case of two *equal* forces P and P acting at O, and with their lines of action inclined at an angle 2θ. It is manifest that the line of action of the resultant bisects the angle contained between the lines of action of the components; because every reason which can be alleged why it should be on one side of this line is equally valid to prove that it should be on the other: and an integral part of the conception of a resultant is that it should be unique both as to line of action and as to magnitude; hence by the principle of sufficient reason we conclude that the line of action of the resultant bisects the angle between the lines of action of the components.

To determine the magnitude of the resultant. Let OP, OP_1 represent, see fig. 3, the two equal forces acting at O; let the angle $POP_1 = 2\theta$; let OR be the line of action of the resultant R, so that $POR = P_1OR = \theta$. Now the magnitude of R can depend on only P and θ; so that if f denotes a function which is to be determined,

$$R = f(P, \theta); \tag{5}$$

in this equation R and P are numbers depending on the arbitrarily chosen unit of force, and varying of course as the unit varies;

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[*] The following proof of the parallelogram of forces is due to M. Poisson, and commonly bears his name. A discussion, more or less complete, on 45 other proofs will be found in 'Præcipuorum inde a Neutono conatuum, compositionem virium demonstrandi, recensio. Auctore Carolo Jacobi. Gottingæ, MDCCCXVIII.'

but the law of relation between R, P, and θ cannot depend on this unit; in other words, the equation must be homogeneous in terms of P and R; therefore (5) must be such that the unit may be divided out, whatever be its magnitude; and this can only be the case when the equation is of the form

$$R = P f(\theta). \tag{6}$$

It remains for us to determine the form of f.

Suppose P to be the resultant of two equal forces Q and Q_1 acting at equal angles on the opposite sides of P's line of action; and let $QOP = Q_1OP = \phi$; therefore by (6),

$$P = Q f(\phi); \tag{7}$$

similarly let P_1 be the resultant of two forces Q and Q_1, equal to each other and to the former Qs, acting at equal angles ϕ on the opposite sides of P_1's line of action; so that

$$P_1 = Q f(\phi); \tag{8}$$

consequently from (6),

$$R = Q f(\theta) f(\phi). \tag{9}$$

Now R is the resultant of P and P_1; and therefore, as P and P_1 are the resultants of Q, Q, Q_1, and Q_1, R is the resultant of these also; let them be taken in pairs, so that R is the resultant of Q, Q, and of Q_1, Q_1; but by (6),

$$\left.\begin{aligned} &\text{the resultant of } Q,\ Q &&= Q f(\theta+\phi), \\ &\text{- - - - - -}\quad Q_1, Q_1 &&= Q f(\theta-\phi); \end{aligned}\right\} \tag{10}$$

therefore substituting in (9),

$$Q f(\theta+\phi) + Q f(\theta-\phi) = Q f(\theta) f(\phi),$$

$$\text{and}\quad f(\theta+\phi) + f(\theta-\phi) = f(\theta) f(\phi); \tag{11}$$

that is, the form of f is such as to satisfy the functional equation (11).

Expanding the left-hand member of (11) by Taylor's series, we have

$$2\left\{f(\theta) + f''(\theta)\frac{\phi^2}{1.2} + f''''(\theta)\frac{\phi^4}{1.2.3.4} + \dots\right\} = f(\theta) f(\phi);$$

$$\therefore\ f(\phi) = 2\left\{1 + \frac{f''(\theta)}{f(\theta)}\frac{\phi^2}{1.2} + \frac{f''''(\theta)}{f(\theta)}\frac{\phi^4}{1.2.3.4} + \dots\right\}; \tag{12}$$

but as no relation exists between θ and ϕ, θ is constant in reference to ϕ: therefore in (12), which is the expansion of $f(\phi)$, we may put, if a is constant,

$$\frac{f''(\theta)}{f(\theta)} = -a^2; \qquad \therefore\ \frac{f''''(\theta)}{f(\theta)} = a^4;$$

and so for the other terms;

$$\therefore \quad f(\phi) = 2\left\{1 - \frac{a^2\phi^2}{1.2} + \frac{a^4\phi^4}{1.2.3.4} - \dots\right\}$$

$$= 2\cos a\phi;$$

$$\therefore \quad f(\theta) = 2\cos a\theta; \tag{13}$$

$$\text{and} \qquad R = 2P\cos a\theta; \tag{14}$$

a is still undetermined; it must however be some uneven number, because $R = 0$, when $\theta = 90^\circ$, that is, when the two equal forces act in the same line and in opposite directions: and the uneven number can be none other than unity, because if it were 3 or 5, or ... or $2n+1$, R would vanish when $\theta = \frac{\pi}{6}, = \frac{\pi}{10}, = \dots, = \frac{\pi}{4n+2}$, and this would be absurd: therefore the functional relation between R, P, and θ is

$$R = 2P\cos\theta^*. \tag{15}$$

The form of function given in (13) evidently satisfies (11), because

$$2\cos a(\theta+\phi) + 2\cos a(\theta-\phi) = 4\cos a\theta\cos a\phi.$$

If I had assumed in the preceding $f''(\theta) = a^2 f(\theta)$, then

$$f(\phi) = 2\left\{1 + \frac{a^2\phi^2}{1.2} + \frac{a^4\phi^4}{1.2.3.4} + \dots\right\}$$

$$= e^{a\phi} + e^{-a\phi};$$

$$\text{so that } f(\theta) = e^{a\theta} + e^{-a\theta},$$

and thus $f(\theta)$ would increase without limit as θ increased without limit; and consequently R would increase indefinitely with θ. This of course cannot be the case, and the solution is accordingly excluded, and (15) is the only solution admissible by the conditions of the problem.

18.] The following is the geometrical interpretation of this theorem; Let OP and OP$_1$, fig. 4, represent the components in line of action, direction, and magnitude, so that POP$_1$ = 2θ; let OR bisect the angle P$_1$OP; from P draw PD perpendicular to OR, and produce OD to R, so that DR = OD; then OR = 2OP$\cos\theta$, and therefore OR by its length and direction represents the resultant of P and P$_1$; join PR, RP$_1$: then P$_1$OPR is manifestly a rhombus, of which OP, OP$_1$ are two adjacent sides, and OR is the diagonal.

If therefore two adjacent sides of a rhombus represent two forces acting at O, the diagonal of the rhombus abutting on O represents the resultant both as to line of action and intensity; hence also, since

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* Another mode of solving (11) is given in Ex. 7, Art. 456, Vol. II. (Integral Calculus).

$$\text{OR}^2 = \text{OP}^2 + \text{OP}_1{}^2 + 2\,\text{OP}.\text{OP}_1 \cos \text{P}_1\text{OP},$$

$$\text{R}^2 = 2\text{P}^2 + 2\text{P}^2 \cos 2\theta. \tag{16}$$

Hence also conversely we infer that a force acting on a particle may be equivalently replaced by two equal forces acting at equal angles on either side of its line of action if, R being the force to be replaced, P being one of the equal components of it, and θ being the angle between the lines of action of R and P,

$$\text{P} = \frac{\text{R}}{2} \sec \theta; \tag{17}$$

P therefore cannot be less than $\frac{\text{R}}{2}$; and increases as θ increases, and lastly becomes infinite when $\theta = 90°$: hence we infer that the effect of R on O cannot be produced by any force whose line of action is perpendicular to that of R; and therefore that two forces whose lines of action are perpendicular to each other do not affect each other's effects. As the theorem admits of the preceding geometrical interpretation, it has received the name of *the parallelogram of forces.*

19.] Let us in the next place take the case of two unequal forces P and Q acting at a point O, fig. 5, and along lines of action perpendicular to each other. Let P and Q be represented by the lines OP and OQ; complete the rectangle OPRQ, and draw the diagonal OR; let the angle ROP $= \alpha$; then the force P may by reason of the preceding Articles be resolved into two forces P′ and P′ acting at equal angles α on either side of OP, and by reason of (17),

$$\text{P}' = \frac{\text{P}}{2} \sec \alpha; \tag{18}$$

and therefore P′ is geometrically and equivalently represented by half of the diagonal OR. Again, let Q be resolved into two equal forces Q′ and Q′ acting at equal angles $90° - \alpha$ on each side of OQ, so that by reason of (17)

$$\text{Q}' = \frac{\text{Q}}{2} \operatorname{cosec} \alpha, \tag{19}$$

and therefore Q′ is geometrically and equivalently represented by half of the diagonal of the rectangle. Hence we have two forces, each of which is represented by half of OR, acting along OR and in the same direction, and of which therefore OR is the resultant both as to line of action and as to magnitude; and also two forces Q' and P' acting at O in the same line and in opposite directions: and as these are equal, both being represented by half of OR, they neutralize each other; and therefore the resultant of the two forces P and Q acting at O is represented by the diagonal of the rectangle of which the containing sides are the representatives of the components. Hence if R is the resultant

$$R^2 = P^2+Q^2; \tag{20}$$

and from (18) and (19),

$$R = P \sec a = Q \operatorname{cosec} a. \tag{21}$$

Hence also conversely, fig. 6; if a force P acts at O, and is represented in line of action, direction, and magnitude by the line OP; it may be resolved into two forces acting along two lines originating at O and perpendicular to each other; so that if X and Y are the resolved forces, and if the angle between the lines of action of P and X is θ, then by (21)

$$X = P\cos\theta, \qquad Y = P\sin\theta; \tag{22}$$

$$P^2 = X^2+Y^2. \tag{23}$$

Hence the resolved part of a force along any line is equal to the product of the force and the cosine of the angle between the given line and the action-line of the given force.

This theorem is most important, and is very frequently employed in subsequent investigations. By virtue of it forces may be resolved, or projected, according to the same law as lines and areas are projected. It is for this, with many other reasons, that the cosine of an angle is called *the projective coefficient.*

20.] Lastly, let us consider the case of two unequal forces P and Q acting on a point O, along lines of action inclined to each other at an angle γ; see fig. 7; let OP and OQ be the geometrical representatives of the forces, and let $QOP = \gamma$; complete the parallelogram QOPR, and draw the diagonal OR. Now resolve P into two forces P' and P'' along OR and perpendicularly to OR, and suppose $ROP = \theta$; then by (22),

$$P' = P\cos\theta, \qquad P'' = P\sin\theta; \tag{24}$$

so that by the geometry of the figure, OP' is the geometrical representative of P', and OP'' of P''. Again, resolve Q into two forces Q' and Q'', in lines along and at right angles to OR; then, by (22),

$$Q' = Q\cos(\gamma-\theta), \qquad Q'' = Q\sin(\gamma-\theta); \tag{25}$$

and therefore OQ' is the geometrical representative of Q', and

OQ'' of Q''. Now P'' and Q'' are manifestly equal, and act in the same line but in opposite directions; they therefore neutralize each other; and there remain P' and Q' acting along OR in the same direction, and therefore the resultant is equal to the sum of them, and is geometrically represented by $OP'+OQ'$, that is, by OR, which is the diagonal of the parallelogram of which OP and OQ are the containing sides; and since

$$\begin{aligned} OR^2 &= OP^2+PR^2-2.OP.PR\cos OPR \\ &= OP^2+OQ^2+2.OP.OQ\cos POQ; \qquad (26) \end{aligned}$$

therefore replacing the geometrical lines by their statical proportionals,

$$R^2 = P^2+Q^2+2PQ\cos\gamma. \qquad (27)$$

Evidently the former two cases are particular instances of this: for if

$$\gamma = 90^\circ, \qquad R^2 = P^2+Q^2;$$

$$\text{if} \quad P = Q, \qquad R = 2P\cos\frac{\gamma}{2}.$$

Hence in all cases we may enuntiate the theorem in the following form:

If two forces acting at a point are represented by two lines meeting at the point, the resultant is represented as to line of action, direction, and magnitude by the diagonal of the parallelogram of which the two lines are adjacent sides.

This theorem is, as above mentioned, called *the parallelogram of forces*, on account of the geometrical interpretation of it.

Hence, conversely, if any force R acts at a point O, it may be resolved into any two forces P and Q, whose lines of action are inclined at an angle γ, if P, Q, and γ satisfy the condition (27). And from (24) and (25), if θ is the angle between the action-lines of R and P, if we resolve P and Q along, and at right-angles to, the action-line of R,

$$\left.\begin{aligned} R &= P\cos\theta+Q\cos(\gamma-\theta), \\ P\sin\theta &- Q\sin(\gamma-\theta) = 0. \end{aligned}\right\} \qquad (28)$$

Hence, fig. 8, if a force R, equal to R', say, the resultant of P and Q, acts on a particle at O in the line OR', but in an opposite direction to R', the three forces P, Q, R are in equilibrium: and either force is equal to the resultant of the other two; and therefore if $QOR = \alpha$, $ROP = \beta$, $POQ = \gamma$,

$$\left.\begin{aligned} P^2 &= Q^2+2QR\cos\alpha+R^2, \\ Q^2 &= R^2+2RP\cos\beta+P^2, \\ R^2 &= P^2+2PQ\cos\gamma+Q^2. \end{aligned}\right\} \qquad (29)$$

Hence also it is plain that a force acting at a given point may be resolved into two forces whose lines of action pass each through a given point, if the three points and the action-line of the given force are in one plane.

21.] Also since the three equilibrating forces P, Q, R are proportional to the three lines OP, OQ, OR, or to OP, PR′, R′O; and since the three sides of a triangle are proportional to the sines of the opposite angles, therefore

$$\frac{\mathrm{P}}{\sin \mathrm{OR'P}} = \frac{\mathrm{Q}}{\sin \mathrm{R'OP}} = \frac{\mathrm{R}}{\sin \mathrm{OPR'}};$$

$$\text{or} \quad \frac{\mathrm{P}}{\sin \alpha} = \frac{\mathrm{Q}}{\sin \beta} = \frac{\mathrm{R}}{\sin \gamma}; \tag{30}$$

that is, if three forces acting at a point are in equilibrium, each is proportional to the sine of the angle contained between the lines of action of the other two.

From (30) we infer that three forces acting at a point are in equilibrium, if they are proportional to the three sides of any triangle whose sides are parallel to the lines of action of the forces, and if their directions are those of a point traversing the perimeter of the triangle. This theorem is known by the name of the triangle of forces.

22.] Also from the second equation in (28) it appears that if p and q are the lengths of the perpendiculars drawn from any point in the line of action of R to the lines of action of P and Q, then

$$\frac{p}{q} = \frac{\sin \theta}{\sin (\gamma - \theta)} = \frac{\mathrm{Q}}{\mathrm{P}};$$

$$\therefore \quad \mathrm{P}p = \mathrm{Q}q. \tag{31}$$

And thus if P_1 and P_2 are forces acting at a given point along lines of action, the equations to which are

$$\left.\begin{aligned} x \cos \alpha_1 + y \sin \alpha_1 - \delta_1 &= 0, \\ x \cos \alpha_2 + y \sin \alpha_2 - \delta_2 &= 0, \end{aligned}\right\} \tag{32}$$

which we may represent by the abridged notation $\alpha_1 = 0$, and $\alpha_2 = 0$; then attaching the proper signs to α_1 and α_2, the equation to the line of action of the resultant is

$$\mathrm{P}_1 \alpha_1 + \mathrm{P}_2 \alpha_2 = 0. \tag{33}$$

The product of a force and the perpendicular from a given point on the action-line of the force is called *the moment of the force* with reference to the given point, and denotes a certain property of the force which will be explained at length hereafter; consequently (33) contains the following theorem;

The moments of the components are equal with reference to any point in the action-line of the resultant.

23.] Let us next consider the general case of many forces acting at a given point, the lines of action of all of which are in one plane.

Let O be the point at which all the forces act: and through it let two lines, as coordinate axes, be drawn perpendicular to each other, and in the plane in which the forces act.

Let the force be $P_1, P_2, \dots P_n$, of which let P be the type-force: and let the angles between the x-axis and their action-lines severally be $a_1, a_2, \dots a_n$, of which let a be the type-angle; and let the several forces be resolved along the axes of x and y: then by equations (22), Art. 19, the resolved parts along the x-axis severally are $P_1 \cos a_1$, $P_2 \cos a_2$, ... $P_n \cos a_n$; and those along the y-axis are $P_1 \sin a_1$, $P_2 \sin a_2$, ... $P_n \sin a_n$; and therefore if X and Y denote the forces along the axes of x and y respectively,

$$\left.\begin{aligned} X &= P_1 \cos a_1 + P_2 \cos a_2 + \dots + P_n \cos a_n \\ &= \Sigma . P \cos a. \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} Y &= P_1 \sin a_1 + P_2 \sin a_2 + \dots + P_n \sin a_n \\ &= \Sigma . P \sin a. \end{aligned}\right\} \tag{35}$$

Let R be the resultant of all the forces acting at O, and θ the angle which its line of action makes with the axis of x; then as R produces at O the same effect as to magnitude, line of action, and direction as all the impressed pressures taken in combination, so are the resolved parts of R along the axes equal severally to X and Y: consequently

$$\left.\begin{aligned} R \cos \theta &= X = \Sigma . P \cos a, \\ R \sin \theta &= Y = \Sigma . P \sin a; \end{aligned}\right\} \tag{36}$$

$$\therefore \quad R^2 = X^2 + Y^2; \qquad \tan \theta = \frac{Y}{X}; \tag{37}$$

$$\frac{\sin \theta}{Y} = \frac{\cos \theta}{X} = \frac{1}{R}; \tag{38}$$

and hereby may the magnitude, line of action, and direction of the resultant of many forces acting in one plane on a given particle be determined.

If the forces are so related that the particle is at rest, then the resultant vanishes; and

$$R^2 = X^2 + Y^2 = 0; \tag{39}$$

$$\therefore \quad X = \Sigma . P \cos a = 0, \quad Y = \Sigma . P \sin a = 0. \tag{40}$$

As the conditions of equilibrium must be independent of the particular system of coordinate axes, we infer that, if many forces acting on a particle in one plane are in equilibrium, the sum of the resolved parts of the forces along every straight line is equal to zero.

24.] The following examples are in illustration of the preceding theorems.

Ex. 1. Four equal forces whose directions are inclined to the axis of x at angles of 15°, 75°, 135° and 225° act at a point: determine the magnitude and direction of their resultant.

Let each pressure be equal to P; then

$$\begin{aligned} \mathrm{X} &= \mathrm{P}\cos 15^{\circ} + \mathrm{P}\cos 75^{\circ} + \mathrm{P}\cos 135^{\circ} + \mathrm{P}\cos 225^{\circ} \\ &= \mathrm{P}\,\frac{3^{\frac{1}{2}}-2}{2^{\frac{1}{2}}}; \\ \mathrm{Y} &= \mathrm{P}\sin 15^{\circ} + \mathrm{P}\sin 75^{\circ} + \mathrm{P}\sin 135^{\circ} + \mathrm{P}\sin 225^{\circ} \\ &= \mathrm{P}\left(\frac{3}{2}\right)^{\frac{1}{2}}; \end{aligned}$$

$$\therefore \quad \mathrm{R} = \mathrm{P}\,\{5-2.3^{\frac{1}{2}}\}^{\frac{1}{2}}; \qquad \tan\theta = \frac{3^{\frac{1}{2}}}{3^{\frac{1}{2}}-2}.$$

Ex. 2. Three forces act perpendicularly to the sides of a triangle at their middle points, and are proportional to the sides; it is required to prove that they are in equilibrium.

Let ABC, fig. 9, be the triangle, and let the forces be P, Q, R, and act in the directions indicated by the arrow-heads; their lines of action meet at the point O; let QOR $= \alpha$, ROP $= \beta$, POQ $= \gamma$; α, β, γ being manifestly the supplements of A, B, C; then by the data

$$\frac{\mathrm{P}}{a} = \frac{\mathrm{Q}}{b} = \frac{\mathrm{R}}{c} = k \text{ (say)}; \tag{41}$$

and since the sides are proportional to the sines of the opposite angles,

$$\frac{\mathrm{P}}{\sin \mathrm{A}} = \frac{\mathrm{Q}}{\sin \mathrm{B}} = \frac{\mathrm{R}}{\sin \mathrm{C}};$$

$$\therefore \quad \frac{\mathrm{P}}{\sin \alpha} = \frac{\mathrm{Q}}{\sin \beta} = \frac{\mathrm{R}}{\sin \gamma};$$

and therefore by (30), P, Q, R are in equilibrium.

Or thus resolving along BC;

$$\begin{aligned} \text{The forces along BC} &= \mathrm{Q}\sin \mathrm{C} - \mathrm{R}\sin \mathrm{B} \\ &= k\,\{b\sin \mathrm{C} - c\sin \mathrm{B}\}, \text{ by (41)}, \\ &= 0; \end{aligned}$$

and similarly will the sum of the resolved parts of the forces along any other line vanish. And therefore the system is in equilibrium.

Ex. 3. If R is the resultant of P and Q acting at O, fig. 7, and A is any point in the plane POQ, from which perpendiculars Ap, Aq, Ar are drawn to OP, OQ, OR respectively, then

$$(1)\quad \text{P}.\text{A}p+\text{Q}.\text{A}q = \text{R}.\text{A}r;$$

$$(2)\quad \text{P}.\text{O}p+\text{Q}.\text{O}q = \text{R}.\text{O}r.$$

Join AO, and let AOP $= \theta$. Let P, Q, R be resolved along and perpendicularly to AO; then as R is in all respects equivalent to P and Q in combination, the component of R along any line is equal to the sum of the components of P and Q: consequently

$$\text{P} \sin \text{AO}p + \text{Q} \sin \text{AO}q = \text{R} \sin \text{AO}r,$$

$$\text{P} \cos \text{AO}p + \text{Q} \cos \text{AO}q = \text{R} \cos \text{AO}r:$$

and replacing the sines and cosines by their geometrical values, we have

$$(1)\quad \text{P}.\text{A}p+\text{Q}.\text{A}q = \text{R}.\text{A}r;$$

$$(2)\quad \text{P}.\text{O}p+\text{Q}.\text{O}q = \text{R}.\text{O}r.$$

(1) is the theorem of the equivalence of moments which has already been proved analytically in Art. 22; and (2) is the theorem of virtual velocities the general investigation of which will be made hereafter.

Hence also if P, Q, R are three forces which equilibrate at O, and A is another point in the plane PQRO from which Ap, Aq, Ar are drawn perpendicular to the action-lines of P, Q, R respectively,

$$\text{P}.\text{A}p+\text{Q}.\text{A}q+\text{R}.\text{A}r = 0,$$

$$\text{P}.\text{O}p+\text{Q}.\text{O}q+\text{R}.\text{O}r = 0.$$

Hence also generally if many forces $\text{P}_1, \text{P}_2, \dots \text{P}_n$ equilibrate at O,

$$\Sigma.\text{P}\times\text{A}p = 0; \qquad \Sigma.\text{P}\times\text{O}p = 0.$$

25.] In the application of the preceding principles, statical forces often arise from (1) the determinate tension of strings, (2) reacting pressures. It is worth while to say a few words on each of these cases.

Suppose in fig. 1 OA to be a string, fastened at O, and pulled at its other extremity with a certain force = P; then it is (experimentally) plain that O is pulled with a force equal to that exerted on the string at A, and that the tension of the string is the same throughout; the line of the string of course expresses the *line* in which the pressure acts on O, but the length of it is not a measure of the intensity of the pull, although a length may be taken along it which shall be proportional to that intensity. One or two examples, in which such pressures are involved, are subjoined.

Ex. 1. A and B, fig. 10, are two fixed points in a horizontal line; at A is fastened a string of length c with a smooth ring at its other extremity C, through which passes another string fastened at one end at B; the other end of which is attached to a given weight W; it is required to determine the position of C.

Let $\text{AB} = 2a$, $\text{AC} = \text{C}$, $\text{CAB} = \theta$, $\text{ABC} = \phi$. Let the tension of the string $\text{AC} = \text{T}$; which is undetermined. Now as the ring at C is smooth, the tension of WCB is the same throughout, and is of course equal to the weight W; and therefore C is kept at rest by three forces, W, W, and T; let us apply equations (40) and resolve the forces horizontally and vertically; and equate those acting towards the right-hand to those acting towards the left; and those acting upwards to those acting downwards. Then

the horizontal forces are, $\text{W} \cos\phi = \text{T} \cos\theta$;

and the vertical forces are, $\text{W} \sin\phi + \text{T} \sin\theta = \text{W}$.

Therefore eliminating T,

$$\cos\theta = \sin(\theta+\phi);$$

$$\therefore \quad 2\theta+\phi = 90°. \tag{42}$$

Also from the geometry

$$\frac{\sin(\theta+\phi)}{\sin\phi} = \frac{2a}{c}; \tag{43}$$

from (42) and (43) θ and ϕ may be found: and thence T may be determined; and thus all the circumstances of the problem are determined.

Ex. 2. A and B are two points in a horizontal line; a string fastened at A, fig. 11, passes over a small pulley at B, and supports at its other end a weight W; a small and smooth heavy ring of weight W′ slides on the string between A and B; determine the position in which the string rests.

Let C be the point at which the heavy ring rests: as the pully is smooth, and has no friction, and as the ring is also smooth, the tension of the string is the same throughout and is equal to the weight of W; hence the point C is kept in equilibrium by three forces, W along CA, W along CB, and W′ which acts vertically downwards: let $\text{CAB} = \theta$, $\text{CBA} = \phi$; therefore, taking horizontal and vertical forces, we have

Horizontal forces; $\quad w \cos\theta = w \cos\phi;$

Vertical forces; $\quad w \sin\theta + w \sin\phi = w';$

$$\therefore \quad \theta = \phi = \sin^{-1}\frac{w'}{2w}.$$

26.] Again, suppose the particle, on which the statical forces act, to be on a smooth plane surface, which is capable of bearing the resultant of the component forces which acts along the normal and in a direction towards the plane; but by reason of its smoothness does not offer any resistance to motion in the direction of its surface; then, since the actual normal pressure of such a plane is equal, and in direction opposite, to that impressed on it by the component forces, this normal reaction of the plane is one of the forces by which such a material particle is kept at rest, and, as such, will enter into the equations of equilibrium.

Ex. 1. A particle of weight w is kept at rest on a smooth inclined plane by a force P acting at a given angle to the plane; determine the pressure on the plane, and the magnitude of P.

Let fig. 12 be a vertical section of the system; AC the inclined plane; $CAB = \alpha$, $PQC = \beta$, $R =$ the reaction of the plane against the particle Q: then, as the lines along which forces may be resolved are arbitrary, let us resolve along, and perpendicularly to, the plane. Then we have

Forces along the plane, $\quad P\cos\beta = W \sin\alpha;$

Forces perpendicular to the plane, $\quad R + P\sin\beta = W\cos\alpha;$

$$\therefore \quad P = W\frac{\sin\alpha}{\cos\beta}; \qquad R = W\frac{\cos(\alpha+\beta)}{\cos\beta}.$$

The force P therefore acts to the greatest advantage, that is, W is the greatest, when $\beta = 0$.

Ex. 2. Two forces P and Q acting respectively parallel to the base and length of an inclined plane will each singly sustain on it a particle of weight W; to determine the weight of W.

Let α be the inclination of the plane to the horizon; then in each case resolving along the plane, so that the normal pressures may not enter into the equations,

$$P\cos\alpha = W\sin\alpha, \qquad Q = W\sin\alpha;$$

$$\therefore \quad W = \frac{PQ}{(P^2 - Q^2)^{\frac{1}{2}}}.$$

The case of this Article is a particular one of the general theory of a constrained particle which is fully discussed in Art. 32.

27.] The resultant of forces acting at a point in one plane must be, as to line of action and intensity, independent of the particular origin and the particular system of coordinates; and we may in the following manner deduce this property from the preceding results:

$$\therefore \quad \left.\begin{aligned} \mathrm{X} &= \Sigma.\mathrm{P}\cos\alpha = \mathrm{P}_1\cos\alpha_1+\mathrm{P}_2\cos\alpha_2+\ldots+\mathrm{P}_n\cos\alpha_n, \\ \mathrm{Y} &= \Sigma.\mathrm{P}\sin\alpha = \mathrm{P}_1\sin\alpha_1+\mathrm{P}_2\sin\alpha_2+\ldots+\mathrm{P}_n\sin\alpha_n, \end{aligned}\right\} \tag{44}$$

$$\begin{aligned} \therefore \quad \mathrm{R}^2 &= \mathrm{X}^2+\mathrm{Y}^2 \\ &= \mathrm{P}_1^2+\mathrm{P}_2^2+\ldots+\mathrm{P}_n^2 \\ &\qquad +2\,\{\mathrm{P}_1\mathrm{P}_2\cos(\alpha_1-\alpha_2)+\mathrm{P}_1\mathrm{P}_3\cos(\alpha_1-\alpha_3)+\ldots \\ &\qquad\qquad \ldots+\mathrm{P}_{n-1}\mathrm{P}_n\cos(\alpha_{n-1}-\alpha_n)\} \qquad (45) \\ &= \Sigma.\mathrm{P}^2+2\,\Sigma.\mathrm{PP}'\cos(\alpha-\alpha'), \qquad (46) \end{aligned}$$

where P, P′ are the symbols for any two of the forces, and $\alpha-\alpha'$ is the angle contained between their lines of action; and the sign of summation prefixed to $\mathrm{PP}'\cos(\alpha-\alpha')$ indicates the sum of the products corresponding to the n forces taken two and two together; and therefore (46) is independent of the system of coordinate axes. The parallelogram of forces which is given in equation (27) is a particular case of (46). A further generalization of this theorem is given in (68), Art. 31.

28.] We have also the following relation between the several components and their lines of action, and any point in the line of action of the resultant.

Let the equations to the lines of action of the components be

$$\left.\begin{aligned} x\cos\alpha_1+y\sin\alpha_1 &= 0 = \alpha_1, \\ x\cos\alpha_2+y\sin\alpha_2 &= 0 = \alpha_2, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x\cos\alpha_n+y\sin\alpha_n &= 0 = \alpha_n; \end{aligned}\right\} \tag{47}$$

the point at which they act being the origin, α being the angle between the axis of x and the normal to the line of action; and the α on the right-hand side of the equation being the length of the perpendicular from the point (x, y) to the line.

Now if the components are $\mathrm{P}_1, \mathrm{P}_2, \ldots \mathrm{P}_n$, and the resultant is R, and α is the angle between the normal to R's direction and the axis of x, then the equation to R's line of action is

$$x\cos\alpha+y\sin\alpha = 0; \tag{48}$$

$$\therefore \quad x\mathrm{R}\cos\alpha+y\mathrm{R}\sin\alpha = 0;$$

but $\quad \mathrm{R}\cos\alpha = \Sigma.\mathrm{P}\cos\alpha, \qquad \mathrm{R}\sin\alpha = \Sigma.\mathrm{P}\sin\alpha;$

$$\therefore \quad x\Sigma.\mathrm{P}\cos\alpha+y\Sigma.\mathrm{P}\sin\alpha = 0;$$

$$\therefore\quad \text{P}_1\{x\cos\alpha_1+y\sin\alpha_1\}+\text{P}_2\{x\cos\alpha_2+y\sin\alpha_2\}+\ldots$$
$$\ldots+\text{P}_n\{x\cos\alpha_n+y\sin\alpha_n\} = 0; \qquad (49)$$
$$\text{P}_1 a_1+\text{P}_2 a_2+\ldots+\text{P}_n a_n = 0, \qquad (50)$$

where $a_1, a_2, \ldots a_n$ are the perpendiculars from (x, y), any point in the line of action of R, on the lines of action of the components; therefore, bearing in mind the meaning of the word moment as given in Art. 22, we have the following theorem;

With reference to any point in the line of action of the resultant, the sum of the moments of the components vanishes.

The theorem given in (33) Art. 22 is a particular case of the preceding.

The following also is a more general theorem; if (x, y) is a point in the plane of the forces but not on the resultant, then $x\cos\alpha+y\sin\alpha =$ the perpendicular distance from (x, y) on the line of action of R: let this $= r$; then from (49),

$$\text{P}_1 a_1+\text{P}_2 a_2+\ldots+\text{P}_n a_n = \text{R}r;$$

that is, with reference to any point in the plane of the forces the sum of the moments of the components is equal to the moment of the resultant.

Hence if two forces only act, as is the case in the parallelogram of forces, with reference to any point in the plane of the forces, the moments of the resultant is equal to the sum of the moments of the components.

As the moment is the product of the line-representative of the force and of the perpendicular on the action-line of the force from a given point, it expresses geometrically twice the area of the triangle of which the given point is the vertex and the line-representative of the force is the base. Hence, in fig. 7, if A is any point in the plane POQR, and if AO, AP, AQ, AR are drawn, the triangle AOR is equal to the sum of the two triangles AOP and AOQ. This is easily demonstrated geometrically.

29.] The following is another geometrical interpretation of the conditions of equilibrium in equations (40).

It is a well-known property of a closed polygon that the sum of the projections of its sides on any given straight line is zero; the projections of the sides being affected with positive or negative signs according as the angles made by them with the given straight line are acute or obtuse, and care being taken to estimate the angles between the given line and the sides of the polygon which are turned all towards the inside or all towards the outside of the figure. Hence, if $l_1, l_2, \dots l_n$ are the lengths of the sides, and $a_1, a_2, \dots a_n$ are the angles between them and the given straight line,

$$\Sigma . l \cos a = 0. \tag{51}$$

Now if n forces act at a point, the condition of equilibrium is

$$\Sigma . \text{P} \cos a = 0. \tag{52}$$

Hence if n forces, having their lines of action parallel to the successive sides of a closed polygon, their directions the same as that of a point traversing the sides of the polygon, and their magnitudes represented by the lengths of those sides, act at a point, (51) assumes the analogous mechanical form (52), and the forces are in equilibrium: hence conversely, if many pressures whose action-lines are in one plane act at a point and are in equilibrium, their action-lines are parallel to the sides of a closed polygon, the sides being proportional to the magnitudes of the forces; or in other words, the line-representatives of a system of forces, acting in equilibrium and in one plane at a point, will form the contour of a closed polygon, the sides of which taken in order are equal and parallel to these line-representatives taken in the same order.

This proposition is known by the name of the polygon of forces, and the triangle of forces proved in Article 21 is a particular case of it.

SECTION 3.—*Composition and resolution of forces acting in any directions on a material particle.*

30.] Here and elsewhere we shall refer the effects of forces acting in space to a system of rectangular coordinates; because the results are not more general, and are much more complicated, when they are referred to a system of oblique axes. And let us in the first place take the case of three forces X, Y, Z acting at the origin O, see fig. 13, and along the coordinate axes. Let the resultant of X and Y, which are at right-angles to each other in the plane of (x, y), be R′; then, by (20), Art. 17,

$$\text{R}'^2 = \text{X}^2 + \text{Y}^2.$$

Again, of R′ and Z, which are at right-angles to each other, let the resultant be R; then

$$\begin{aligned} \text{R}^2 &= \text{R}'^2 + \text{Z}^2 \\ &= \text{X}^2 + \text{Y}^2 + \text{Z}^2; \end{aligned} \tag{53}$$

and R is the resultant of the three forces. Let the direction-angles of its line of action be a, b, c; then, by equation (22),

$$\text{X} = \text{R}\cos a, \qquad \text{Y} = \text{R}\cos b, \qquad \text{Z} = \text{R}\cos c. \tag{54}$$

Hence, conversely, any force P, acting at O, the direction-angles of whose line of action are α, β, γ, may be resolved into three forces X, Y, Z acting along the coordinate axes, such that

$$\text{X} = \text{P}\cos \alpha, \qquad \text{Y} = \text{P}\cos \beta, \qquad \text{Z} = \text{P}\cos \gamma. \tag{55}$$

31] Next let us take the case of many forces acting in any lines at the point O.

Let the forces be $\text{P}_1, \text{P}_2, \ldots \text{P}_n$; and let the direction-angles of their lines of action be $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; $\ldots \alpha_n, \beta_n, \gamma_n$; let these be resolved severally along the coordinate axes, and let X, Y, Z be the sums of the resolved parts along the axes respectively of x, y, and z; then

$$\left.\begin{aligned} \text{X} &= \text{P}_1 \cos\alpha_1 + \text{P}_2\cos\alpha_2 + \ldots + \text{P}_n\cos\alpha_n \\ &= \Sigma.\text{P}\cos\alpha; \\ \text{Y} &= \Sigma.\text{P}\cos\beta; \\ \text{Z} &= \Sigma.\text{P}\cos\gamma. \end{aligned}\right\} \tag{56}$$

Let R be the resultant of all the impressed forces; and let the direction-angles of its line of action be a, b, c; then as the resolved parts of R along the three coordinate axes are equal to the sum of the resolved parts of the several components along the same lines,

$$\text{R}\cos a = \text{X}, \qquad \text{R}\cos b = \text{Y}, \qquad \text{R}\cos c = \text{Z}; \tag{57}$$

and squaring and adding,

$$\text{R}^2 = \text{X}^2 + \text{Y}^2 + \text{Z}^2; \tag{58}$$

$$\cos a = \frac{\text{X}}{\text{R}}, \qquad \cos b = \frac{\text{Y}}{\text{R}}, \qquad \cos c = \frac{\text{Z}}{\text{R}}; \tag{59}$$

and the equations to the line of action of the resultant are

$$\frac{x}{\Sigma.\text{P}\cos\alpha} = \frac{y}{\Sigma.\text{P}\cos\beta} = \frac{z}{\Sigma.\text{P}\cos\gamma}. \tag{60}$$

Also from (58),
$$\begin{aligned} \text{R} &= \text{X}\frac{\text{X}}{\text{R}} + \text{Y}\frac{\text{Y}}{\text{R}} + \text{Z}\frac{\text{Z}}{\text{R}} \\ &= \text{X}\cos a + \text{Y}\cos b + \text{Z}\cos c, \end{aligned}$$

that is, R is equal to the sum of the forces along the coordinate axes resolved along the line of action of R.

If the point at which all the forces act is (x', y', z'), so that the equations to the lines of action of the components are

$$\left.\begin{array}{l}
\dfrac{x-x'}{\cos\alpha_1}=\dfrac{y-y'}{\cos\beta_1}=\dfrac{z-z'}{\cos\gamma_1},\\[2ex]
\dfrac{x-x'}{\cos\alpha_2}=\dfrac{y-y'}{\cos\beta_2}=\dfrac{z-z'}{\cos\gamma_2},\\[1ex]
\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\\[1ex]
\dfrac{x-x'}{\cos\alpha_n}=\dfrac{y-y'}{\cos\beta_n}=\dfrac{z-z'}{\cos\gamma_n};
\end{array}\right\}\qquad(61)$$

then the equations to the line of action of the resultant are

$$\frac{x-x'}{\Sigma.\text{P}\cos\alpha}=\frac{y-y'}{\Sigma.\text{P}\cos\beta}=\frac{z-z'}{\Sigma.\text{P}\cos\gamma}.\qquad(62)$$

32.] Now from the point at which the forces act, let straight lines be drawn, which are in length and direction geometrical representatives of the forces: and let the extremities of these lines be (x_1, y_1, z_1), (x_2, y_2, z_2), ... (x_n, y_n, z_n), and let their lengths be s_1, s_2, ... s_n; then

$$\left.\begin{array}{l}
x'-x_1=s_1\cos\alpha_1,\\
y'-y_1=s_1\cos\beta_1,\\
z'-z_1=s_1\cos\gamma_1;
\end{array}\right\}\quad
\left.\begin{array}{l}
x'-x_2=s_2\cos\alpha_2,\\
y'-y_2=s_2\cos\beta_2,\\
z'-z_2=s_2\cos\gamma_2;
\end{array}\right\}\cdots\cdots\qquad(63)$$

and

$$\left.\begin{array}{l}
\Sigma.\text{P}\cos\alpha=\Sigma(x'-x)=nx'-(x_1+x_2+\ldots+x_n),\\
\Sigma.\text{P}\cos\beta=\Sigma(y'-y)=ny'-(y_1+y_2+\ldots+y_n),\\
\Sigma.\text{P}\cos\gamma=\Sigma(z'-z)=nz'-(z_1+z_2+\ldots+z_n),
\end{array}\right\}\qquad(64)$$

and therefore (62) become

$$\frac{x-x'}{x'-\dfrac{x_1+x_2+\ldots+x_n}{n}}=\frac{y-y'}{y'-\dfrac{y_1+y_2+\ldots+y_n}{n}}=\frac{z-z'}{z'-\dfrac{z_1+z_2+\ldots+z_n}{n}},\qquad(65)$$

which are the equations to the line of action of the resultant. The point whose coordinates are

$$\frac{x_1+x_2+\ldots+x_n}{n},\quad\frac{y_1+y_2+\ldots+y_n}{n},\quad\frac{z_1+z_2+\ldots+z_n}{n},$$

is that which is known by the name of the geometrical *centre of mean distances* of the points which are the extremities of the line-representatives of the forces: and therefore from (65) it appears that the line of action of the resultant passes through this point.

33.] Also the magnitude of the resultant of the pressures, which is of course independent of the particular system of coordinate axes, may thus be found; since

$$\left.\begin{array}{l}
\text{X}=\text{P}_1\cos\alpha_1+\text{P}_2\cos\alpha_2+\ldots+\text{P}_n\cos\alpha_n,\\
\text{Y}=\text{P}_1\cos\beta_1+\text{P}_2\cos\beta_2+\ldots+\text{P}_n\cos\beta_n,\\
\text{Z}=\text{P}_1\cos\gamma_1+\text{P}_2\cos\gamma_2+\ldots+\text{P}_n\cos\gamma_n;
\end{array}\right\}\qquad(66)$$

$$
\begin{aligned}
\therefore\ \mathrm{R}^2 = {} & \mathrm{P}_1^2+\mathrm{P}_2^2+\ldots+\mathrm{P}_n^2 \\
& +2\mathrm{P}_1\mathrm{P}_2\{\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2\} \\
& + \ldots\ldots\ldots\ldots\ldots\ldots \\
& +2\mathrm{P}_{n-1}\mathrm{P}_n\{\cos\alpha_{n-1}\cos\alpha_n+\cos\beta_{n-1}\cos\beta_n+\cos\gamma_{n-1}\cos\gamma_n\} \qquad (67) \\
= {} & \Sigma.\mathrm{P}^2+2\Sigma.\mathrm{PP}'\cos(\mathrm{P},\mathrm{P}'), \qquad (68)
\end{aligned}
$$

where P, P' are the symbols for any two of the forces, and $\cos(\mathrm{P},\mathrm{P}')$ is the cosine of the angle between their lines of action. And from the forms, which the resolved parts of R take in equations (64), it follows that the geometrical representative of it is n times the length of the line joining the point of application of the forces and the centre of mean distances of the extremities of the geometrical representatives of them. This theorem is due to M. Chasles, and is the true generalization of the parallelogram of forces.

34.] If the forces are in equilibrium, $\mathrm{R} = 0$; in which case, by reason of (58), $\mathrm{X} = 0$, $\mathrm{Y} = 0$, $\mathrm{Z} = 0$; or,

$$\Sigma.\mathrm{P}\cos\alpha = 0, \qquad \Sigma.\mathrm{P}\cos\beta = 0, \qquad \Sigma.\mathrm{P}\cos\gamma = 0; \qquad (69)$$

that is, the sum of the resolved parts of the forces along each of three coordinate axes is equal to zero.

35.] We have thus far employed rectangular coordinate axes, and have in reference to them proved that a force may be resolved into three components whose lines of action are at right angles to each other, and that these three forces equivalently replace the given force. A force may however be equivalently replaced by three forces whose action-lines meet on a point in its action-line, provided that the action-lines of these three forces are not in one and the same plane. To demonstrate this theorem, let P be the force, and let X, Y, Z be its axial components; and let (l_1, m_1, n_1) (l_2, m_2, n_2) (l_3, m_3, n_3) be the three straight lines along which the forces $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$ are to act, and which are equivalently to replace P; then

$$
\begin{aligned}
\mathrm{X} &= \mathrm{P}_1 l_1 + \mathrm{P}_2 l_2 + \mathrm{P}_3 l_3, \\
\mathrm{Y} &= \mathrm{P}_1 m_1 + \mathrm{P}_2 m_2 + \mathrm{P}_3 m_3, \\
\mathrm{Z} &= \mathrm{P}_1 n_1 + \mathrm{P}_2 n_2 + \mathrm{P}_3 n_3;
\end{aligned}
$$

from which equations, if they are independent of each other, $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$ can be determined. If however the action-lines of $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$ are in the same plane, and the action-line of P does not lie in this plane, then, employing the symbols of determinants, $\Sigma \pm l_1\, m_2\, n_3 = 0$, and $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$ are infinite, and the proposed equivalent substitution is impossible. The values of P_1, P_2, P_3 are indeterminate if their action-lines and that of P are in the same plane.

SECTION 4.—*Conditions of equilibrium of many forces acting on a particle which is in contact with a smooth surface or a smooth curve.*

36.] Let us first take the case of a smooth surface, and suppose a particle acted on by many forces to be in contact with it at a given point. As the surface is smooth, the only direction along which it can offer any resistance to the particle's motion is that of its normal; and as it is conceived to have no active power of its own, but only a capacity of resisting any force that acts against it along its normal, so must the resultant of the impressed forces act along the normal and towards the surface: these conditions therefore are sufficient for the equilibrium of the particle.

Let the equation to the surface be

$$F(x, y, z) = 0; \qquad (70)$$

and employing the same notation as in Art. 332, Vol. I. (Differential Calculus), and Art. 236, Vol. II. (Integral Calculus), let

$$\left(\frac{dF}{dx}\right) = U, \quad \left(\frac{dF}{dy}\right) = V, \quad \left(\frac{dF}{dz}\right) = W, \qquad (71)$$

$$U^2 + V^2 + W^2 = Q^2;$$

so that if λ, μ, ν are the direction-cosines of the normal at (x, y, z),

$$\cos\lambda = \frac{U}{Q}, \quad \cos\mu = \frac{V}{Q}, \quad \cos\nu = \frac{W}{Q};$$

then as this line is to be coincident with the line of action of the resultant of the acting forces, whose direction-cosines are proportional to X, Y, Z, the conditions of equilibrium are

$$\frac{X}{U} = \frac{Y}{V} = \frac{Z}{W}; \qquad (72)$$

and if these equations are not, and cannot be, satisfied, equilibrium on the surface cannot exist. Consequently the point on a given surface, at which a particle under the action of given forces will rest in equilibrium, is the point on the surface at which the preceding equations are satisfied.

The normal pressure of the surface, which arises from the action of the impressed forces, may thus be determined. Let

N represent the normal pressure; then the resolved parts of it along the coordinate axes are

$$\text{N}\frac{\text{U}}{\text{Q}}, \quad \text{N}\frac{\text{V}}{\text{Q}}, \quad \text{N}\frac{\text{W}}{\text{Q}};$$

and these together with the acting forces must be in equilibrium: therefore

$$\left.\begin{aligned} \Sigma.\text{P}\cos\alpha &= \text{X} = \text{N}\frac{\text{U}}{\text{Q}}, \\ \Sigma.\text{P}\cos\beta &= \text{Y} = \text{N}\frac{\text{V}}{\text{Q}}, \\ \Sigma.\text{P}\cos\gamma &= \text{Z} = \text{N}\frac{\text{W}}{\text{Q}}; \end{aligned}\right\}$$

whence, squaring and adding,

$$\text{N}^2 = \text{X}^2 + \text{Y}^2 + \text{Z}^2.$$

If the normal resistance of the surface acts in only one direction, the resultant of the acting forces must act in the direction opposite to that of the resistance. We subjoin some examples of the preceding formulae.

Ex. 1. A particle is placed on the surface of an ellipsoid and is acted on by attracting forces which vary directly as the distance of the particle from the principal planes of section; it is required to determine the position of equilibrium.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 = \text{F}(x, y, z);$$

$$\therefore \quad \text{U} = \frac{2x}{a^2}, \quad \text{V} = \frac{2y}{b^2}, \quad \text{W} = \frac{2z}{c^2};$$

let $\text{X} = \mu_1 x, \quad \text{Y} = \mu_2 y, \quad \text{Z} = \mu_3 z;$

then equations (72) become

$$\frac{\mu_1}{a^{-2}} = \frac{\mu_2}{b^{-2}} = \frac{\mu_3}{c^{-2}} = \frac{\mu_1 + \mu_2 + \mu_3}{a^{-2} + b^{-2} + c^{-2}};$$

if these conditions are fulfilled, the particle will rest at all points of the surface.

Ex. 2. Again, take the same surface, and let the forces vary inversely as the distances of the point from the principal planes: it is required to determine the position of equilibrium.

$$\text{X} = \frac{\mu_1}{x}, \quad \text{Y} = \frac{\mu_2}{y}, \quad \text{Z} = \frac{\mu_3}{z};$$

therefore (72) become

$$\frac{\frac{x^2}{a^2}}{\mu_1} = \frac{\frac{y^2}{b^2}}{\mu_2} = \frac{\frac{z^2}{c^2}}{\mu_3} = \frac{1}{\mu_1+\mu_2+\mu_3} = \frac{1}{\mu}, \text{ (say)};$$

$$\therefore \quad x = a\left(\frac{\mu_1}{\mu}\right)^{\frac{1}{2}}, \qquad y = b\left(\frac{\mu_2}{\mu}\right)^{\frac{1}{2}}, \qquad z = c\left(\frac{\mu_3}{\mu}\right)^{\frac{1}{2}};$$

$$\begin{aligned} N^2 &= \frac{\mu_1^2}{x^2} + \frac{\mu_2^2}{y^2} + \frac{\mu_3^2}{z^2} \\ &= \mu\left\{\frac{\mu_1}{a^2} + \frac{\mu_2}{b^2} + \frac{\mu_3}{c^2}\right\}. \end{aligned}$$

Ex. 3. A heavy particle is placed inside a smooth sphere on the concave surface, and is acted on by a repulsive force varying inversely as the square of the distance from the lowest point of the sphere: find the position of rest of the particle.

Let the lowest point of the sphere be taken for the origin, and let the axis of z be vertical; then the equation of the sphere, whose radius is a, is

$$x^2+y^2+z^2-2az = 0.$$

Let W = the weight of the particle, and r = the distance of it from the lowest point; then

$$\begin{aligned} r^2 &= x^2+y^2+z^2 \\ &= 2az. \end{aligned}$$

Also let the repulsive force $= \frac{\mu}{r^2} = \frac{\mu}{2az}$;

$$\therefore \quad X = \frac{\mu}{2az}\frac{x}{r}, \qquad Y = \frac{\mu}{2az}\frac{y}{r}, \qquad Z = \frac{\mu}{2az}\frac{z}{r} - W.$$

Let N = the normal pressure of the curve; then

$$\frac{\mu}{2az}\frac{x}{r} = N\frac{x}{a}, \qquad \frac{\mu}{2az}\frac{y}{r} = N\frac{y}{a},$$

$$\frac{\mu}{2az}\frac{z}{r} - W = N\frac{z-a}{a};$$

from which we have

$$r^3 = \frac{\mu a}{W}; \qquad z = \frac{\mu^{\frac{2}{3}}}{2a^{\frac{1}{3}}W^{\frac{2}{3}}};$$

whence the position of the particle is known for a given weight of it, and for a given value of μ.

If another force of the same kind, and in which μ is replaced by μ', makes the particle to rest at a distance r' from the lowest point, then

$$r'^3 = \frac{\mu' a}{\mathrm{W}}; \qquad \therefore \frac{\mu}{\mu_1} = \frac{r^3}{r'^3};$$

that is, the absolute values of the repulsive forces at an unit-distance vary as the cubes of the distances from the lowest point of their positions of rest.

37.] Next let us consider the circumstances of pressure of a particle resting, or (to fix our thoughts) of a small ring sliding, on a given curved line which is smooth and offers no resistance to motion along itself.

As the curve is smooth, the resultant of the impressed forces is manifestly perpendicular to the tangent of the curve at the point of equilibrium; therefore if the curve is of double curvature, so that the direction-cosines of its tangent are proportional to dx, dy, dz, the required condition is

$$\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz = 0; \tag{73}$$

and if N is the normal pressure, and λ, μ, ν are the direction-angles of its line of action,

$$\mathrm{N}\cos\lambda = \mathrm{X}, \qquad \mathrm{N}\cos\mu = \mathrm{Y}, \qquad \mathrm{N}\cos\nu = \mathrm{Z}; \tag{74}$$

$$\mathrm{N}^2 = \mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2; \tag{75}$$

whence N, λ, μ, ν are known. If the equation (73) cannot be satisfied at any point of the curve, equilibrium is impossible; and if the forces are given, the point, at which equilibrium takes place, may be determined by means of (73) and the equations to the curve.

If the curve is a plane curve, (73) becomes

$$\mathrm{X}\,dx + \mathrm{Y}\,dy = 0. \tag{76}$$

And if $\mathrm{F}(x, y) = 0$ is the equation to the curve, this may be expressed in the form

$$\frac{\mathrm{X}}{\left(\frac{d\mathrm{F}}{dx}\right)} = \frac{\mathrm{Y}}{\left(\frac{d\mathrm{F}}{dy}\right)}. \tag{77}$$

Also (75) becomes

$$\mathrm{N}^2 = \mathrm{X}^2 + \mathrm{Y}^2. \tag{78}$$

Ex. 1. A ring is capable of sliding on a smooth helix, and is acted on by a constant force perpendicular to the axis; shew that equilibrium is impossible, unless the force parallel to the axis of z is zero.

The equations to the helix are

$$x = a\cos\phi, \qquad \therefore \quad dx = -y\,d\phi,$$
$$y = a\sin\phi, \qquad dy = x\,d\phi,$$
$$z = ka\phi; \qquad dz = ka\,d\phi;$$

and if μa is the constant force which acts towards and perpendicular to the axis,

$$X = -\mu x, \qquad Y = -\mu y;$$

and therefore substituting in (73), we have

$$\mu xy - \mu xy + Zka = 0;$$

which can be satisfied only when $Z = 0$.

Ex. 2. A small ring, capable of sliding on a smooth ellipse, whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

is acted on by forces parallel to the axes of x and y represented by μx^n and μy^n; find the position of equilibrium.

In this case (77) becomes

$$a^2 x^{n-1} = b^2 y^{n-1};$$

$$\therefore \quad x = ab^{\frac{n+1}{n-1}} \left\{ a^{2\frac{n+1}{n-1}} + b^{2\frac{n+1}{n-1}} \right\}^{-\frac{1}{2}};$$

and a similar value may be found for y.

Ex. 3. Two weights P and Q are fastened to the ends of a string, fig. 14, which passes over a pulley O; and Q hangs freely when P rests on a plane curve AP in a vertical plane; it is required to find the position of rest when the curve is given.

The forces which act on P are, (1) the tension of the string in the line OP, and which is equal to the weight of Q, (2) the weight of P acting vertically downwards, (3) the normal reaction of the curve, viz. R.

Let $F(x, y) = c$ be the equation to the plane curve, O being the origin, and the axis of x being vertical. Let $OM = x$, $MP = y$, $OP = r$, $POM = \theta$, $OA = a$. Then

$$X = P - Q\cos\theta - R\frac{dy}{ds}, \qquad Y = -Q\sin\theta + R\frac{dx}{ds};$$

therefore from (76),

$$(P - Q\cos\theta)\,dx - Q\sin\theta\,dy = 0,$$

$$P\,dx - Q\,\frac{x\,dx + y\,dy}{r} = 0;$$

but since $x^2+y^2 = r^2$; $\therefore$ $x\,dx+y\,dy = r\,dr$;

$$\therefore \quad \text{P}\,dx-\text{Q}\,dr = 0; \tag{79}$$

and this condition must be satisfied by P, Q, and the equation to the curve. Also

$$\text{R}^2 = \text{P}^2-2\text{PQ}\cos\theta+\text{Q}^2. \tag{80}$$

(1) Let the curve AP be a hyperbola of which O is the centre; then

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1;$$

$$\therefore \quad r^2 = x^2+y^2 = e^2x^2-b^2; \qquad r\,dr = e^2x\,dx;$$

$$\therefore \quad r\text{P}\,dx-\text{Q}e^2x\,dx = 0;$$

$$\therefore \quad x = \frac{b\text{P}}{e\{\text{P}^2-e^2\text{Q}^2\}^{\frac{1}{2}}}.$$

(2) Let it be required to find the equation to the curve, on all points of which P will rest. In this case (79) must be satisfied at all points of the curve; therefore

$$\begin{aligned}\text{P}x-\text{Q}r &= \text{a constant}\\ &= (\text{P}-\text{Q})\,a, \text{ (say)},\end{aligned}$$

if the curve passes through A, and OA $= a$; therefore

$$r = \frac{\left(1-\frac{\text{P}}{\text{Q}}\right)a}{1-\frac{\text{P}}{\text{Q}}\cos\theta}; \tag{81}$$

which is the equation to a conic section, of which the focus is the pole; and is an ellipse, parabola, or hyperbola, according as P is less than, equal to, or greater than, Q.

(3) Let the curve be a circular quadrant, convex downwards, with a horizontal radius passing through O, which is also a point on the circle, and let P $= 2$Q; then the equation to the circle is, if a is the radius,

$$r = 2a\sin\theta;$$

and therefore (79) becomes

$$4(\cos\theta)^2-\cos-2 = 0;$$

whence θ may be determined.

(4) Another form of the problem is, The length of the string being given, and Q always resting on a given curve, to find the curve on which P shall rest in all positions.

Let the tension of the string be equal to T, and let r' and θ'

refer to the curve BQ, fig. 15, on which Q rests, and of which let the equation be

$$r' = f(\theta'), \tag{82}$$

where f is the symbol of a known function: then we have from (79)

$$Q\,dx' - T\,dr' = 0,$$

also

$$P\,dx - T\,dr = 0;$$

and since

$$r + r' = 2c = \text{length of the string}; \tag{83}$$

$$\therefore \quad dr + dr' = 0; \qquad \therefore \quad Q\,dx' + P\,dx = 0; \tag{84}$$

and by means of (82), (83), and (84), r' and θ' are to be eliminated, and the resulting equation in terms of r and θ will be that required.

Let the curve on the left-hand side in the diagram be a parabola of which O is the focus; then

$$r' = \frac{2b}{1-\cos\theta'}; \tag{85}$$

and from (84),

$$Qx' + Px = 2kQ,$$

where k is an arbitrary constant; therefore from (85),

$$r' - r'\cos\theta' = 2b; \qquad 2c - r - \frac{2kQ - Pr\cos\theta}{Q} = 2b;$$

$$\therefore \quad r = \frac{2(c-b-k)}{1 - \dfrac{P}{Q}\cos\theta}; \tag{86}$$

which is the equation to a conic section of which the focus is O.

38.] In review of the preceding results it appears that, (1) if the particle on which certain forces act is entirely free, so that three variables are independent, the forces must satisfy three conditions; (2) if the particle is constrained to be on a given surface, there are two equations of equilibrium; and (3) only one condition is requisite, when the particle is on a given curve.

That is, if a particle is entirely unconstrained it has three degrees of freedom; if it is constrained to a given surface it has only two degrees of freedom, one degree being lost because the particle cannot move in the line of the normal to the surface; and if it is constrained to a given curve, it has only one degree of freedom, as it can move from an assigned point in the direction of the tangent of the curve, and along that line only.

CHAPTER III.

COMPOSITION AND RESOLUTION OF STATICAL FORCES ACTING ON A RIGID BODY.

SECTION 1.—*Composition of two forces acting on a rigid body in one plane.*

39.] Before we enter on the formal inquiry into the mode and results of the composition of forces acting on a rigid body it is necessary to explain some properties of such bodies, with the view of obtaining a principle which is necessary to the discussion.

A rigid body is such that its component particles are in a state of relative rest by the action of unknown molecular forces, such as attractions, cohesions, &c.: and the intensity of these forces is so great, that the relative equilibrium of the particles, which is due to them, is not disturbed by the forces which act on the body.

When a force acts at a definite point of a body and along a definite line it produces a pressure of the particle on which it acts *against* the contiguous particle in the line of its action, and *from* the contiguous particle in the opposite direction: and this pressure on these particles, although infinitesimal in comparison of the molecular forces, is propagated from one particle to another along the whole line of action of the acting force; and is the same at all points in this line. Hence we infer that the effect of a force on a rigid body, acting in a definite line, is unaltered, whatever is the point in its line of action at which it is applied. This principle is called that of Transmissibility of Pressure, and the truth of it depends on the rigidity of the body which involves such a mode of action as that described above.

Now two equal forces acting on a particle in the same line and in opposite directions neutralize each other; and this property may be extended by means of the preceding principle, so that, Two equal forces acting in the same line and in opposite directions at any points of a rigid body in that line neutralize each other. Hence we infer, that when many forces are acting on a rigid body, any two which are equal and have the same line of action and act in opposite directions may be omitted; and similarly the introduction of two equal forces along the same line of action and in opposite directions does not change the circumstances of the system as to resultant pressure.

The effects of the forces which have been considered in the preceding chapter are a tendency to motion in a given straight line, and, so far as we have considered them, along that straight line only: these are called *pressures or forces of translation.* But suppose a point O, fig. 16, of a rigid body to be fixed, so that there cannot be any motion of translation of the whole body; and suppose a force P to act on the body at a definite point M in the line MP; join OM, and resolve P into two parts, one along, and the other perpendicular to, OM; then the part along OM produces a pressure at O, which being fixed is capable of bearing it without the body having thereby any tendency to motion: but the other component causes a pressure on M in a direction at right angles to OM; but as O is fixed, M can only describe a circle about O as the centre; the effect therefore of this latter component is a tendency to *circular* motion of M, or, as it is commonly called, to rotation about O; a force producing such an effect is called *a pressure or force of rotation* about or in reference to a given point; and we have now to consider these, their measures, and their laws at length, and fully discuss them. Single particles are subject to forces of translation, but, having neither magnitude nor parts, not to pressures of rotation.

40.] Composition of two forces acting at definite points on a rigid body in one plane.

Let the two forces be P and Q, and let them act in the plane of the paper at the points A and B, fig. 17; join AB, and let us assume that the lines of action of P and Q are not parallel; let the angles between AB and the lines of action of P and Q be respectively α and β; produce the lines of action to meet in O, O being supposed to be in the rigid body or to be rigidly connected with it; then by virtue of the principle of transmissibility, we may suppose P and Q to be applied at O. Let R be the resultant of them so transferred, and let the line of action of R intersect AB in the point C; then we have to determine the magnitude of R, its line of action, and a point in that line; these last two will be conveniently known, if we find AC, and the angle between AB and CO.

Let $\quad AC = x, \quad CB = y, \quad AB = a; \qquad \therefore \quad x+y = a;$

$$OAB = \alpha, \quad OBA = \beta, \quad OCB = \theta;$$

then, by the parallelogram of forces,

$$R^2 = P^2 - 2PQ\cos(\alpha+\beta) + Q^2; \tag{1}$$

whereby the magnitude of the resultant is known. And resolving P, Q, R at O along lines through O, parallel, and perpendicular to, AB, we have

$$\left.\begin{aligned} R\cos\theta &= P\cos\alpha - Q\cos\beta, \\ R\sin\theta &= Q\sin\beta + P\sin\alpha; \end{aligned}\right\} \tag{2}$$

$$\therefore \quad \tan\theta = \frac{Q\sin\beta + P\sin\alpha}{P\cos\alpha - Q\cos\beta}; \tag{3}$$

and by reason of equations (30) Art. 21,

$$\frac{P}{\sin(\theta+\beta)} = \frac{Q}{\sin(\theta-\alpha)} = \frac{R}{\sin(\alpha+\beta)}. \tag{4}$$

Let p and q be the lengths of the perpendiculars on the lines of action of P and Q from any point in the line of action of R, say, from C; then

$$\left.\begin{aligned} p &= CO\sin(\theta-\alpha), \\ q &= CO\sin(\theta+\beta); \end{aligned}\right\} \tag{5}$$

therefore from the first two terms of (4),

$$Pp = Qq; \tag{6}$$

and therefore, since $p = x\sin\alpha$, $q = y\sin\beta$,

$$Px\sin\alpha = Qy\sin\beta, \tag{7}$$

$$\frac{x}{Q\sin\beta} = \frac{y}{P\sin\alpha} = \frac{a}{P\sin\alpha + Q\sin\beta}$$

$$= \frac{a}{R\sin\theta}; \tag{8}$$

whereby x and y are given in terms of known quantities: the magnitude, line of action, and point of application on the line AB of the resultant are therefore determined.

41.] The equation (6) requires especial consideration with reference to the properties of moments which have been mentioned in the previous chapters; two forces, P and Q, act on the body, each of which alone produces a pressure of translation along its line of action: but the resultant of the two taken in combination is a single force R, the position of whose line of action is given by (3); a force therefore equal to R, along the same line of action, and opposite in direction to R, will with

P and Q produce equilibrium. Now this force may be applied at *any* point in the line of action of R; let C be the point of application; and thus the system is in equilibrium, and is as if C were a fixed point. Let us consider this in the light of the remarks of Art. 39; P and Q severally produce a pressure of rotation about C, and manifestly in opposite directions; and they neutralize each other, for the body is at rest: therefore their rotatory effects are equal. But what relation exists between them? because we may thence infer a measure of their rotatory effects with reference to the point or centre C. P and Q balance when (6) is satisfied; that is, the rotatory effect due to one force is equal to, and neutralized by, that due to the other, when the products of the force and the perpendicular distance from C on its line of action are equal. This product therefore may be taken as the measure of the rotatory effect of a force. And as it is desirable to have a distinctive name for such an effect, it is called a force's *moment;* and therefore we define as follows:

DEF. *Moment* of a force with reference to a given point is the rotatory effect of it with reference to that point; and is measured by the product of the numbers which represent the force and the perpendicular distance from the point on the line of action of the force. This is the algebraical measure of the moment.

Two forces are said to be *equimomental* with respect to a point when their moments with respect to that point are equal.

As the forces act in one plane we have spoken of the moments with respect to a point: it is more correct to say, with respect to an axis passing through the point and perpendicular to the plane in which the forces act, because it is about this *line* that the forces *per se,* and all other things neglected, tend to make the body turn. However, when the body, on which the forces act, moves, we shall have a modification of this statement.

A force may tend to make a body turn about an axis in either one or the other of two directions; it is necessary therefore to distinguish these, and to affect them with different signs: let therefore the moment of a force be positive if it tends to turn a body from right to left, that is, in the direction in which the hands of a clock revolve, when it is opposite to us; and let the moment of a force be negative, when it turns a body in the opposite direction.

As the moment of a force in reference to a point is the product of the perpendicular from that point on the line-representative of the force and that line-representative, its geometrical representative, as we have observed in Art. 28, is twice the area of the triangle, of which the given point is the vertex, and the line-representative is the base. Hence as properties of forces of translation have their geometrical analogues in lines, so properties of moments are translated directly geometrically into theorems concerning areas. We shall however see hereafter that moments are also frequently represented by lines whose lengths are proportionals to the moments.

Moments of forces, being quantities measurable by number, are capable of addition and subtraction. Thus if three forces are proportional to, and act along, the sides of a plane triangle in the same direction, as to translation they neutralize each other, and the result is zero. But as to rotation, the resultant moment with reference to any point in the plane of the triangle is equal to twice the area of the triangle.

42.] Let us return to equation (7), and consider C as a point at rest, by means of the force R acting on it which is in equilibrium with P and Q: then resolving P and Q along and perpendicular to AB, we have $P \sin \alpha$ and $Q \sin \beta$ perpendicular to AB, and $P \cos \alpha$ and $Q \cos \beta$ along AB: these latter forces produce a pressure on C which is equal to their difference; but the former components produce a rotatory pressure about C, and equilibrate when the moments of the two are equal, that is, when

$$xP \sin \alpha = yQ \sin \beta;$$

and this is equation (7).

Again, suppose that the components are P_1 and P_2, and that the equations to their lines of action are given; and let it be required to find that of the line of action of the resultant R.

Let the equations to the lines of action of the components be

$$\left.\begin{array}{l} x \cos \alpha_1 + y \sin \alpha_1 - p_1 = 0 = a_1, \\ x \cos \alpha_2 + y \sin \alpha_2 - p_2 = 0 = a_2, \end{array}\right\} \qquad (9)$$

a_1 and a_2 being symbols of notation for the left-hand members of the equations: then, if x and y refer to any point in the line of action of the resultant, by equation (6) we have

$$P_1 a_1 + P_2 a_2 = 0;$$

$$\therefore \quad (P_1 \cos \alpha_1 + P_2 \cos \alpha_2) x + (P_1 \sin \alpha_1 + P_2 \sin \alpha_2) y$$
$$- p_1 P_1 - p_2 P_2 = 0; \qquad (10)$$

which is the equation to the line of action of R.

Hence if r is the perpendicular from the origin on the line of action of R,

$$r = \frac{p_1 P_1 + p_2 P_2}{\{(P_1 \cos\alpha_1 + P_2 \cos\alpha_2)^2 + (P_1 \sin\alpha_1 + P_2 \sin\alpha_2)^2\}^{\frac{1}{2}}}$$

$$= \frac{p_1 P_1 + p_2 P_2}{\{P_1^2 + 2 P_1 P_2 \cos(\alpha_1 - \alpha_2) + P_2^2\}^{\frac{1}{2}}}$$

$$= \frac{p_1 P_1 + p_2 P_2}{R};$$

$$\therefore \quad R r = p_1 P_1 + p_2 P_2; \qquad (11)$$

that is, the moment of the resultant is equal to the sum of the moments of the components.

43.] Let us consider the subject from another point of view, and take two forces, whose lines of action are parallel, acting in the same direction on a rigid body.

Let P, Q be the two parallel forces acting at A and B, fig. 18: join AB, and let α be the angle between AB and the lines of action of P and Q; at A and B introduce two equal forces S and S which act along AB, and in opposite directions: the circumstances of pressure are not hereby altered. Let P′ be the resultant of P and S at A, and Q′ the resultant of Q and S at B; let the lines of action of P′ and Q′ be produced to meet in O, O being supposed to be rigidly connected with the body: at O resolve P′ and Q′ into the forces of which they were compounded; the components along the line parallel to AB manifestly cancel each other, and there remains P+Q acting in a line parallel to the lines of action of P and Q. Let this resultant be R, so that

$$R = P + Q; \qquad (12)$$

that is, the resultant is the sum of the two parallel forces.

Let AC $= x$, CB $= y$, AB $= a$; therefore $x + y = a$; then P′ is the resultant of P and S, and these pressures are parallel to the sides of the triangle ACO;

$$\therefore \quad \frac{S}{x} = \frac{P}{CO}; \qquad \text{similarly} \quad \frac{S}{y} = \frac{Q}{CO};$$

$$\therefore \quad P x = Q y. \qquad (13)$$

Let p and q be the perpendicular distances from C on the lines of action of P and Q: then $p = x \sin\alpha$, $q = y \sin\alpha$, and thus (13) becomes

$$P p = Q q; \qquad (14)$$

that is, the moments of P and Q about C are equal.

Again, from (13),

$$\frac{x}{\text{Q}} = \frac{y}{\text{P}} = \frac{x+y}{\text{P}+\text{Q}} = \frac{a}{\text{R}}; \tag{15}$$

whence x and y are known; and are reciprocally proportional to the forces at their extremities. Hence also when three parallel forces are in equilibrium, each is proportional to the distance between the action-lines of the other two.

$$\text{If} \quad \text{P} = \text{Q}, \quad y = x = \frac{a}{2}, \quad \text{R} = 2\text{P};$$

that is, the resultant is equal to twice one of the forces, and is applied at the point of bisection of the line joining the points of application of the forces.

As (14) is independent of the angle between AB and the direction of the forces, C is the same whatever that angle is; C is for this reason called the centre of the two parallel forces.

44.] Suppose one of the parallel forces of the preceding Article to act in a direction contrary to that of the other: then fig. 19, introducing as before two equal forces S, S acting along AB and in opposite directions, and compounding P and S into P′, and Q and S into Q′, let us suppose the lines of action of P′ and Q′ to meet at O, O being rigidly connected with the body; and at O let P′ and Q′ be resolved into the forces of which they were compounded; the forces parallel to the line AB cancel each other, and there remain P and Q acting in a line parallel to the original lines of action of P and Q, the resultant of which is equal to their difference: let us suppose Q to be the greater, then

$$\text{R} = \text{Q} - \text{P}. \tag{16}$$

Let AB $= a$, AC $= x$, BC $= y$; therefore $x - y = a$; and let α be the angle between AB and the lines of action of P and Q. Since P′ is the resultant of P and Q,

$$\frac{\text{S}}{x} = \frac{\text{P}}{\text{CO}}; \qquad \text{similarly} \quad \frac{\text{S}}{y} = \frac{\text{Q}}{\text{CO}},$$

$$\therefore \quad \text{P}x = \text{Q}y. \tag{17}$$

Let p and q be the perpendicular distances from C on the lines of action of P and Q; then $p = x \sin \alpha$, $q = y \sin \alpha$; therefore (17) becomes

$$\text{P}p = \text{Q}q; \tag{18}$$

that is, the moments of P and Q about C, and similarly about every point in the line of action of R, are equal.

Again, from (17)

$$\frac{x}{\mathrm{Q}} = \frac{y}{\mathrm{P}} = \frac{x-y}{\mathrm{Q}-\mathrm{P}} = \frac{a}{\mathrm{R}}; \tag{19}$$

whence x and y are known, and are reciprocally proportional to the forces acting at their extremities.

This theorem of the equality of moments, whether of parallel forces as I have demonstrated in this and the preceding articles, or of forces whose lines of action are not parallel, has been called the *principle of the lever*, and has been by many writers on mechanics made fundamental; and other mechanical theorems, including that of the parallelogram of forces, have been derived from it. I, on the other hand, have derived the equality of moments from the parallelogram of forces, in the conviction that the latter proposition is more simple, and that the former follows more directly from it. The immediate application of the theorem is so easy, that it is unnecessary to insert examples at this stage of the work.

45.] The equation to the line of action of the resultant of two parallel forces P_1 and P_2 may be determined as follows:—

Let the equations to the lines of actions of the components be

$$\left.\begin{aligned} x\cos\alpha + y\sin\alpha - \delta_1 &= 0 = \alpha_1, \\ x\cos\alpha + y\sin\alpha - \delta_2 &= 0 = \alpha_2; \end{aligned}\right\} \tag{20}$$

therefore by (14) or (18) the equation to the line of action of the resultant is

$$(\mathrm{P}_1+\mathrm{P}_2)\,x\cos\alpha + (\mathrm{P}_1+\mathrm{P}_2)\,y\sin\alpha - (\delta_1\mathrm{P}_1 + \delta_2\mathrm{P}_2) = 0;$$

that is, since $\mathrm{P}_1 + \mathrm{P}_2 = \mathrm{R}$,

$$x\mathrm{R}\cos\alpha + y\mathrm{R}\sin\alpha - (\delta_1\mathrm{P}_1 + \delta_2\mathrm{P}_2) = 0. \tag{21}$$

If $\mathrm{P}_1 + \mathrm{P}_2 = 0$; that is, if the forces are equal and act in opposite directions, then

$$(\delta_1 - \delta_2)\mathrm{P}_1 = 0, \tag{22}$$

which is the equation to a straight line at an infinite distance; consequently the resultant of two equal and opposite forces acts at an infinite distance.

SECTION 2.—*On couples—their laws and composition.*

46.] These results arising from the simultaneous action of two equal forces, working in opposite directions along two parallel straight lines which are at a finite distance apart, require closer consideration; for they open to us a series of theorems in themselves and in their inferences of very great use in the simplification of mechanical propositions. It is indeed on these theorems that a large and distinct part of our subject has been raised; and it is consequently necessary to investigate them at considerable length. I will start from the results of Art. 43 which refer to the composition of two unequal forces P and Q, which act in opposite directions along parallel straight lines, and I will suppose Q to be the larger of the two; let us suppose the difference between Q and P gradually to diminish, and Q ultimately to become equal to P; then R becomes less; and x becomes greater; and ultimately, when $Q=P$, $R=0$, and $x=y=\infty$; that is, there is no single force of translation which will be equivalent to such a pair of forces; and therefore there is no one force of translation which will be in equilibrium with them. It is also by the principle of sufficient reason manifest that such a system cannot have a single resultant of translation; because such a resultant is *unique;* and whatever is the process of reasoning by which its line of action is assigned in respect of one of the forces, by the same will it be assigned in a similar position with respect to the other force.

Such a pair of forces, equal and acting in parallel lines and in opposite directions, is called *a couple**; its effect is evidently *a pressure of rotation* about a line perpendicular to the plane in which the forces act, and which line is called *the axis of the couple.* Now in statics, as the motion is only *virtual* and not *actual,* the *direction* of the axis is fixed, but not the *position* of it; it is *some* line perpendicular to the plane in which the forces act. If motion takes place the position of the axis, as well as its direction, becomes fixed, as we shall see hereafter. If the axes of couples are parallel, that is, if the planes of these forces are parallel, the couples are *coaxal.*

The perpendicular distance between the lines of action of the forces is called the *arm of the couple.*

The rotatory effect of a couple is called the *moment* of the couple. In estimating its measure we must examine all possible positions of the axis. Let the couple be that indicated in fig. 20;

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* See Poinsot, "Mémoire sur la composition des Moments et des Aires dans la Mecanique." The tract is appended to "Élémens de Statique" of the same author, 8me edition, Paris, 1842.

and (1) let us suppose the axis to pierce the plane of the couple at the point O which lies between the forces; then

$$\begin{aligned}\text{the moment of the couple} &= P \times OA + P \times OB\\ &= P \times AB. \qquad (23)\end{aligned}$$

(2) Suppose the axis to pass through A, one of the extremities of the arm: then the force which acts at A produces no pressure of rotation, and we have

$$\text{the moment of the couple} = P \times AB. \qquad (24)$$

(3) Suppose the axis to pierce the plane of the couple at a point O, fig. 21, in the arm produced: then

$$\begin{aligned}\text{the moment of the couple} &= P \times OB - P \times OA\\ &= P \times AB. \qquad (25)\end{aligned}$$

In all cases therefore the moment of the couple is equal to the product of the numbers expressing the force and the length of the arm. Thus if the force contains 6 units of pressure, and the arm 3 units of linear length, the moment of the couple is expressed by 18; that is,

$$\text{the moment of couple} = \text{the force} \times \text{the length of the arm}. \qquad (26)$$

A couple may evidently tend to make a body revolve in either one or the other of two opposite directions; that is, in the direction of the hands of a watch, as we face it, or in the opposite direction; and it is desirable to affect these different directions with different signs; for the present, let the former be positive or right-handed couples, and the latter, negative or left-handed couples. In figs. 20 and 21 right-handed couples are represented.

Two couples whose moments are equal are said to be *equimomental*.

The forces applied in turning the handle of a corkscrew, of a gimlet and of an auger, are familiar instances of couples.

47.] The following three theorems concern the transference of couples:—

THEOREM I. The effect of a couple on a rigid body is not altered, if the length of the arm and the force being the same, the arm is turned about its extremity through any angle in the plane of the couple.

Let AB, fig. 22, be the arm of the original couple, and P, P its forces; through A draw *any* straight line AB′ in the plane of the couple equal to AB, and at A and B′ respectively introduce in the plane of the couple two forces equal to P, with their lines of action perpendicular to the arm AB′, and opposite in direction to each other; then the original circumstances of pressure are not altered by the introduction of these forces. Let $BAB' = 2\theta$; then the resultant of P acting at B, and of P acting at B′, whose lines of action meet at Q, is $2P \sin\theta$, and acts along the line AQ: similarly the resultant of P acting at A perpendicularly to AB, and of P perpendicularly to AB′, is $2P \sin\theta$, and acts along the line AQ in a direction opposite to that of the former resultant: these two resultants therefore neutralize each other, and there remains the couple whose arm is AB′ and the forces P, P: and this is equimomental with the original couple and replaces it, and consequently the theorem is true.

THEOREM II. The effect of a couple on a rigid body is not altered, if the plane of the forces is transferred to any other parallel plane, the arm being parallel to its original line, and of an equal length, and the forces being unaltered in magnitude.

Let AB, fig. 23, be the arm, and P, P the forces of the given couple: let A′B′ be an arm equal and parallel to AB; at A′ and B′ respectively introduce two forces equal to P, acting perpendicularly to A′B′, and in opposite directions, and in a plane parallel to the plane of the original couple: the original circumstances of pressure are not altered by the introduction of these new forces. Join AB′, A′B; these lines evidently intersect and bisect each other in O; then P at A and P at B′, acting in parallel lines and in the same direction, are equivalent to a force 2P acting at O: similarly P at B and P at A′, acting in parallel lines and in the same direction, are equivalent to 2P acting at O in a line parallel to their original lines of action: at O therefore these two resultants, being equal and opposite, neutralize each other; and there remains the couple whose arm is A′B′, and whose forces are P, P, acting in the same direction as those of the original couple, in a parallel plane, and with an equal arm: it is therefore coaxal and equimomental, and may equivalently replace the original couple.

The proof which is here given for a parallel plane is of course valid for the less general case of the same plane: and therefore from this and Theorem I. we infer, that the effect of a couple on a rigid body is not changed whatever is the position of its plane, if the direction of the axis is unaltered, and the arm and the forces are equal.

THEOREM III. The effect of a couple on a rigid body is not altered, whatever is the position of its plane, arm, and force, provided that its axis and moment are unaltered.

In fig. 24, let AB be the arm, and P, P the forces of the given couple; at A and B introduce any equal forces S and S acting along AB and in opposite directions. Let P′ be the resultant of P and S at A, and let P′ also be the resultant of P and S at B: the lines of action of P′ and P′ are of course parallel; produce P′A backwards, and from B draw BA′ perpendicular to AA′: then the forces P′ and P′ form a couple whose arm is BA′, and each of whose forces is P′; let $BAA' = \theta$; then $A'B = AB \sin\theta$; $P' = P \operatorname{cosec}\theta$; $S = P'\cos\theta = P\cot\theta$; and

$$\begin{aligned} \text{the moment of the new couple} &= P' \times A'B \\ &= P\operatorname{cosec}\theta \times AB\sin\theta \\ &= P \times AB \\ &= \text{the moment of the original couple.} \quad (27) \end{aligned}$$

It will be observed that S is arbitrary, and that θ and consequently the length of the new arm, as also the force of the new couple, depend on it: consequently they are also arbitrary; but they are subject to the condition (27), which requires the new couple to be equimomental with the original one. And thus it appears that a couple is equivalent to, and may be replaced by, another couple, of which the moment is the same, the forces are in the same plane, and the arms have a common extremity.

Combining this theorem with the preceding, we conclude that a couple is equivalent to, and may be replaced by, any other equimomental and coaxal couple.

48.] Now in all these transformations, the axis of the couple, that is, the direction of the line about which the couple tends to make the body rotate, has not been altered; the arm and the force have been altered in position, in length, in magnitude; and the plane in which the forces act has been changed from any one into any other parallel plane; but the normal to the plane, which is the axis, has continued unaltered; and the moment has continued the same; and these quantities cannot be changed without changing the effect of the couple; the former of these then has a fixed direction, and the latter is a fixed quantity. It is convenient, as of forces of translation, so of these forces of rotation, to have geometrical lengths as adequate representatives; and such we shall obtain, if along the *axis* we take lengths containing the same number of linear units as the moment of the couple contains units of pressure. Thus if the force of a couple is 4 and the length of the arm is 3, the moment is represented by the number 12; and if along the axis 12 linear units are measured, this length is a full and adequate representative of the couple; and moreover as couples may be right-handed or left-handed, that is, have positive or negative signs, so from a fixed point (the origin) on the axis may the line be taken in one or the other direction, and thus indicate the sign of the couple. Now this line is technically called *the axis of the couple,* the word being used in a sense different to the former one: *there* it indicated line of rotation only; *here* it indicates three things, viz. the line of rotation, a finite length of that line measured from a given point on it, and the direction in which it is measured. This axis therefore fully determines all the circumstances of the couple. Some confusion may arise from the ambiguous use of the word, and therefore I shall always take care to specify axis as to rotation, and axis as to rotation and moment, by calling the former *rotation-axis,* and the latter *moment-axis,* bearing in mind however that the latter is indicative of direction as well as the former; and when couples are said to be coaxal, it is with respect to the former meaning of the word only; and when two couples are statically equivalent they are coaxal and equimomental.

49.] The following theorems concern the composition of couples:—

THEOREM IV. The resultant of many coaxal couples is a coaxal couple whose moment is equal to the algebraical sum of the moments of the component couples.

Let the forces of the several couples be $P_1, P_2, \ldots P_n$; and the lengths of the arms $p_1, p_2, \ldots p_n$; so that their moments are $P_1 p_1, P_2 p_2, \ldots P_n p_n$. Let all, by virtue of Theorem II, be transferred to the same plane, and let all the arms have a common extremity; again, by virtue of Theorem III, let all be transformed into equivalent couples with arms of the same length, equal to r, and let the forces thereby changed be $P_1', P_2', \ldots P_n'$; so that

$$P_1' r = P_1 p_1, \qquad P_2' r = P_2 p_2, \ldots\ldots P_n' r = P_n p_n; \tag{28}$$

and lastly, by virtue of Theorem I, let all the arms be turned about their common extremity, and become coincident; then the length of it is r, and at each extremity there are equal and opposite forces, of which let the sum be R, where

$$\mathrm{R} = \mathrm{P}_1' + \mathrm{P}_2' + \ldots \mathrm{P}_n'; \tag{29}$$

so that the moment of the resultant couple is

$$\begin{aligned} \mathrm{R}r &= \mathrm{P}_1'r + \mathrm{P}_2'r + \ldots + \mathrm{P}_n'r \\ &= \mathrm{P}_1 p_1 + \mathrm{P}_2 p_2 + \ldots + \mathrm{P}_n p_n \\ &= \Sigma . \mathrm{P}p; \end{aligned} \tag{30}$$

that is, the moment of the resultant couple is equal to the sum of the moments of the several component couples.

If some of the couples are negative, the forces belonging to them will in (29) have negative signs, and R will be equal to the difference of the forces which have positive signs and of those which have negative signs: and the same result will appear in (30), so that the right-hand member denotes the algebraical sum.

The moment-axis of the resultant is equal to the sum of the moment-axes of the component couples.

Two equimomental and coaxal couples acting in opposite directions evidently neutralize each other.

A close analogy exists between parallel forces of translation applied at the same point and coaxal couples: in either case the effect of the resultant is equal to the algebraical sum of the effects of the components. We shall trace this analogy further in the succeeding Article. As to the geometrical representatives of the effects, in the case of couples the moment-axis may be transferred parallel to itself in any manner; in the case of forces of translation, the representative line can, by the principle of transmissibility, be transferred only along its own line of action.

50.] THEOREM V. If two lines meeting at a point represent the moment-axes of two couples, the diagonal of the parallelogram originating at the same point, and of which the two lines are adjacent sides, will represent the moment-axis of a single equivalent couple.

Suppose two couples to act in planes which are inclined to each other at an angle γ; let the couples be transferred in their own planes so as to have the same arm lying along the line of intersection of the two planes; let the forces of the couples thus transferred be P and Q. And, fig. 25, let AB be the common arm, and let us suppose it to lie in the plane of the paper: then compounding P and Q at A into a single force R, and P and Q at B in the same way, since PAQ $= \gamma$, we have

$$R^2 = P^2 + 2PQ\cos\gamma + Q^2; \tag{31}$$

and the R at B is equal and parallel to the R at A. At A draw A*a*, A*b* perpendicular respectively to the planes PBAP, QBAQ, and of lengths equal to the moment-axes of the couples; complete the parallelogram A*acb*, and draw the diagonal A*c*; then A*c* is the moment-axis of the resultant couple whose arm is AB and whose force is R. For since Aa = P × AB, and Ab = Q × AB, therefore A*a* and A*b* are proportional to P and Q, that is, to AP and AQ; and they are also perpendicular to these lines, and are in the same plane with them; therefore the diagonal A*c* is perpendicular, and proportional in the same ratio, to AR; therefore Ac = R × AB, and is the moment-axis of the resultant couple. Therefore, if A*a* and A*b* are the moment-axes of two couples, A*c* the diagonal of the parallelogram of which A*a* and A*b* are the two adjacent sides is the moment-axis of the resultant couple. Hence if L and M are the moment-axes of two couples, and are inclined to each other at an angle γ, and if G is the moment-axis of the resultant couple,

$$G^2 = L^2 + 2LM\cos\gamma + M^2. \tag{32}$$

Attention must of course be paid to the direction of the couple; thus, if A*a* is the moment-axis, to an eye placed at A and looking along A*a*, the couple is right-handed.

Hereby also we are authorized to resolve a couple whose moment-axis is given into any two couples, such that their moment-axes are the sides of the parallelogram of which the given moment-axis is the diagonal. And the number of ways in which such resolution can be effected is infinite.

51.] If the moment-axes of two couples are perpendicular to each other, then $\gamma = 90°$; and

$$G^2 = L^2 + M^2; \tag{33}$$

if λ is the angle between the rotation-axes of G and L, then

$$L = G\cos\lambda, \qquad M = G\sin\lambda, \tag{34}$$

$$\tan\lambda = \frac{M}{L}; \tag{35}$$

a couple therefore whose moment-axis is G may be resolved into any two couples such that their moment-axes are the sides of the rectangle whose diagonal is the given moment-axis.

Hence also a couple, whose moment-axis is equal to G, but is in an opposite direction, neutralizes L and M, and the whole system is in equilibrium.

Also from (32) by a process analogous to that of Article 21 we can shew that if, fig. 26, OL, OM, ON represent the moment-axes of three couples L, M, N; and if MON $=\alpha$, NOL $=\beta$, LOM $=\gamma$, and if

$$\frac{\text{L}}{\sin\alpha} = \frac{\text{M}}{\sin\beta} = \frac{\text{N}}{\sin\gamma},$$

then the three couples are in equilibrium; and conversely, if three couples are in equilibrium, the moment-axis of each is proportional to the sine of the angle contained between the rotation-axes of the other two.

Hence also if many couples acting on a rigid body are in equilibrium, their rotation-axes are parallel to the sides of a closed polygon, the sides themselves being the moment-axes.

And finally we conclude that couples may by means of their moment-axes, which are their geometrical representatives, be resolved and compounded according to the same laws as forces of translation by means of their equivalent lines of action. And whatever is true of pressures of translation is also true, mutatis mutandis, of pressures of rotation as exhibited by the moment-axes of the couples which are their geometrical representatives.

52.] The analogy which has been traced between the moment-axes of couples and the line-representatives of the forces of translation also holds good when there are many couples of which the moment-axes are not all parallel and are not all in one plane. And to take the most general case, let us consider the composition of couples whose rotation-axes have any position in space.

Take any point O in space for an origin of coordinate-axes, and at it let three straight lines originate, forming a system of rectangular axes.

Let the axis of every component couple be shifted, and pass through O, and let the moment-axis of each component couple be resolved into two moment-axes, one of which coincides with the z-axis, and the other lies in the plane of (x, y); also let this latter moment-axis be resolved into two others which coincide with the axes of x and y respectively; then when every component couple has been resolved in this way, we have three series of coaxal couples, whose axes are the coordinate axes of

x, y, z respectively. Let the sum of these coaxal couples be taken; and let L, M, N be the moment-axes of the sums which respectively have their rotation-axes coincident with the axes of x, y, z. Thus all the component couples are reduced to three couples whose rotation-axes are perpendicular, each to every other two, and of which the moment-axes are L, M, N.

Let us further compound these three couples. Let G′ be the resultant moment-axis of L and M; then by (33),

$$G'^2 = L^2 + M^2.$$

Also again compounding G′ and N which are perpendicular to each other, if G is the resultant moment-axis,

$$\begin{aligned} G^2 &= L^2 + G'^2 \\ &= L^2 + M^2 + N^2. \qquad (36) \end{aligned}$$

Let λ, μ, ν be the direction-angles of the rotation-axis of G: then

$$L = G\cos\lambda, \qquad M = G\cos\mu, \qquad N = G\cos\nu;$$

$$\therefore \quad \cos\lambda = \frac{L}{G}, \qquad \cos\mu = \frac{M}{G}, \qquad \cos\nu = \frac{N}{G}; \qquad (37)$$

so that if L, M, N are given, we can find G and the line of its rotation-axis; and if a moment-axis is given, we can resolve it into three component moment-axes, which are at right angles to each other. It is to M. Poinsot that we are indebted for this great simplification of a problem which it is very difficult to follow in its complex form.

The analogy which has thus been traced to composition and resolution between couples as expressed by their moment-axes and forces of translation by means of their line-representatives establishes a real and a large principle of duality, and of which we shall hereafter have many illustrations. Every theorem hereby becomes double. It admits of interpretation with respect to couples, that is, with respect to pressure of rotation, as well as with respect to pressure of translation; and the proof of a theorem of one class authorizes the inference of the analogous theorem in the other class.

SECTION 3.—*On the composition and resolution of forces acting on a rigid body, the lines of action of which are in one plane.*

53.] I propose in the first place to investigate the composition of those forces, the action-lines of which are parallel to each other, and which are consequently called parallel forces.

Let the plane in which the forces act be the plane of (x, y); and let the origin O be, fig. 27, any point which is in, or rigidly connected with, the body; and let the forces be $\mathrm{P}_1, \mathrm{P}_2, \ldots \mathrm{P}_n$, of which let P be the type: let $p_1, p_2, \ldots p_n$ be the perpendiculars from the origin on their lines of action, of which let p be the type-perpendicular: let (x, y) be any point in the line of action of the type-force P, and let a be the angle between the line of action of P and the axis of x: then the equation to the line of action of P is

$$x \sin a - y \cos a - p = 0.$$

Let two forces each equal to P, with their lines of action parallel to that of P, and acting in opposite directions, be introduced at the origin O; so that instead of the original force P, we have P acting at O in a parallel line and the same direction, and a couple whose moment is $\mathrm{P}p$ and whose rotation-axis is perpendicular to the plane of the forces.

Let P at O be resolved into two forces along the coordinate axes, viz. $\mathrm{P} \cos a$, and $\mathrm{P} \sin a$; and let all the forces be similarly transformed; then, if X and Y are the resultants of the forces severally along the axes of x and y,

$$\begin{aligned} \mathrm{X} &= \mathrm{P}_1 \cos a + \mathrm{P}_2 \cos a + \ldots + \mathrm{P}_n \cos a \\ &= \cos a \, \Sigma.\mathrm{P}; \qquad (38) \\ \mathrm{Y} &= \mathrm{P}_1 \sin a + \mathrm{P}_2 \sin a + \ldots + \mathrm{P}_n \sin a \\ &= \sin a \, \Sigma.\mathrm{P}. \qquad (39) \end{aligned}$$

Also the moment of the couple arising from P is equal to $\mathrm{P}p$, the tendency of which is to turn the body from the axis of x towards that of y; and, as a similar couple and moment will arise from every one of the forces, if G is the moment of the resultant couple, by reason of Art. 49,

$$\begin{aligned} \mathrm{G} &= \Sigma.\mathrm{P}p \\ &= \Sigma.\mathrm{P}(x \sin a - y \cos a) \\ &= \sin a \, \Sigma.\mathrm{P}x - \cos a \, \Sigma.\mathrm{P}y, \qquad (40) \end{aligned}$$

placing $\sin a$ and $\cos a$ outside the signs of summation, because they are the same for all the forces: and observing that x and y refer to some point in the line of action of each pressure, which will generally be different for each. G in (40) consists of two parts, which are affected with different signs; the resultant couple therefore is the difference between the resultants of two systems of coaxal couples acting in contrary directions: $\sin a \, \Sigma.\mathrm{P}x$ tend to turn the body from the axis of x towards that of y, and $\cos a \, \Sigma.\mathrm{P}y$ act in the opposite direction.

54.] Suppose now that all the forces are capable of being reduced to a single force R; or, in other words, suppose that one force R will have the same effect on the rigid body as all the impressed forces taken in combination. Let a be the angle at which the line of action of R is inclined to the axis of x, and let $(\bar{x}, \bar{y})$ be any point in the line of action of R, and $\bar{r}$ the perpendicular distance from the origin on it. Then introducing at O two forces, each equal to R, with their lines of action parallel to that of R, and acting in opposite directions, we have the force of translation R acting at the origin, and a couple $\mathrm{R}\bar{r}$; whence, resolving R at the origin along the coordinate axes, and equating the resolved parts to the sum of the resolved parts of the impressed forces, we have

$$\left.\begin{aligned} \mathrm{R}\cos a &= \Sigma.\mathrm{P}\cos\alpha = \cos\alpha\,\Sigma.\mathrm{P}, \\ \mathrm{R}\sin a &= \Sigma.\mathrm{P}\sin\alpha = \sin\alpha\,\Sigma.\mathrm{P}; \end{aligned}\right\} \tag{41}$$

therefore $$\mathrm{R} = \Sigma.\mathrm{P}, \qquad a = \alpha; \tag{42}$$

that is, the resultant is equal to the algebraical sum of the components, and its line of action is parallel to those of the several components.

Also the couple $\mathrm{R}\bar{r}$, due to the resultant R, must be equal to G; so that

$$\bar{r} = \frac{\mathrm{G}}{\mathrm{R}} = \frac{\Sigma.\mathrm{P}p}{\Sigma.\mathrm{P}}. \tag{43}$$

and thus the force R is determined as to magnitude, line of action, and direction.

The equation to its line of action may thus be found. Replacing $\bar{r}$ in (43) in terms of $\bar{x}$ and $\bar{y}$, the current coordinates of the line of action of R, we have

$$\mathrm{R}(\bar{x}\sin\alpha - \bar{y}\cos\alpha) = \mathrm{G}; \tag{44}$$

$$\therefore \quad \bar{x}\sin a - \bar{y}\cos a = \frac{\mathrm{G}}{\mathrm{R}}; \tag{45}$$

which is the equation required.

We may however employ the abridged form of the equation to a straight line; in which case let the equations to the lines of action of $\mathrm{P}_1, \mathrm{P}_2, \dots \mathrm{P}_n$ be

$$a_1 = 0, \qquad a_2 = 0, \; \dots \; a_n = 0, \tag{46}$$

where a is the length of the perpendicular from any point (x, y) on the line of action of P. Now since $\mathrm{R}\bar{r} = \mathrm{G}$, it is plain that in reference to any point in the line of action of the resultant, $\mathrm{G} = 0$; therefore

$$\mathrm{P}_1 a_1 + \mathrm{P}_2 a_2 + \dots + \mathrm{P}_n a_n = \Sigma.\mathrm{P}a = 0, \tag{47}$$

which is the equation to the line of action of R; and written at length is

$$x \cos \alpha\, \Sigma.\mathrm{P} + y \sin \alpha\, \Sigma.\mathrm{P} - \Sigma.\mathrm{P}p = 0; \qquad (48)$$

and therefore the perpendicular distance from the origin on the line of action of R is $\dfrac{\Sigma.\mathrm{P}p}{\Sigma.\mathrm{P}}$.

Thus if the equations of the lines of actions of the several parallel forces are given, that of the line of action of the resultant is given by (45) or (48): and it is the locus of point in the plane of the forces with reference to which the sum of the moments of the component couples vanishes.

55.] If the forces are in equilibrium, that is, if the system is what we shall call an equilibrium-system, whatever point is taken as the origin, the particle at that point is at rest, and the moment of the couple producing rotation about that point vanishes. If this is the case we must have the two following conditions; viz.

$$\mathrm{R} = \Sigma.\mathrm{P} \;\; = 0; \qquad (49)$$

$$\mathrm{G} = \Sigma.\mathrm{P}p = 0; \qquad (50)$$

and these are the conditions of equilibrium of a system of parallel forces.

If $\Sigma.\mathrm{P} = 0$, and $\Sigma.\mathrm{P}p$ is a finite quantity, then $\mathrm{R} = 0$, $\bar{r} = \infty$, and the forces are reducible to a couple whose moment is $\Sigma.\mathrm{P}p$.

If $\Sigma.\mathrm{P}p = 0$, and $\Sigma.\mathrm{P}$ is a finite quantity, the forces are reduced to a single force of translation, the line of action of which passes through the origin.

It will be observed that $\Sigma.\mathrm{P}$ which is equal to R is a quantity independent of the position of the origin and of the coordinate axes; and is accordingly an invariant. Not so is $\Sigma.\mathrm{P}p$ or G; it depends on the position of the origin, although it is independent of that of the coordinate axes. The law of dependence will be considered at length in a more general case hereafter.

56.] In the preceding Articles the line of action, the direction, and the magnitude of the resultant of a system of parallel forces have been determined, when the lines of action, direction, and magnitudes of the component forces have been given: that is, we have considered the forces with reference to only three out of the four incidents as stated in Art. 14. The problem which I have now to investigate will require the fourth incident also, viz. the point of application of each force. The problem is this. Suppose that an equilibrium-system consists of n parallel forces, of each of which the four incidents are given; what conditions must it fulfil, so that it should be an equilibrium-system, when, the direction and points of application being unchanged, the lines of action are all turned in the same direction in the plane of the forces through the same angle?

As the action-lines of the forces are all turned through the same angle, the system after the displacement is also one of parallel forces. Let $\mathrm{P}_1, \mathrm{P}_2, \dots \mathrm{P}_n$ be the forces, and let (x_1, y_1) $(x_2, y_2) \dots (x_n, y_n)$ be their points of application, and let α' be the angle between the new lines of action and the x-axis. Then the conditions of equilibrium of the displaced system are (1) $\Sigma.\mathrm{P} = 0$; (2) $\Sigma.\mathrm{P}p = \Sigma.\mathrm{P}(x \sin \alpha' - y \cos \alpha') = 0$; the former of which is satisfied because the system was originally in equilibrium; and as α' in the latter is indeterminate, we must have

$$\Sigma.\mathrm{P}x = 0, \qquad \Sigma.\mathrm{P}y = 0; \tag{51}$$

and these together with $\Sigma.\mathrm{P} = 0$ are the conditions requisite that an equilibrium-system of parallel forces should also be an equilibrium-system when the lines of action of the forces are all turned through the same angle in the plane of the forces.

From these conditions we have the following results. Let us suppose the equilibrium-system to consist of n forces $\mathrm{P}_1, \mathrm{P}_2, \dots \mathrm{P}_n$ whose points of applications are $(x_1, y_1), (x_2, y_2) \dots (x_n y_n)$ and of a force $-\mathrm{R}$, whose point of application is $(\bar{x}, \bar{y})$; then R, acting along the action-line of $-\mathrm{R}$, will neutralize $-\mathrm{R}$, and is consequently the resultant of the n forces $\mathrm{P}_1, \mathrm{P}_2, \dots \mathrm{P}_n$; and the preceding conditions become

$$\mathrm{R} = \Sigma.\mathrm{P}, \qquad \mathrm{R}\bar{x} = \Sigma.\mathrm{P}x, \qquad \mathrm{R}\bar{y} = \Sigma.\mathrm{P}y.$$

$$\left.\begin{aligned} \therefore \quad \bar{x} &= \frac{\Sigma.\mathrm{P}x}{\mathrm{R}} = \frac{\Sigma.\mathrm{P}x}{\Sigma.\mathrm{P}}; \\ \bar{y} &= \frac{\Sigma.\mathrm{P}y}{\mathrm{R}} = \frac{\Sigma.\mathrm{P}y}{\Sigma.\mathrm{P}}; \end{aligned}\right\} \tag{52}$$

which are the coordinates of the point of application of the resultant of the n components, and are the same whatever is the angle through which the action-lines of the forces are turned in the plane of the forces. It is for this reason that the point $(\bar{x}, \bar{y})$ is called the centre of parallel forces. We shall hereafter have many applications in which the position of it is of great importance.

If the centre of parallel forces is at the origin, then in that system of forces, and in that reference, $\Sigma.\mathrm{P}x = \Sigma.\mathrm{P}y = 0$.

If the system consists of two forces P_1 and P_2 applied at the points (x_1, y_1) (x_2, y_2) respectively, then

$$\bar{x} = \frac{P_1 x_1 + P_2 x_2}{P_1 + P_2}, \qquad \bar{y} = \frac{P_1 y_1 + P_2 y_2}{P_1 + P_2};$$

and if $P_2 = -P_1$, $\bar{x} = \bar{y} = \infty$; consequently, as in this case the system is a couple, the centre of a couple is at an infinite distance.

If the forces are all equal, viz. $P_1 = P_2 = \ldots = P_n$, then

$$\left.\begin{aligned} \bar{x} &= \frac{P\Sigma.x}{nP} = \frac{x_1 + x_2 + \ldots + x_n}{n}, \\ \bar{y} &= \frac{P\Sigma.y}{nP} = \frac{y_1 + y_2 + \ldots + y_n}{n}; \end{aligned}\right\} \qquad (53)$$

and the centre of parallel forces is the centre of mean distances of the points at which the forces are applied.

The following are examples in which the centre of parallel forces is determined.

Ex. 1. Suppose six parallel pressures proportional to the numbers 1, 2, ... 6 to act at points whose coordinates are severally $(-2, -1)$, $(-1, 0)$, $(0, 1) \ldots (3, 4)$; find the resultant, and the centre of these parallel forces.

$$\begin{aligned} R = \Sigma.P &= 1 + 2 + \ldots + 6 \\ &= 21; \\ \Sigma.Px &= -2 - 2 + 4 + 10 + 18 \\ &= 28; \\ \Sigma.Py &= -1 + 3 + 8 + 15 + 24 \\ &= 49; \end{aligned}$$

$$\therefore \quad \bar{x} = \frac{28}{21}; \qquad \bar{y} = \frac{49}{21}.$$

Ex. 2. At the three angular points of a triangle parallel forces are applied severally proportional to the opposite sides of the triangle; it is required to find the centre of these forces.

Let (x_1, y_1) (x_2, y_2) (x_3, y_3) be the angular points of the triangle, and let a, b, c be the sides severally opposite to them; then

$$\bar{x} = \frac{ax_1 + bx_2 + cx_3}{a + b + c}; \qquad \bar{y} = \frac{ay_1 + by_2 + cy_3}{a + b + c}.$$

57.] Composition of many forces acting in one plane on a rigid body or a rigid system of material particles.

Let the plane in which the forces act be that of (x, y); and let O, the origin, fig. 27, be a point of the body, or rigidly connected with it: let the forces be $P_1, P_2, \ldots P_n$: let $a_1, a_2, \ldots a_n$ be the angles between their lines of action and the axis of x: let $p_1, p_2, \dots p_n$ be the lengths of the perpendiculars drawn from the origin on the lines of action: and of these quantities let P, a, and p be the types: so that

$$p = x \sin a - y \cos a. \tag{54}$$

At O let there be introduced two forces equal to P, with their lines of action parallel to that of P, and in opposite directions; so that, in the place of the original force P, we have P acting at O in a parallel line and the same direction, and a couple whose moment is $\text{P}p$, and whose rotation-axis is perpendicular to the plane of the forces. Let P at O be resolved into parts along the coordinate axes, so that $\text{P} \cos a$ acts along the axis of x, and $\text{P} \sin a$ along that of y; and let all the forces be similarly replaced. Then if X and Y are the sums of the resolved parts of the forces along the axes of x and y respectively,

$$\begin{aligned} \text{X} &= \text{P}_1 \cos a_1 + \text{P}_2 \cos a_2 + \dots + \text{P}_n \cos a_n, \\ &= \Sigma . \text{P} \cos a, \end{aligned} \tag{55}$$

$$\begin{aligned} \text{Y} &= \text{P}_1 \sin a_1 + \text{P}_2 \sin a_2 + \dots + \text{P}_n \sin a_n, \\ &= \Sigma . \text{P} \sin a; \end{aligned} \tag{56}$$

and if R is the resultant of X and Y, and a is the angle between the action-line of R and the x-axis,

$$\text{R}^2 = \text{X}^2 + \text{Y}^2; \tag{57}$$

$$\cos a = \frac{\text{X}}{\text{R}}, \qquad \sin a = \frac{\text{Y}}{\text{R}}. \tag{58}$$

Also the moment of the couple arising from P is $\text{P}p$; the tendency of which is to turn the body from the axis of x towards that of y; and as a similar couple will arise from every one of the forces, and as all these couples are coaxal, the moment of their resultant is equal to the sum of the moments of the components. Let G be the moment of the resultant couple; then

$$\begin{aligned} \text{G} &= \text{P}_1 p_1 + \text{P}_2 p_2 + \dots + \text{P}_n p_n \\ &= \Sigma . \text{P} p \\ &= \Sigma . \text{P} (x \sin a - y \cos a) \\ &= \Sigma . \text{P} x \sin a - \Sigma . \text{P} y \cos a. \end{aligned} \tag{59}$$

58.] From these results four cases arise: (1) that in which R and G have both finite values; (2) that in which R is finite, and $\text{G} = 0$; (3) that in which $\text{R} = 0$, and G is finite; (4) that in which $\text{R} = 0$, and $\text{G} = 0$. These cases severally require consideration.

The first case in which R and G have both finite values is that in which these resultants are equivalent to a single force of translation which acts along a definite line of action. For let the couple whose moment is G be turned about its rotation-axis until its arm is perpendicular to the action-line of R; and let the length of the arm of $G = r$, and the force $= R$, so that $rR = G$. Also let the couple be so placed that one of its forces acts along the action-line of the resultant of translation, and in a direction opposite to that of that resultant; and the other acts along a line parallel to the resultant, and at a distance r from it. Then one force of the couple is neutralized by the resultant of translation, but the other force remains, and is the final single resultant of translation; and as its action-line is parallel to that of the original resultant and at a distance r from it, where $Rr = G$, if x and y are its current coordinates, $r = x \sin a - y \cos a$; and either

$$xR \sin a - yR \cos a = G, \tag{60}$$

or

$$xY - yX = G, \tag{61}$$

is the equation to the action-line of R.

If the equations of the action-lines of the several components are given in the ordinarily abridged forms of notation; that is, if $a_1 = 0$, $a_2 = 0$, ... $a_n = 0$ are the equations to the lines along which $P_1, P_2, \dots P_n$ act, then the equation to the action-line of R is

$$P_1 a_1 + P_2 a_2 + \dots + P_n a_n = 0, \tag{62}$$

or

$$x \Sigma . P \cos a + y \Sigma . P \sin a = \Sigma . Pp; \tag{63}$$

either of which equations states that the action-line of the resultant is the locus of points in reference to which the moment of the resultant couple vanishes.

59.] The second case is that in which R is finite, and $G = 0$. This is that particular case of the preceding Article, in which the forces have a resultant of translation, on the action-line of which the origin has been taken.

In the third case, $R = 0$, and G is finite. Here the forces are equivalent to a couple whose moment is G, and the value of which is independent of the position of the origin in the plane of the forces.

In the fourth case $R = 0$, and $G = 0$; that is, no force acts at the origin, and there is no force of rotation tending to turn the body about an axis perpendicular to the plane of the forces; that is, there is no pressure of translation on the origin, and no pressure of rotation about it; in other words the forces are in equilibrium and the body is at rest. And since by reason of (57), when $R=0$, $X=0$, $Y=0$, three conditions must be satisfied by a system of forces, whose action-lines are in one plane, which are in equilibrium; viz.

$$\left.\begin{aligned} X &= \Sigma . P\cos\alpha = 0, \\ Y &= \Sigma . P\sin\alpha = 0; \end{aligned}\right\} \qquad (64)$$

$$G = \Sigma . Pp \quad = 0. \qquad (65)$$

As the origin is arbitrary and the directions of the axes are also arbitrary, a system of forces acting in one plane on a body is in equilibrium, if the sums of the resolved parts of the forces along any two straight lines in the plane perpendicular to one another vanish, and if the sum of the moments of the forces about an axis perpendicular to the plane also vanishes.

As the three conditions given in (64) and (65) are all that can in the most general case be required for the equilibrium of a system of forces in one plane, they show that the body on which the forces act has at the most three degrees of freedom; which have to be severally neutralized. There are two displacements of translation along any two lines which are perpendicular to each other, and a displacement of rotation about an axis perpendicular to the plane of the forces.

If one point of the body in which the forces act is fixed, and the point is in the plane of the forces, the body can have no displacement of translation, and this circumstance satisfies the first two conditions, viz. (64); and this effect is also otherwise manifest, inasmuch as the determination of a point requires two conditions, and these may be the first two of (64).

If two points of the body are fixed in the plane in which the forces act, the body is entirely fixed. These circumstances indeed give one condition in excess of those which are requisite; they give four conditions, whereas three are sufficient to satisfy (64) and (65).

The four preceding cases show that when a body is acted on by a system of forces whose action-lines are in one plane, the system is either one of equilibrium, or is reducible to a single force of translation, or to a single couple of rotation.

60.] The examples in which the equations of equilibrium (64) and (65) are applied are extremely numerous; and a large supply will be found in any of the ordinary collections; it is desirable however to insert a few, that the reader may understand the mode of application.

Ex. 1. A heavy uniform beam AB rests in a vertical plane, fig. 28, with one end A on a smooth horizontal plane and the other end B against a smooth vertical wall: the end A is prevented from sliding by a horizontal string of given length fastened to the end of the beam and to the wall: determine the tension of the string and the pressures against the horizontal plane and the wall.

Let the length of the beam be $2a$, and let W be its weight; which, as the beam is uniform, we may suppose to act at its middle point G; let R be the vertical pressure of the horizontal plane against the beam; and R′ the horizontal pressure of the vertical wall, and T the tension of the horizontal string AC; let BAC $= \alpha$, which is a known angle, as the lengths of the beam and the string are given. Then equations (64) and (65) become,

for horizontal forces, $\quad T = R'$;

for vertical forces, $\quad W = R$;

for moments about A, $\quad Wa \cos\alpha = R' 2a \sin\alpha$;

$$\therefore \quad R' = T = \frac{W}{2}\cot\alpha.$$

Ex. 2. A heavy uniform beam rests on two given smooth inclined planes: it is required to find the position of the beam, and the pressures on the planes.

Let AB, fig. 29, be the beam, whose length is $2a$, and whose weight is W acting at the centre of gravity G: let the inclinations of the planes AC and BC to the horizon be respectively α and β; and let the inclination of the beam be θ; let R and R′ be the pressures of the planes on the beam, and the lines of action of which are perpendicular to the planes by reason of their smoothness. Then we have

for horizontal forces, $\quad R\sin\alpha = R'\sin\beta$;

for vertical forces, $\quad W = R\cos\alpha + R'\cos\beta$;

for moments about G, $\quad Ra\cos(\alpha-\theta) = R'a\cos(\beta+\theta)$;

$$\therefore \quad \tan\theta = \frac{\sin(\alpha-\beta)}{2\sin\alpha\sin\beta};$$

$$R = \frac{W\sin\beta}{\sin(\alpha+\beta)}; \qquad R' = \frac{W\sin\alpha}{\sin(\alpha+\beta)}.$$

Ex. 3. A heavy uniform beam AB, fig. 30, rests with one end A against a smooth vertical wall, and the other B is fastened by a string BC of given length to a point C in the wall; the beam and the string are in a vertical plane: it is required to determine the pressure against the wall, the tension of the string, and the position of the beam and the string.

Let $\text{AG} = \text{GB} = a, \quad \text{AC} = x, \quad \text{BC} = b,$

weight of beam $=$ W, tension of string $=$ T, pressure of wall $=$ R,

$$\text{BAE} = \theta, \qquad \text{BCA} = \phi;$$

then for horizontal forces, $\quad \text{R} = \text{T} \sin\phi;$

for vertical forces, $\quad \text{W} = \text{T} \cos\phi;$

for moments about A, $\quad \text{W}a \sin\theta = \text{T}x \sin\phi;$

$$\therefore \quad a\sin\theta = x\tan\phi;$$

and, by the geometry of the figure,

$$\frac{b}{\sin\theta} = \frac{2a}{\sin\phi} = \frac{x}{\sin(\theta-\phi)},$$

$$\therefore \quad x = \left\{\frac{b^2-4a^2}{3}\right\}^{\frac{1}{2}}; \qquad \cos\phi = \frac{2}{b}\left\{\frac{b^2-4a^2}{3}\right\}^{\frac{1}{2}};$$

$$\sin\theta = \frac{1}{2a}\left\{\frac{16a^2-b^2}{3}\right\}^{\frac{1}{2}};$$

whence R and T are known.

Ex. 4. A system of forces acting on a rigid body in one plane is represented by the sides of a plane closed polygon taken in order; it is required to determine the resultant.

Let some point within the polygon be taken for the origin, and two lines drawn perpendicularly to each other for coordinate axes. Let the lengths of the sides of the polygon be $s_1, s_2, \dots s_n$; and let their angles of inclination to the axis of x be $\alpha_1, \alpha_2, \dots \alpha_n$, and the perpendiculars from the origin on the lines of action be $p_1, p_2, \dots p_n$: at the origin let pairs of equal and opposite forces be introduced, equal and parallel to those along the sides of the polygon: so that the system is changed into (1) a system of forces acting at the origin, which are in equilibrium by reason of Article 29, and (2) a system of coaxal couples, the moment of the resultant of which is equal to $s_1 p_1 + s_2 p_2 + \dots + s_n p_n$; that is, to a moment of which the geometrical representative is twice the area of the polygon.

A particular case is that of a triangle, whose sides are geometrical representatives of three forces: of which the resultant of translation vanishes, and the moment of the resultant couple is represented by twice the area of the triangle. See Art. 41.

Ex. 5. A heavy and smooth circular ring rests on two horizontal bars, which are not in the same horizontal plane: determine the pressure on each bar.

Let fig. 31 represent a vertical section of the system; P and Q being the two bars, R and R′ the pressures of the ring against them, W the weight of the ring acting at its centre O; let the angle POQ $= \alpha$, which is known; and let the angles of inclination to the vertical of the lines of action of R and of R′ be β and γ; then, as the three forces meet in the centre of the ring, we have

$$\frac{\mathrm{R}}{\sin\gamma} = \frac{\mathrm{R}'}{\sin\beta} = \frac{\mathrm{W}}{\sin\alpha}.$$

Ex. 6. A parabolic curve, fig. 32, is placed in a vertical plane with its axis vertical and vertex downwards, and inside of it and against a peg in the focus a smooth uniform and heavy beam rests: required the position of rest.

Let PQ be the beam, of length $2c$ and of weight W; let SA $=a$, SP $= r$, PSA $= \theta$;

$$\therefore \quad r = \frac{2a}{1+\cos\theta};$$

$$\text{also} \qquad \mathrm{SPT} = \mathrm{STP} = 90^\circ - \frac{\theta}{2}; \qquad \mathrm{PG} = \mathrm{GQ} = c,$$

$$\left.\begin{array}{ll} \text{for forces along PQ,} & \mathrm{R}\sin\mathrm{STP} = \mathrm{W}\cos\theta; \\ \text{for moments about S,} & \mathrm{R}r\cos\mathrm{SPT} = \mathrm{W}(r-c)\sin\theta; \end{array}\right\} \qquad (66)$$

$$\therefore \quad \theta = 2\cos^{-1}\left(\frac{a}{2c}\right)^{\frac{1}{4}}.$$

Suppose that it were required to find the curve AP such that the beam should rest in all positions; then $\tan \mathrm{SPT} = r\dfrac{d\theta}{dr}$; therefore from (66),

$$\frac{d\theta}{dr} = \frac{\cos\theta}{(r-c)\sin\theta}; \qquad \therefore \quad r = c + a\sec\theta; \qquad (67)$$

where a is an arbitrary constant; and this is the equation to the conchoid with an arbitrary modulus.

Ex. 7. To discuss the properties and conditions of equilibrium of a balance; fig. 33.

Let AB be the arm of the balance; AC $=$ CB $= a$; and let the balance be suspended by a point O in a line perpendicular to AB at its middle point C, and let OC $= c$; let the balance be symmetrical with respect to the line OC, and let the centre of gravity of the beam, scales, &c. be at G; let OG $= h$, and let the weight of the whole machine, short of the weights in the scales, $=$ W; and to consider the general case suppose the weights in the scales P and Q to be unequal, Q being greater than P; and let the arm of the balance be inclined to the horizontal line at an angle θ. Then the vertical pressure on O $=$ P $+$ Q $+$ W; and taking moments about O,

$$\text{Q}(a\cos\theta - c\sin\theta) = \text{P}(a\cos\theta + c\sin\theta) + \text{W}h\sin\theta;$$

$$\therefore \quad \tan\theta = \frac{(\text{Q}-\text{P})a}{c(\text{P}+\text{Q})+\text{W}h}. \tag{68}$$

Now the conditions required in a balance are (1) horizontality of the beam, when the arms and weights are equal; (2) sensibility, which is estimated by the angle through which the arm is turned when the weights are unequal; (3) stability, or the tendency to return after the cause of displacement is removed.

Condition (1) is fulfilled when Q $=$ P, since, by (68), in that case, $\theta = 0$.

Condition (2) is more or less satisfied according as θ is larger or smaller for a small difference between P and Q; now in (68), if Q $-$ P is very small, $\tan\theta$, and therefore θ, is large,

(1) when a is large, that is, when the arms of the balance are long;

(2) when c is small, that is, when the point of suspension is not far above the beam;

(3) when P $+$ Q is small, that is, when the weights are small;

(4) when W is small, that is, when the weight of the whole balance is small;

(5) when h is small, that is, when the centre of gravity of the machine is not far below the beam;

and either c or h or both may be negative; and then as a limiting case we may have $\tan\theta = \infty$, and $\theta = 90°$; in which case the beam becomes vertical when it is displaced at all, and may have no tendency to return to its horizontal position; and thus the sensibility of the balance may be very great, but there may be no stability, and one of the necessary conditions is not satisfied: this last condition therefore may be inconsistent with the second, and the two must be adjusted as is practically most convenient.

61.] Although in all cases it is possible, and in most cases scarcely less general, to refer forces and conditions of equilibrium to rectangular coordinates, yet it is desirable to indicate the forms which the reduced resultants take, if the coordinate axes are oblique.

Let the angle of ordination be ω; let the forces be $P_1, P_2, \dots P_n$; $(x_1, y_1), (x_2, y_2), \dots (x_n, y_n)$ their points of application; $p_1, p_2, \dots p_n$ the perpendiculars from the origin on their lines of action; $\alpha_1 \beta_1, \alpha_2 \beta_2, \dots \alpha_n \beta_n$ the angles between the perpendiculars to the lines of action and the axes of x and y respectively; then, employing the symbols without any subscripts as the type-symbols, we have for the line of action of P

$$x \cos \alpha + y \cos \beta - p = 0. \tag{69}$$

Let two equal and opposite forces, each of which is equal to P and has its line of action parallel to that of P, be introduced at the origin; so that, instead of the one force P applied at (x, y), there are (1) a parallel and equal force at the origin, (2) a couple whose arm is p and whose force is P. Let the former be resolved into parts along the coordinate axes, viz. $-P \sin \alpha$, and $-P \sin \beta$; and let all the forces be similarly reduced; let X and Y be the sums of the resolved parts along the axes of x and y respectively; then

$$\begin{aligned} -X &= P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \dots + P_n \sin \alpha_n \\ &= \Sigma . P \sin \alpha; \end{aligned} \tag{70}$$

$$\begin{aligned} -Y &= P_1 \sin \beta_1 + P_2 \sin \beta_2 + \dots + P_n \sin \beta_n \\ &= \Sigma . P \sin \beta; \end{aligned} \tag{71}$$

and therefore if R is the resultant of X and of Y,

$$R^2 = X^2 + 2XY \cos \omega + Y^2. \tag{72}$$

And let G be the moment of the resultant couple: then

$$\begin{aligned} G &= P_1 p_1 + P_2 p_2 + \dots + P_n p_n \\ &= \Sigma . P p \\ &= \Sigma . P (x \cos \alpha + y \cos \beta). \end{aligned} \tag{73}$$

If the impressed forces are in equilibrium, $R = 0$, and $G = 0$;

$$\therefore \quad \Sigma . P \sin \alpha = 0, \qquad \Sigma . P \sin \beta = 0, \qquad \Sigma . P p = 0.$$

If the equations to the lines of action of the impressed forces are given, that to the line of action of the resultant may thus be found; let the equation to the lines of action of the forces be

$$\left.\begin{aligned} x \cos \alpha_1 + y \cos \beta_1 - p_1 &= 0, \\ x \cos \alpha_2 + y \cos \beta_2 - p_2 &= 0, \\ \dots \dots \dots & \\ x \cos \alpha_n + y \cos \beta_n - p_n &= 0; \end{aligned}\right\} .$$

then in reference to any point in the line of action of the resultant, $\Sigma.\text{P}p = 0$; therefore we have

$$\Sigma.\text{P}\,(x\cos\alpha + y\cos\beta - p) = 0,$$

$$x\,\Sigma.\text{P}\cos\alpha + y\,\Sigma.\text{P}\cos\beta - \Sigma.\text{P}p = 0. \tag{74}$$

62.] On referring to Arts. 58 and 59 it will be seen that the effects of a system of forces acting in one plane as to translation and as to rotation depend on R and G, since these are respectively the resultant of translation and the moment of the resultant couple with respect to an arbitrarily chosen origin. It will be observed that R is independent of the origin and of the coordinate axes, being the same whatever they are; it is accordingly an invariant. But not so is G, which is equal to $\Sigma.\text{P}p$, and consequently depends on the origin, though it is independent of the coordinate axes; thus the value of it varies according as the point varies in reference to which it is estimated. The general value of it is determined as follows:

Let G_0 be the value of the moment of the resultant couple with reference to (x_0, y_0); and let (x', y') be a point in the action-line of P with respect to (x_0, y_0); so that $x = x_0 + x'$, $y = y_0 + y'$; then from (61),

$$\begin{aligned} \text{G}_0 &= \text{Y}x' - \text{X}y' \\ &= \text{Y}(x - x_0) - \text{X}(y - y_0) \\ &= \text{Y}x - \text{X}y - \text{Y}x_0 + \text{X}y_0 \\ &= \text{G} - \text{Y}x_0 + \text{X}y_0. \end{aligned} \tag{75}$$

The following are theorems deduced from this equation:

(1) On comparing (75) with the equation of the action-line of the resultant given in (61), it is seen that if the right-hand member vanishes, that is, if the point (x_0, y_0) is on the line of action of the resultant of translation, $\text{G}_0 = 0$; that is, the moment of the resultant couple vanishes for all points on the line of action of the resultant, and this is the absolutely least value of G.

(2) If G_0 is a constant, the locus of (x_0, y_0) is a straight line parallel to the action-line of the resultant; hence for all points in a straight line parallel to the action-line of the resultant, the moment of the resultant couple is the same.

(3) If the forces are in equilibrium, so that $\text{X} = \text{Y} = \text{G} = 0$, $\text{G}_0 = 0$; so that if a system of forces is in equilibrium, the moment of the resultant couple vanishes for all points in the plane of the forces.

(4) If the system of forces is reducible to a couple, in which case $R = 0$, that is, $X = Y = 0$, $G_0 = G$; consequently the moment of that couple is the same for all points in the plane of the forces.

(5) If the moment of the resultant couple vanishes for three points in the plane of the forces which are not in the same straight line, the system is in equilibrium. For if (x_1, y_1), (x_2, y_2), (x_3, y_3) are three points in the plane of the forces, and with reference to them we have

$$\left.\begin{array}{l} G - Yx_1 + Xy_1 = 0, \\ G - Yx_2 + Xy_2 = 0, \\ G - Yx_3 + Xy_3 = 0; \end{array}\right\} \tag{76}$$

then eliminating X and Y we have

$$G\{x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1\} = 0:$$

but the second factor of the left-hand member of this equation is twice the area of the triangle of which the three given points are the angular points; and as they are not in the same straight line, it does not vanish: consequently $G = 0$; and similarly $X = 0$, $Y = 0$; and therefore the system is in equilibrium.

(6) Hence if the moment of the resultant couple of the system vanishes for three points in the plane which are not in the same straight line, it also vanishes for all points in the plane.

(7) If the moments of the resultant couples of a system are given for three points not in the same straight line, the moment G_0 is given for every other point (x_0, y_0) of the plane. The given equations are

$$\left.\begin{array}{l} G_1 = G + Yx_1 - Xy_1, \\ G_2 = G + Yx_2 - Xy_2, \\ G_3 = G + Yx_3 - Xy_3; \end{array}\right\} \tag{77}$$

from which G, X, Y may be determined; and consequently G_0, of which the value is given in (75), may be found.

63.] The preceding investigations on the composition of forces in one plane have depended on the magnitude, line of action, and direction of the acting forces; but, the principle of transmissibility having been applied, have been independent of the points of application of the forces. I come now to the problem analogous to that of Art. 56, and propose to consider a case in which the last incidents are required; viz. to investigate the circumstances under which an equilibrium-system of forces in a plane will also be in equilibrium, when the body is displaced in the most general manner in the plane; the magnitudes, points of application in the body, and directions of the forces being the same as before the displacement, and the lines of action in the new position of the body being parallel to those in the former position; or, in other words, when the action-lines of the forces are all turned in the same direction through the same angle in the plane of the forces.

Let us take two systems of rectangular coordinate axes, one of (x, y) fixed in the body, and the other of (x', y') fixed in the plane of the forces; and let these coincide in the original position of the body. Let the body be shifted through distances (x_0, y_0) respectively, parallel to the original fixed axes, so that the origin of the axes fixed in the body is brought to the point (x_0, y_0); and let the body be turned through the angle θ about an axis perpendicular to the plane of the forces, and passing through (x_0, y_0): then, if (x', y') is in reference to the axes fixed in space the same point as (x, y) in reference to the axes fixed in the body,

$$\left.\begin{aligned} x' &= x_0 + x \cos\theta - y \sin\theta, \\ y' &= y_0 + x \sin\theta + y \cos\theta. \end{aligned}\right\} \tag{78}$$

Now as the system of forces is in equilibrium in the original and in the new positions of the body, and as the lines of action of a force in the new position is parallel to that in the former position, we have

$$\Sigma . \mathrm{P} \cos a = \Sigma . \mathrm{P} \sin a = \Sigma . \mathrm{P} (x \sin a - y \cos a) = 0, \tag{79}$$

$$\Sigma . \mathrm{P} (x' \sin a - y' \cos a) = 0; \tag{80}$$

let the values of x', y' which are given in (78) be substituted in (80): then

$$\begin{aligned} &x_0 \, \Sigma . \mathrm{P} \sin a - y_0 \, \Sigma . \mathrm{P} \cos a \\ &+ \cos\theta \, \Sigma . \mathrm{P} (x \sin a - y \cos a) - \sin\theta \, \Sigma . \mathrm{P} (x \cos a + y \sin a) = 0. \end{aligned} \tag{81}$$

As the first three terms of this expression vanish by reason of (79), we must have also

$$\Sigma . \mathrm{P} (x \cos a + y \sin a) = 0; \tag{82}$$

and as this is independent of x_0, y_0, and θ, it holds good for all displacements of the body, and gives a fourth relation to be satisfied by the forces and the points of application, when the system is in equilibrium, whatever is the displacement of the body, so long as the plane of the forces is the same and the displacement of rotation is about an axis perpendicular to the plane of the forces. Hence four conditions must be satisfied, three in (79), and one in (82) when the equilibrium-system satisfies the stated requirement.

The condition (82) admits of the following interpretation. Let the point of application of each force be referred to polar coordinates, the original origin being the pole, and the fixed x-axis the prime radius. Let (r, θ) be the point of application of P, and let P be resolved along and perpendicularly to the radius vector. Let U be the component along the radius vector and acting from the pole, and let U be called the *central component;* let V be the component acting perpendicularly to the radius vector, and tending to increase θ, and let it be called *the transversal component;* all these being type-symbols, and type-names. Then

$$\begin{aligned} \text{U} &= \text{P} \cos(a-\theta) \\ &= \frac{\text{P}(x\cos a + y \sin a)}{r}; \\ \text{V} &= \text{P}\sin(a-\theta) \\ &= \frac{\text{P}(x\sin a - y\cos a)}{r}; \end{aligned}$$

$$\therefore \quad \Sigma.\text{P}\,(x\cos a + y\sin a) = \Sigma.\text{U}r = \text{H, say}: \tag{83}$$

$$\Sigma.\text{P}\,(x\sin a - y\cos a) = \Sigma.\text{V}r = \text{G}. \tag{84}$$

Thus H, which represents (82), is the sum of the products of each central component and the distance from the origin of its point of application. Let H be called *the radial moment**. As the lines of action of all the central components pass through the origin, they produce no pressure of rotation about that point; consequently the moment of the resultant couple is due to the transversal components only; and evidently, as in (84), $\text{G} = \Sigma.\text{V}r$.

Thus if an equilibrium-system of forces in one plane is also in equilibrium after the displacement of the body, subject to the stated conditions, the requisite relations of the forces are given by the four conditions

$$\text{X} = \text{Y} = \text{G} = \text{H} = 0. \tag{85}$$

The first three being requisite so that the system should be an equilibrium-system in its original position; and the last being an additional condition so that it should be an equilibrium-system after displacement.

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* German writers on Mechanics call H "Fliehmomente;" see Dr. Schweins in Crelle's Journal, Vol. XXXVIII, p. 77.

64.] Suppose now one force to be taken out of this equilibrium-system, and to be replaced by an equal one acting at the same point of application and along the same line of action but in an opposite direction; then this new force is the resultant of all the other remaining forces. Let us slightly modify the system as before conceived, and suppose it to consist of $(n+1)$ forces, viz. n forces, $\text{P}_1, \text{P}_2, \dots \text{P}_n$, of which the points of application are $(x_1, y_1), (x_2, y_2), \dots (x_n, y_n)$, and of $-\text{R}$, of which the point of application is $(\bar{x}, \bar{y})$, and a the angle at which its line of action is inclined to the x-axis. Let this be an equilibrium-system, then R is the resultant of the other n forces; let it also be an equilibrium-system after an arbitrary displacement; then the four conditions (85) become

$$\text{X} = \Sigma.\text{P} \cos a = \text{R} \cos a; \quad \text{Y} = \Sigma.\text{P} \sin a = \text{R} \sin a; \tag{86}$$

$$\text{G} = \Sigma.\text{P}\,(x \sin a - y \cos a) = \text{R}\,(\bar{x} \sin a - \bar{y} \cos a); \tag{87}$$

$$\text{H} = \Sigma.\text{P}\,(x \cos a + y \sin a) = \text{R}\,(\bar{x} \cos a + \bar{y} \sin a). \tag{88}$$

Now $(\bar{x}, \bar{y})$ is the point of application of R, and is the same for all positions of the body; that is, the magnitudes of the forces and their points of applications being unaltered, if these lines of action are all turned in the same direction through equal angles in the plane of the forces, the resultant will always be applied at $(\bar{x}, \bar{y})$, its magnitude being unaltered, and its line of action being turned in the plane of the forces through the same angle as the lines of action of the other forces. The point $(\bar{x}, \bar{y})$ is for this reason called *the centre of the forces*, and its position is determined by means of (87) and (88). Thus let the moment of the resultant couple of the n forces $\text{P}_1, \text{P}_2, \dots \text{P}_n$ be G, and let the radial moment of the same forces be H; then we have

$$\text{G} = \text{R}\,(\bar{x} \sin a - \bar{y} \cos a), \tag{89}$$

$$\text{H} = \text{R}\,(\bar{x} \cos a + \bar{y} \sin a): \tag{90}$$

whence

$$\left.\begin{aligned} \bar{x} &= \frac{\text{H} \cos a + \text{G} \sin a}{\text{R}} = \frac{\text{HX} + \text{GY}}{\text{R}^2}, \\ \bar{y} &= \frac{\text{H} \sin a - \text{G} \cos a}{\text{R}} = \frac{\text{HY} - \text{GX}}{\text{R}^2}; \end{aligned}\right\} \tag{91}$$

and these assign the position of the centre of the forces.

If the system consists of parallel forces,

$$\text{H} = \cos a\, \Sigma.\text{P}x + \sin a\, \Sigma.\text{P}y, \qquad \text{G} = \sin a\, \Sigma.\text{P}x - \cos a\, \Sigma.\text{P}y,$$

and consequently

$$\bar{x} = \frac{\Sigma.\text{P}x}{\text{R}}, \qquad \bar{y} = \frac{\Sigma.\text{P}y}{\text{R}}, \tag{92}$$

which are the same values as (52).

65.] The centre of two forces acting in a plane on two given points may be determined in the following manner by a geometrical construction. Let the forces be P, Q, and let their points of application be A and B; let the lines of action of the forces meet in O; describe a circle passing through O, A, B; and let OC be the line of action of the resultant R, and let it cut the circle in C; then C is the centre of P, Q. Whatever is the position of O in the circumference of the circle between A and B, and suppose it to be at O′, the angles AO′B, BO′C, CO′A are equal severally to AOB, BOC, COA; so that the action-lines of all the forces are turned through equal angles in the plane of the forces, as long as O is on the circumference of the circle; and as the equilibrating relation between P, Q, R depends on these angles only, it is the same whatever is the position of O′: but in all cases C remains the same; therefore C is the centre of the forces.

66.] The radial moment of which the value is given in (83) has the following properties:

(1) Since $\text{H} = \Sigma.\text{U}r = \Sigma.\text{P}x \cos a + \Sigma.\text{P}y \sin a$, it appears that the radial moment of the whole system is equal to the sum of the radial moments of the two systems of the resolved forces along the axes.

(2) It is evident that the value of the radial moment is not altered, whatever is the position of the coordinate axes, if the origin remains the same.

(3) If the origin be moved to the point (x_0, y_0); so that, if x', y' are the coordinates at the new origin,

$$x = x_0 + x', \qquad y = y_0 + y';$$

then

$$\begin{aligned} \Sigma.r\text{U} &= \Sigma.\text{P}(x' \cos a + y' \sin a) + x_0\, \Sigma.\text{P} \cos a + y_0\, \Sigma.\text{P} \sin a \\ &= \Sigma.r'\text{U}' + x_0\, \Sigma.\text{P} \cos a + y_0\, \Sigma.\text{A} \sin a; \end{aligned} \tag{93}$$

so that if H_0 is its value at (x_0, y_0),

$$\text{H} = \text{H}_0 + x_0 \text{X} + y_0 \text{Y};$$

$$\therefore \quad \text{H}_0 = \text{H} - \text{X}x_0 - \text{Y}y_0; \tag{94}$$

and thus the radial moment varies with the position of the origin to which it is referred.

If (x_0, y_0) is a point at which the central moment vanishes, that is, at which $\mathrm{H}_0 = 0$, then

$$\mathrm{X}x_0 + \mathrm{Y}y_0 = \mathrm{H}; \tag{95}$$

which is the equation to a straight line, of which x_0, y_0 are the coordinates; and consequently at any point in this line the radial moment vanishes. This line is called *the line of radial moments.*

On comparing this equation with (88) and (90) it appears that the centre of the forces lies on this line of radial moments; and as it also, as it appears from (87) and (89), lies in the line of action of the resultant; the centre of forces is at the intersection of these two lines, and these two lines intersect, as their equations shew, at right angles.

From (94) a series of theorems may be inferred similar to those which have been inferred in Art. 62, from (75).

67.] If the system of forces in its original condition is reducible to a couple, so that $\Sigma.\mathrm{P}\cos a = 0$, $\Sigma.\mathrm{P}\sin a = 0$, but that $\mathrm{G} = \Sigma.\mathrm{P}(x\sin a - y\cos a)$ does not vanish; and if after the displacement the system is an equilibrium-system, then from (81),

$$\tan\theta = \frac{\Sigma.\mathrm{P}(x\sin a - y\cos a)}{\Sigma.\mathrm{P}(x\cos a + y\sin a)} = \frac{\mathrm{G}}{\mathrm{H}}; \tag{96}$$

and thus the angle is assigned through which the system must be turned, so as to be brought into an equilibrium-system. This result is also manifest from the following reasoning.

Let the forces of the couple to which the original system is equivalent be $\mathrm{P}_1, -\mathrm{P}_1$; and let their points of applications be (x_1, y_1) (x_2, y_2) and let a be the angle between their action-lines and the x-axis; and let r be the distance between their points of application, and θ the angle between this line and the action-lines of the forces. Then if the lines of action of the forces are turned through an angle θ towards the line r, these lines will lie along r and the two forces will neutralize each other, and the system will become an equilibrium-system. Now $\mathrm{G} = \mathrm{P}_1 r\sin\theta$, $\mathrm{H} = \mathrm{P}_1\{(x_1 - x_2)\cos a + (y_1 - y_2)\sin a\} = \mathrm{P}_1 r\cos\theta$:

$$\therefore\ \tan\theta = \frac{\mathrm{G}}{\mathrm{H}};$$

which is the same result as (96).

SECTION 4.—*Composition and resolution of forces acting on a rigid body or system of material particles in any directions.*

68.] We proceed now to the most general case of statical forces acting in any directions on a rigid body or system of material particles in space.

Let any point, either of the body, or rigidly connected with it, be taken as the origin, and let a system of rectangular coordinate axes originate at it. Let the forces be $\text{P}_1, \text{P}_2, \dots \text{P}_n$; the direction-angles of their lines of action, $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \dots \alpha_n, \beta_n, \gamma_n$; a point in the line of application of each (x_1, y_1, z_1), $(x_2, y_2, z_2), \dots (x_n, y_n, z_n)$; the perpendiculars from the origin on their lines of action, $p_1, p_2, \dots p_n$; and of these quantities let the types be $\text{P}, \alpha, \beta, \gamma, (x, y, z), p$. At the origin O, fig. 35, let there be introduced a pair of equal and opposite forces, each of which is equal to P, and has its line of action parallel to that of P; from O let the perpendicular OD $(= p)$ be drawn to the line of action of P: then, instead of the original P, we have P at O equal to the former force and acting in the same direction along a parallel line of action, and a couple each of whose forces is P, whose arm is OD, and whose rotation-axis is perpendicular to the plane PODP: and let a similar process be performed on all the other forces. As to the force of translation at O, let P be resolved into three components $\text{P}\cos\alpha$, $\text{P}\cos\beta$, $\text{P}\cos\gamma$ along the axes of x, y, z respectively; and let X, Y, Z be the sums of the resolved parts of all the forces along these axes; then

$$\begin{aligned} \text{X} &= \text{P}_1 \cos\alpha_1 + \text{P}_2 \cos\alpha_2 + \dots + \text{P}_n \cos\alpha_n \\ &= \Sigma.\text{P}\cos\alpha; \qquad (97) \\ \text{Y} &= \text{P}_1 \cos\beta_1 + \text{P}_2 \cos\beta_2 + \dots + \text{P}_n \cos\beta_n \\ &= \Sigma.\text{P}\cos\beta; \qquad (98) \\ \text{Z} &= \text{P}_1 \cos\gamma_1 + \text{P}_2 \cos\gamma_2 + \dots + \text{P}_n \cos\gamma_n \\ &= \Sigma.\text{P}\cos\gamma; \qquad (99) \end{aligned}$$

and consequently, if R is the resultant of these three forces,

$$\text{R}^2 = \text{X}^2 + \text{Y}^2 + \text{Z}^2; \qquad (100)$$

and if a, b, c are the direction-angles of the line of action of R,

$$\cos a = \frac{\text{X}}{\text{R}}, \qquad \cos b = \frac{\text{Y}}{\text{R}}, \qquad \cos c = \frac{\text{Z}}{\text{R}}; \qquad (101)$$

so that the magnitude, the line of action, and the direction of R are known.

As to the couple which arises from P, its moment is Pp: and as p is the perpendicular distance from the origin on a line passing through a point (x, y, z), and having direction-angles, a, β, γ,

$$p^2 = (y\cos\gamma - z\cos\beta)^2 + (z\cos a - x\cos\gamma)^2 + (x\cos\beta - y\cos a)^2;$$

and as the rotation-axis of the couple is perpendicular to the plane passing through the origin and containing this line, its direction-cosines are

$$\frac{y\cos\gamma - z\cos\beta}{p}, \quad \frac{z\cos a - x\cos\gamma}{p}, \quad \frac{x\cos\beta - y\cos a}{p}; \tag{102}$$

in accordance with the law of Article (52) let us resolve the moment-axis of the couple along the three coordinate axes; then the resolved parts are $\text{P}(y\cos\gamma - z\cos\beta)$, $\text{P}(z\cos a - x\cos\gamma)$, $\text{P}(x\cos\beta - y\cos a)$, which are the moment-axes of the three component couples, and whose rotation-axes are along the three coordinate axes. Let the couples corresponding to all the impressed forces be similarly resolved, and let L, M, N be the sums of the moment-axes of those couples whose rotation-axes are severally along the three coordinate axes: so that by reason of (30) Article 49,

$$\text{L} = \text{P}_1(y_1\cos\gamma_1 - z_1\cos\beta_1) + \ldots + \text{P}_n(y_n\cos\gamma_n - z_n\cos\beta_n); \tag{103}$$

$$\left.\begin{aligned} \therefore\quad \text{L} &= \Sigma.\text{P}(y\cos\gamma - z\cos\beta); \\ \text{similarly}\quad \text{M} &= \Sigma.\text{P}(z\cos a - x\cos\gamma); \\ \text{N} &= \Sigma.\text{P}(x\cos\beta - y\cos a); \end{aligned}\right\} \tag{104}$$

and if G is the resultant moment-axis of these three couples,

$$\text{G}^2 = \text{L}^2 + \text{M}^2 + \text{N}^2; \tag{105}$$

and if the direction-angles of the resultant rotation-axis are λ, μ, ν,

$$\cos\lambda = \frac{\text{L}}{\text{G}}, \quad \cos\mu = \frac{\text{M}}{\text{G}}, \quad \cos\nu = \frac{\text{N}}{\text{G}}; \tag{106}$$

so that both the moment-axis and the rotation-axis of the resultant couple are determined. Thus the forces are reduced to a force of translation, viz. R, acting at the origin, and to a couple G, whose axis is determined by (105) and (106).

69.] The formulae (104) require closer consideration; the right-hand member of each of the equations consists of two parts, one of which is affected with a positive, and the other with a negative sign. Thus L is composed of two sets of coaxal couples, viz. $\Sigma.\text{P}y\cos\gamma$ and $-\Sigma.\text{P}z\cos\beta$; the former of which is the sum of a system of couples, the force in each of which is the z-component of the impressed force, and the arm is the y-ordinate of its point of application; and in the latter system, the force of each couple is the y-component of the impressed force, and the arm is the z-ordinate of its point of application. Imagine therefore the force P to be, at its point of application, resolved into three components along lines parallel to the coordinate axes; and let these be $\mathrm{P}\cos\alpha$, $\mathrm{P}\cos\beta$, $\mathrm{P}\cos\gamma$; and let couples be considered positive, which having for their rotation-axes severally the axes of x, y, and z, tend to turn the body from the y-axis to the z-axis, from the z-axis to the x-axis, from the x-axis to the y-axis; and let those couples be negative which act in a contrary direction: which arrangement, it will be observed, is cyclical. Now consider $\mathrm{P}\cos\gamma$; and, fig. 36, introduce at M and at O two equal and opposite forces, equal to it and acting parallel to its line of action; so that we have a parallel and equal force acting at O, and two couples, of one of which the arm is OM, and of the other the arm is MN; of which the former has the axis of y for its rotation-axis and is negative, and the latter has the axis of x for its rotation-axis and is positive; hence $\mathrm{P}\cos\gamma$ acting at P is replaced by

A parallel and equal force, $= \mathrm{P}\cos\gamma$, acting at O,

And a couple whose moment is $\mathrm{P}\cos\gamma\, y$, and whose rotation-axis is the axis of x,

And a couple whose moment is $-\mathrm{P}\cos\gamma\, x$, and whose rotation-axis is the axis of y.

By a similar process will $\mathrm{P}\cos\alpha$ and $\mathrm{P}\cos\beta$ be replaced: and the same process having been performed on all the impressed forces, we have ultimately

$\Sigma.\mathrm{P}\cos\alpha$ acting at O along the axis of x,
$\Sigma.\mathrm{P}\cos\beta$ - - - - - - - - - - y,
$\Sigma.\mathrm{P}\cos\gamma$ - - - - - - - - - - z;

and the couples whose moments are

$\Sigma.\mathrm{P}\,(y\cos\gamma - z\cos\beta)$, the rotation-axis of which is the axis of x,
$\Sigma.\mathrm{P}\,(z\cos\alpha - x\cos\gamma)$, - - - - - - - - - - - - - - y,
$\Sigma.\mathrm{P}\,(x\cos\beta - y\cos\alpha)$, - - - - - - - - - - - - - - z,

which results are the same as those investigated in the preceding Article.

The principle on which signs are affixed to couples is of course arbitrary; we have chosen one depending on the order of the letters which distinguish the coordinate axes; the conventionality of the sign and direction is involved in the sign of p in (102), which may be either positive or negative.

70.] The system of forces being thus reduced to a force of translation R, the line of action of which passes through the origin, an arbitrarily chosen point, and to a couple whose moment is G, there are four cases separately to be considered: (1) when $R = G = 0$, and the body is at rest because there is neither a force of translation nor a couple acting on it; in which case we have an equilibrium-system; (2) when $R = 0$, and G has a finite magnitude, in which case the system is reducible to a couple the direction of whose rotation-axis is assigned by (106); (3) when $G = 0$, and B has a finite magnitude, in which case the system is reduced to a single force of translation the line of action of which passes through the origin; (4) when R and G are both of finite magnitude; in this last case also if the line of action of R lies in the plane of the forces of G, R and these two forces having lines of action in the same plane are reducible to a single force = R, and we have the third case. All these cases will be considered in the following pages.

Let us first take the case when $R = G = 0$; that is, when the particle at the arbitrarily chosen origin is at rest, and when there is no tendency to rotation about any axis passing through that point, so that the whole system is in equilibrium: and by reason of (100) and (105) we have

$$X = 0, \qquad Y = 0, \qquad Z = 0; \tag{107}$$

$$L = 0, \qquad M = 0, \qquad N = 0; \tag{108}$$

or,
$$\Sigma.P\cos\alpha = 0, \qquad \Sigma.P\cos\beta = 0, \qquad \Sigma.P\cos\gamma = 0; \tag{109}$$

$$\left.\begin{aligned} \Sigma.P\,(y\cos\gamma - z\cos\beta) &= 0, \\ \Sigma.P\,(z\cos\alpha - x\cos\gamma) &= 0, \\ \Sigma.P\,(x\cos\beta - y\cos\alpha) &= 0; \end{aligned}\right\} \tag{110}$$

which are six independent conditions to be satisfied for an equilibrium-system; that is, the sums of the resolved parts of the forces along any three rectangular axes vanish; and the sums of the moments of the couples whose rotation-axes coincide with the axes of any system of rectangular coordinates also vanish.

The following is an example in which the preceding conditions are required:

Three planes, whose equations are

$$A_1x + B_1y + C_1z = 0,$$

$$A_2x + B_2y + C_2z = 0,$$

$$A_3x + B_3y + C_3z = 0,$$

meet at the origin, and support between them a heavy sphere of weight W: determine the pressure on each of the planes.

Let the axis of z be taken in a vertical direction; and let the pressures on the planes be R_1, R_2, R_3; the lines of action of which are of course normal to the planes; and let the equations of the planes be such as to satisfy in each the condition, $A^2+B^2+C^2=1$: then (109) become

$$\begin{aligned} R_1A_1+R_2A_2+R_3A_3 &= 0, \\ R_1B_1+R_2B_2+R_3B_3 &= 0, \\ R_1C_1+R_2C_2+R_3C_3 &= W; \end{aligned}$$

from which we have, using the notation of determinants,

$$R_1 = W\frac{\Sigma.\pm A_2B_3}{\Sigma.\pm A_1B_2C_3}, \quad R_2 = W\frac{\Sigma.\pm A_3B_1}{\Sigma.\pm A_2B_3C_1}, \quad R_3 = W\frac{\Sigma.\pm A_1B_2}{\Sigma.\pm A_3B_1C_2};$$

and $$\frac{R_1}{A_2B_3-B_2A_3} = \frac{R_2}{A_3B_1-B_3A_1} = \frac{R_3}{A_1B_2-B_1A_2}.$$

As the six conditions given in (107) and (108) are all that can be required in the most general case for the equilibrium of a body under the action of given forces, they shew that such a body has six degrees of freedom, which they severally neutralize. These are three displacements of translation along any three lines which are perpendicular to each other, and three displacements of rotation about three lines which are also perpendicular to each other. These conditions are also equivalent to three displacements of translation along any three lines which are not all in the same plane, and to three rotations about any three lines which are similarly not in the same plane.

71.] These conditions of an equilibrium-system admit of the following geometrical interpretation. Let (x, y, z) be any point in space; then since $X = Y = Z = 0$,

$$Xx+Yy+Zz = 0; \tag{111}$$

and replacing X, Y, Z by those equivalents given in (109), we have

$$\begin{aligned} &P_1(x\cos\alpha_1+y\cos\beta_1+z\cos\gamma_1) \\ &+P_2(x\cos\alpha_2+y\cos\beta_2+z\cos\gamma_2) \\ &+\cdots\cdots\cdots\cdots \\ &+P_n(x\cos\alpha_n+y\cos\beta_n+z\cos\gamma_n) = 0. \end{aligned} \tag{112}$$

Now as x, y, z are the coordinates of any point in space, $x\cos\alpha_1+y\cos\beta_1+z\cos\gamma_1$ is the projection on the line of action of P_1 of the distance of (x, y, z) from the origin; and therefore, as the origin also is an arbitrary point, this equation expresses the following theorem. If the resultant of translation of a system of forces vanishes, the sum of the products of each force and of the projection on its line of action of a line joining two given points (fixed arbitrarily) is equal to zero.

Also as one of the forces of this system is, when taken in an opposite direction along its action-line, the resultant of all the others, we have the following theorem:

In a system of forces acting on a rigid body, the sum of the products of each force and of the projection on its line of action of a line joining two given points fixed arbitrarily, is equal to the product of the resultant of translation and of the projection on its line of action of the same straight line.

Also if $\text{L} = 0$, $\text{M} = 0$, $\text{N} = 0$, then multiplying these severally by x, y, z, we have

$$\text{L}x + \text{M}y + \text{N}z = 0; \tag{113}$$

and replacing them by their values given in (110), we have

$$\begin{aligned} &\text{P}_1 \{(y_1 \cos\gamma_1 - z_1 \cos\beta_1)x + (z_1 \cos\alpha_1 - x_1 \cos\gamma_1)y \\ &\qquad\qquad + (x_1 \cos\beta_1 - y_1 \cos\alpha_1)z\} \\ &+ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ &+ \text{P}_n \{(y_n \cos\gamma_n - z_n \cos\beta_n)x + (z_n \cos\alpha_n - x_n \cos\gamma_n)y \\ &\qquad\qquad + (x_n \cos\beta_n - y_n \cos\alpha_n)z\} = 0. \end{aligned} \tag{114}$$

Now this expression admits of the following interpretation. The equations to the planes passing through the origin and the lines of action of the forces are

$$\left.\begin{aligned} &(y_1\cos\gamma_1 - z_1\cos\beta_1)\xi + (z_1\cos\alpha_1 - x_1\cos\gamma_1)\eta + (x_1\cos\beta_1 - y_1\cos\alpha_1)\zeta = 0, \\ &\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ &(y_n\cos\gamma_n - z_n\cos\beta_n)\xi + (z_n\cos\alpha_n - x_n\cos\gamma_n)\eta + (x_n\cos\beta_n - y_n\cos\alpha_n)\zeta = 0; \end{aligned}\right\} \tag{115}$$

and if $p_1, p_2, \ldots p_n$ are the lengths of the perpendiculars from the origin on the lines of action of the forces, then

$$p_1^2 = (y_1\cos\gamma_1 - z_1\cos\beta_1)^2 + (z_1\cos\alpha_1 - x_1\cos\gamma_1)^2 + (x_1\cos\beta_1 - y_1\cos\alpha_1)^2, \tag{116}$$

with similar values for $p_2 \ldots p_n$; so that, if $\delta_1, \delta_2 \ldots \delta_n$ are the lengths of the perpendiculars from (x, y, z) on the planes whose equations are given in (115),

$$\delta_1 = \frac{(y_1\cos\gamma_1 - z_1\cos\beta_1)x + (z_1\cos\alpha_1 - x_1\cos\gamma_1)y + (x_1\cos\beta_1 - y_1\cos\alpha_1)z}{p_1}, \tag{117}$$

with similar values for $\delta_2, \delta_3 \ldots \delta_n$; and thus (114) becomes,

$$\text{P}_1 p_1 \delta_1 + \text{P}_2 p_2 \delta_2 + \ldots + \text{P}_n p_n \delta_n = 0. \tag{118}$$

Suppose that along the lines of action of the forces lengths are taken proportional to the magnitude of the forces, and thus proportional to $\text{P}_1, \text{P}_2, \ldots \text{P}_n$: then $\text{P}p$ is twice the area of the triangle whose vertex is at the origin, and of which the base is the straight line represented by P: and as δ is the perpendicular distance from (x, y, z) on the plane of the triangle, $\mathrm{P}p\delta$ is six times the volume of the tetrahedron whose base is the triangle and whose vertex is (x, y, z); that is, whose four vertices are at the origin, the point (x, y, z), and the two extremities of the line representative of P; and as the first two points, viz. the origin and (x, y, z), are arbitrary, this equation expresses the following theorem:

If at any point the resultant couple of a system of forces vanishes, the sum of the volumes of the tetrahedra which have for one edge lines along the action-lines of the forces proportional to the forces and for the opposite edge the line joining the given point and any other fixed point in space, is equal to zero.

This and the former theorem are of course true for any system of forces in equilibrium; and in the latter theorem it is to be observed that the base of each tetrahedron is proportional to the moment of the couple which corresponds to the force.

72.] When the number of forces of which an equilibrium-system consists does not exceed six, equations (109) and (110) contain some remarkable theorems concerning their lines of action and points of application. The equations of equilibrium are six in number, and the symbols of the forces enter into them homogeneously and symmetrically in the first degree, the coefficients being functions of the direction-cosines and current coordinates of the action-lines of the forces. Consequently if the number of forces does not exceed six, relations exist among these coefficients; that is, amongst the elements of their action-lines; and these relations express geometrical theorems.

To abridge the notation I shall take l, m, n to be the direction-cosines of the action-line of P, and I shall employ the notation of determinants. In consequence of the former assumption, the equations of equilibrium become

$$\Sigma.\mathrm{P}\,l = \Sigma.\mathrm{P}\,m = \Sigma.\mathrm{P}\,n = 0; \tag{119}$$

$$\Sigma.\mathrm{P}(ny-mz) = \Sigma.\mathrm{P}(lz-nx) = \Sigma.\mathrm{P}(mx-ly) = 0. \tag{120}$$

If the equilibrium-system consists of only two forces, these equations become

$$\mathrm{P}_1 l_1 + \mathrm{P}_2 l_2 = \mathrm{P}_1 m_1 + \mathrm{P}_2 m_2 = \mathrm{P}_1 n_1 + \mathrm{P}_2 n_2 = 0; \tag{121}$$

$$\left.\begin{array}{l} \mathrm{P}_1(n_1 y_1 - m_1 z_1) + \mathrm{P}_2(n_2 y_2 - m_2 z_2) = 0, \\ \mathrm{P}_1(l_1 z_1 - n_1 x_1) + \mathrm{P}_2(l_2 z_2 - n_2 x_2) = 0, \\ \mathrm{P}_1(m_1 x_1 - l_1 y_1) + \mathrm{P}_2(m_2 x_2 - l_2 y_2) = 0; \end{array}\right\} \tag{122}$$

from which groups, by the elimination of P_1 and P_2, we have

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = -1\,;$$

$$\frac{x_1-x_2}{l_1} = \frac{y_1-y_2}{m_1} = \frac{z_1-z_2}{n_1}\,;$$

whence it is evident that the action-lines of the forces are coincident, the forces being equal and acting along them in opposite directions.

73.] If the equilibrium-system consists of three forces, then (119) and (120) become

$$\begin{vmatrix} l_1, & l_2, & l_3 \\ m_1, & m_2, & m_3 \\ n_1, & n_2, & n_3 \end{vmatrix} = 0\,; \tag{123}$$

$$\begin{vmatrix} n_1 y_1 - m_1 z_1, & n_2 y_2 - m_2 z_2, & n_3 y_3 - m_3 z_3 \\ l_1 z_1 - n_1 x_1, & l_2 z_2 - n_2 x_2, & l_3 z_3 - n_3 x_3 \\ m_1 x_1 - l_1 y_1, & m_2 x_2 - l_2 y_2, & m_3 x_3 - l_3 y_3 \end{vmatrix} = 0\,; \tag{124}$$

(123) shews that the action-lines of the three forces are parallel to the same plane; and (124) shews that they meet in a point; consequently these lines meet in a point and are in the same plane. These equations are also satisfied when the action-lines are parallel and lie in the same plane. Hence three straight lines can be the action-lines of an equilibrium-system only when they meet in a point and lie in the same plane.

74.] If the equilibrium-system consists of four forces; then we have the following equations:

$$\left.\begin{array}{l} \mathrm{P}_1\, l_1 + \mathrm{P}_2\, l_2 + \mathrm{P}_3\, l_3 + \mathrm{P}_4\, l_4 = 0, \\ \mathrm{P}_1\, m_1 + \;-\;-\;-\;-\;-\; = 0, \\ \mathrm{P}_1\, n_1 + \;-\;-\;-\;-\;-\; = 0\,; \end{array}\right\} \tag{125}$$

$$\left.\begin{array}{l} \mathrm{P}_1(n_1 y_1 - m_1 z_1) + \mathrm{P}_2(n_2 y_2 - m_2 z_2) + \mathrm{P}_3(n_3 y_3 - m_3 z_3) + \mathrm{P}_4(x_4 y_4 - m_4 z_4) = 0, \\ \mathrm{P}_1(l_1 z_1 - n_1 x_1) + \;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\; = 0, \\ \mathrm{P}_1(m_1 x_1 - l_1 y_1) + \;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\; = 0. \end{array}\right\} (12$$

Let the ratios of $\mathrm{P}_1 : \mathrm{P}_2 : \mathrm{P}_3 : \mathrm{P}_4$ be determined from (125) and be denoted by the letters q_1, q_2, q_3, q_4; then substituting these in the several equations of (126) we have

$$\left.\begin{array}{l} n_1 q_1 y_1 + n_2 q_2 y_2 + n_3 q_3 y_3 + n_4 q_4 y_4 = m_1 q_1 z_1 + m_2 q_2 z_2 + m_3 q_3 z_3 + m_4 q_4 z_4 \\ l_1 q_1 z_1 + \;-\;-\;-\;-\;-\;-\;-\; = n_1 q_1 x_1 + \;-\;-\;-\;-\;-\;-\;- \\ m_1 q_1 x_1 + \;-\;-\;-\;-\;-\;-\;-\; = l_1 q_1 y_1 + \;-\;-\;-\;-\;-\;-\;- \end{array}\right\} (12$$

Let us suppose three action-lines to be given, and consider the fourth as that which is to be determined; so that x_4, y_4, z_4 are variables and l_4, m_4, n_4 are undetermined in the preceding equations. Then the product of the left-hand members equated to that of the right-hand members is, in terms of these variables, the equation to a hyperboloid of one sheet, the three equations in (127) being those of three fixed lines on which each of the lines (l_1, m_1, n_1), ... (l_4, m_4, n_4) rests; and consequently these four lines are generators of the surface of the same class; the three lines given in (127) being generators of the surface of the other class. Hence we have the following theorem: If an equilibrium-system consists of four forces, their lines of action must be generators of the same class of a hyperboloid of one sheet.

This is also otherwise evident; as the system consists of four forces, and three enter homogeneously into the six equations of equilibrium, we have three different and independent relations which contain the elements of the lines of action only. Let us consider three of the action-lines to be given; then the action-line of the fourth must satisfy these three conditions. Now the equations of a straight line in space contain four independent constants; three of these may be satisfied by the three preceding conditions, but one other is still required for the complete determination of the line. Such a condition might be that the line should meet a given line. Then this condition leads to the following result: Let the four action-lines of the forces be called p_1, p_2, p_3, p_4; and let q be any straight line which meets the first three; then as the moments of the forces vanish about any straight line, and as the moments of the first three vanish about q which meets their action-lines; the moment of P_4 also vanishes about it; and consequently p_4 meets q. Let four several positions of q be taken, and let these be q_1, q_2, q_3, q_4; then the line p_4 lies on all these lines. But this relation between the p's and the q's is that which we know to exist between the generators of the two classes of the hyperboloid of one sheet; viz. every line of one class of generating lines intersects every line of the other class of generators. Hence any four lines which are the action-lines of an equilibrium-system of four forces lie on the surface of a hyperboloid of one sheet.

As the cone is a degenerate form of a hyperboloid, so does it give a particular case of the preceding theorem. In it the action-lines of the forces pass through the same point, and they are the generating lines of the cone.

75.] If the equilibrium-system consists of five forces, only two independent conditions can be derived from the six equations of equilibrium; and consequently if the action-lines of four forces are supposed to be given, we have only two conditions for the determination of that of the fifth force; and accordingly two others are required; these may be that the line should pass through a given point or lie in a given plane.

If six forces constitute an equilibrium-system, then only one condition can be obtained from the six equations of equilibrium; and consequently if the action-lines of five forces are given, that of the sixth force must satisfy three other conditions; that is, it may lie on three given straight lines, or it may pass through a given point and intersect a given straight line.

Six straight lines fulfilling the condition requisite that they should be the action-lines of forces of an equilibrium-system are said, by Professor Sylvester*, to be in involution; and certain geometrical relations concerning them have been discovered by him, whereby he has arrived at a geometrical construction of the sixth, when five are given. M. Chasles has added to Professor Sylvester's paper some remarks which well deserve attention.

If an equilibrium system consists of seven forces, the ratios of the forces can be determined from the six equations of equilibrium in terms of the elements of the action-lines of the forces; and if an arbitrary magnitude is given to one of the forces those of all the other forces will also be given.

76.] We now come to the second case mentioned in Art. 70, viz. when $\text{R} = 0$ and G has a finite value. Here it is to be observed that R is independent of the origin and of the coordinate axes; and consequently if $\text{R} = 0$ at any one point, this circumstance holds good for all places of the origin and for all positions of the coordinate axes; and accordingly R is an invariant. G, however, generally depends on the position of the origin; but is an invariant when $\text{R} = 0$; because the system of forces is in this case reducible to a couple of which G is the moment; and theorems already demonstrated shew that the effect of a couple is the same so long as its moment is unaltered and its rotation-axis is parallel to a given straight line.

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* Comptes Rendus, Tome LII. p. 741. 1861.

The following process also proves that if $\text{R} = 0$, G is an invariant:

Let the origin be transferred to (x_0, y_0, z_0), and let $\text{L}_0, \text{M}_0, \text{N}_0, \text{G}_0$ be the values of $\text{L}, \text{M}, \text{N}, \text{G}$ corresponding to the new origin; then

$$\begin{aligned} \text{L}_0 &= \Sigma.\text{P}\,\{(y-y_0)\cos\gamma-(z-z_0)\cos\beta\} \\ &= \Sigma.\text{P}\,(y\cos\gamma-z\cos\beta)-y_0\,\Sigma.\text{P}\cos\gamma+z_0\,\Sigma.\text{P}\cos\beta; \end{aligned}$$

$$\therefore\quad \left.\begin{aligned} \text{L}_0 &= \text{L}-\text{Z}y_0+\text{Y}z_0; \\ \text{M}_0 &= \text{M}-\text{X}z_0+\text{Z}x_0; \\ \text{N}_0 &= \text{N}-\text{Y}x_0+\text{X}y_0; \end{aligned}\right\} \qquad (128)$$

and since $\text{R} = 0$, $\text{X} = \text{Y} = \text{Z} = 0$; consequently $\text{L}_0 = \text{L}$, $\text{M}_0 = \text{M}$, $\text{N}_0 = \text{N}$, $\text{G}_0 = \text{G}$, and the moment of the resultant couple is the same for all points in space; and thus the system is always equivalent to a couple whose moment is G.

77.] The third case is that in which the system is reducible to a single force of translation. If at the arbitrarily chosen origin, $\text{G}=0$, and R has a finite value, in reference to that origin the system has a single resultant of translation acting at that origin; but since G depends on the position of the origin, as (128) shew, some condition or conditions are required so that the reduction may hold good for all origins.

In reference to any arbitrarily chosen origin let R be the single force of translation to which the system is reducible; let (x, y, z) be its point of application; a, b, c the direction-angles of its line of action; r the perpendicular distance from the origin on that line; so that

$$r^2 = (y\cos c-z\cos b)^2+(z\cos a-x\cos c)^2+(x\cos b-y\cos a)^2.$$

Let there be introduced at the origin two equal and opposite forces, each of which is equal to R, and whose line of action is parallel to that of R: so that we have now R acting at the origin, and a couple whose moment is $\text{R}r$; and resolving each of these along the three coordinate axes, and equating the resolved parts to the corresponding parts of the aggregate of the impressed forces, we have

$$\left.\begin{aligned} \text{R}\cos a &= \Sigma.\text{P}\cos\alpha = \text{X}, \\ \text{R}\cos b &= \Sigma.\text{P}\cos\beta = \text{Y}, \\ \text{R}\cos c &= \Sigma.\text{P}\cos\gamma = \text{Z}; \end{aligned}\right\}$$

$$\therefore\quad \text{R}^2 = \text{X}^2+\text{Y}^2+\text{Z}^2;$$

$$\left.\begin{aligned} \text{R}\,(y\cos c - z\cos b) &= \text{L} = \text{Z}y - \text{Y}z, \\ \text{R}\,(z\cos a - x\cos c) &= \text{M} = \text{X}z - \text{Z}x, \\ \text{R}\,(x\cos b - y\cos a) &= \text{N} = \text{Y}x - \text{X}y\,; \end{aligned}\right\} \quad : \quad (129)$$

These equations are not independent, and so do not assign definite values to x, y, z: they are subject to a condition; for if we multiply them severally by X, Y, Z, we have

$$\text{LX}+\text{MY}+\text{NZ} = 0\,; \qquad (130)$$

and this relation is one which the forces must satisfy if they are reducible to a single resultant of translation.

Now LX + MY + NZ is an invariant; being independent of the position of the origin, and of any particular system of coordinate axes. From (128) it is evident that it is independent of the position of the origin; for from those equations

$$\text{L}_0\text{X}+\text{M}_0\text{Y}+\text{N}_0\text{Z} = \text{LX}+\text{MY}+\text{NZ}.$$

It is also independent of the position of the coordinate axes; for let a new system of axes, say of x', y', z', originating at the same point be connected with the former by the system of direction-cosines given in the following scheme:

	x	y	z
x'	a_1	a_2	a_3
y'	b_1	b_2	b_3
z'	c_1	c_2	c_3

(131)

Let X′, Y′, Z′, L′, M′, N′ be the values of X, Y, Z, L, M, N respectively in reference to the new coordinate axes; so that

$$\left.\begin{aligned} \text{X} &= a_1\text{X}' + b_1\text{Y}' + c_1\text{Z}', \\ \text{Y} &= a_2\text{X}' + b_2\text{Y}' + c_2\text{Z}', \\ \text{Z} &= a_3\text{X}' + b_3\text{Y}' + c_3\text{Z}'\,; \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{L} &= a_1\text{L}' + b_1\text{M}' + c_1\text{N}', \\ \text{M} &= a_2\text{L}' + b_2\text{M}' + c_2\text{N}', \\ \text{N} &= a_3\text{L}' + b_3\text{M}' + c_3\text{N}'\,; \end{aligned}\right\}$$

with also corresponding inverse systems; so that

$$\begin{aligned} \text{LX}+\text{MY}+\text{NZ} &= (a_1\text{X}' + b_1\text{Y}' + c_1\text{Z}')\,\text{L} + \ldots\ldots + \\ &= \text{X}'(a_1\text{L}+a_2\text{M}+a_3\text{N}) + \ldots\ldots + \\ &= \text{X}'\text{L}' + \text{Y}'\text{M}' + \text{Z}'\text{N}'\,; \end{aligned}$$

and thus $LX+MY+NZ$ is an invariant for all positions of the origin and of the coordinate axes; and if it vanishes, the system is reducible to a single force of translation.

78.] Let this invariant be denoted by KR, so that, R being constant, K is also constant: that is, let

$$LX+MY+NZ = KR; \tag{132}$$

then we have the following interpretation of K. Replacing L, M, N, X, Y, Z by their values given in (101) and (106), we have

$$RG\{\cos\alpha\cos\lambda+\cos\beta\cos\mu+\cos\gamma\cos\nu\} = KR;$$

consequently if ϕ is the angle between the action-line of R and the rotation-axis of G,

$$K = G\cos\phi; \tag{133}$$

that is, K is the component of G along the action-line of R; and this is consequently constant for all origins and for all systems of coordinate axes.

As $K = 0$, when the system of forces is reducible to a single resultant, therefore, from (133), $\phi = 90°$; that is, the rotation-axis of the resultant-couple is perpendicular to the action-line of R, and consequently the action-line of the resultant of translation lies in the plane of the forces of the resultant couple; which is the circumstance alluded to in the fourth case in Art. 70. Thus there are three forces acting in the plane of the couple: viz. R, and the two forces of the couple. These may evidently be compounded into a single force. As the arm of the couple may be turned round in its own plane without altering the effect of the couple, let it be so arranged that the line of action of each of its forces may be parallel to that of R; and thus if R' is a force and a an arm such that $R'a = G$, we shall have three parallel forces R', $-R'$, and R acting in one and the same plane, and these manifestly have a single resultant, whose magnitude is R.

Its position, or the equations which determine the position of the action-line of this resultant, may be found as follows: As (x, y, z) in (129) is *any* point in the line of action of R, (129) are the equations to that line; and they may be put into the following form:

$$\begin{aligned} M-N &= X(y+Z)-x(Y+Z) \\ &= X(x+y+z)-x(X+Y+Z); \end{aligned}$$

$$\therefore \quad \frac{x+\dfrac{M-N}{X+Y+Z}}{X} = \frac{x+y+z}{X+Y+Z};$$

and therefore from the symmetry of the right-hand member

$$\frac{x+\dfrac{M-N}{X+Y+Z}}{X}=\frac{y+\dfrac{N-L}{X+Y+Z}}{Y}=\frac{z+\dfrac{L-M}{X+Y+Z}}{Z}.$$

Or the equations (129) may be put into the following form: multiplying the second by Z and the third by Y, and subtracting, we have

$$\begin{aligned} MZ-NY &= X(Zz+Yy)-(Z^2+Y^2)x \\ &= X(Xx+Yy+Zz)-x(X^2+Y^2+Z^2) \\ &= X(Xx+Yy+Zz)-xR^2; \end{aligned}$$

$$\therefore\quad \frac{x+\dfrac{MZ-NY}{R^2}}{X}=\frac{Xx+Yy+Zz}{R^2};$$

$$\therefore\quad \frac{x+\dfrac{MZ-NY}{R^2}}{X}=\frac{y+\dfrac{NX-LZ}{R^2}}{Y}=\frac{z+\dfrac{LY-MX}{R^2}}{Z}; \qquad (134)$$

and either system is that of the equations to the line of action of the single resultant, which is plainly parallel to that of the resultant R acting at the origin.

If $L=M=N=0$, that is, if $G=0$, then $K=0$ identically, and the condition requisite for a single resultant of translation is satisfied; in this case the resultant passes through the origin.

79.] If the impressed forces are parallel, the condition (130) is satisfied, and the system admits of a single resultant of translation. Let the forces be $P_1, P_2, \dots P_n$, and be applied at the points $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots (x_n, y_n, z_n)$; then

$$\left.\begin{aligned} X &= \Sigma.P\cos\alpha = \cos\alpha\,\Sigma.P, \\ Y &= \Sigma.P\cos\beta = \cos\beta\,\Sigma.P, \\ Z &= \Sigma.P\cos\gamma = \cos\gamma\,\Sigma.P; \end{aligned}\right\} \qquad (135)$$

$$\therefore\quad R^2 = X^2+Y^2+Z^2 = (\Sigma.P)^2;$$

$$\therefore\quad R = \Sigma.P;$$

and consequently from (135),

$$\cos a = \frac{X}{R} = \cos\alpha, \qquad \cos b = \cos\beta, \qquad \cos c = \cos\gamma; \qquad (136)$$

that is, the resultant of translation at the origin is equal to the sum of all the impressed forces, and acts along a line which is parallel to the lines of action of the components. Again,

$$\left.\begin{aligned} \text{L} &= \cos c\, \Sigma.\text{P}y - \cos b\, \Sigma.\text{P}z, \\ \text{M} &= \cos a\, \Sigma.\text{P}z - \cos c\, \Sigma.\text{P}x, \\ \text{N} &= \cos b\, \Sigma.\text{P}x - \cos a\, \Sigma.\text{P}y; \end{aligned}\right\} \tag{137}$$

$$\begin{aligned} \therefore \quad \text{G}^2 = (\Sigma.\text{P}x)^2 + (\Sigma.\text{P}y)^2 + (\Sigma.\text{P}z)^2 \\ - \{\cos a\, \Sigma.\text{P}x + \cos b\, \Sigma.\text{P}y + \cos c\, \Sigma.\text{P}z\}^2; \end{aligned} \tag{138}$$

and hereby may $\cos\lambda$, $\cos\mu$, $\cos\nu$, equations (106), be found.

From (137) we have

$$\text{L} \cos a + \text{M} \cos b + \text{N} \cos c = 0;$$

and therefore from (136),

$$\text{LX} + \text{MY} + \text{NZ} = 0,$$

which is the condition requisite that the system should be reducible to a single force of translation. Let R be this force; a, b, c the direction-angles of its line of action; $(\bar{x}, \bar{y}, \bar{z})$ its point of application; then introducing at the origin two equal and opposite forces, each of which is equal to R and acts along a line of action parallel to that of R, we have a force acting at the origin equal to R, and in a parallel line of action, and a couple each of whose forces is R, and whose arm is r, where

$$r^2 = (\bar{y}\cos c - \bar{z}\cos b)^2 + (\bar{z}\cos a - \bar{x}\cos c)^2 + (\bar{x}\cos b - \bar{y}\cos a)^2,$$

and the direction-cosines of the rotation-axis of which are

$$\frac{\bar{y}\cos c - \bar{z}\cos b}{r}, \quad \frac{\bar{z}\cos a - \bar{x}\cos c}{r}, \quad \frac{\bar{x}\cos b - \bar{y}\cos a}{r}; \tag{139}$$

then, as this system is to produce the same effect as the aggregate of all the impressed parallel forces, we have

$$\left.\begin{aligned} \text{R}\cos a &= \cos \alpha\, \Sigma.\text{P}, \\ \text{R}\cos b &= \cos \beta\, \Sigma.\text{P}, \\ \text{R}\cos c &= \cos \gamma\, \Sigma.\text{P}; \end{aligned}\right\} \tag{140}$$

whence squaring and adding,

$$\text{R}^2 = (\Sigma.\text{P})^2, \text{ and therefore } \text{R} = \Sigma.\text{P}. \tag{141}$$

$$\therefore \quad \cos a = \cos \alpha, \quad \cos b = \cos \beta, \quad \cos c = \cos \gamma;$$

$$\therefore \quad a = \alpha, \quad b = \beta, \quad c = \gamma. \tag{142}$$

Also

$$\left.\begin{aligned} \text{L} &= \text{R}(\bar{y}\cos\gamma - \bar{z}\cos\beta) = \cos\gamma\, \Sigma.\text{P}y - \cos\beta\, \Sigma.\text{P}z, \\ \text{M} &= \text{R}(\bar{z}\cos\alpha - \bar{x}\cos\gamma) = \cos\alpha\, \Sigma.\text{P}z - \cos\gamma\, \Sigma.\text{P}x, \\ \text{N} &= \text{R}(\bar{x}\cos\beta - \bar{y}\cos\alpha) = \cos\beta\, \Sigma.\text{P}x - \cos\alpha\, \Sigma.\text{P}y. \end{aligned}\right\} \tag{143}$$

Thus (141) and (142) assign the magnitude and direction-angles of the line of action of the single resultant; and as $(\bar{x}, \bar{y}, \bar{z})$ is any point in that line of action, (143) are the equations to it; and the resultant is defined in all its incidents.

80.] Another property of a system of parallel forces requires notice. In the preceding Article the magnitude, line of action, and direction of the resultant have been deduced from the similar incidents of the acting parallel forces; and the fourth incident, viz. the points of application, have not been brought under consideration. In (143), which are the equations to the line of action of R, $(\bar{x}, \bar{y}, \bar{z})$ is any point in that line. Suppose, however, all the forces to act at definite points, so that (x, y, z) in the right-hand members of (143) have given values; also suppose the lines of action of all the forces at their points of applications to be turned through equal angles in the same or in parallel planes, so that the system consists of parallel forces after the change of line of action; and consequently has a single resultant. Now the magnitude of this resultant is equal to the sum of those of the given forces, and the line of action is parallel to those of the acting forces; and both these quantities are independent of the particular system of coordinate axes, consequently α, β, γ are indeterminate, and the point of application of R must be consistent with this condition. But from (143)

$$\frac{\mathrm{R}\bar{x} - \Sigma.\mathrm{P}x}{\cos\alpha} = \frac{\mathrm{R}\bar{y} - \Sigma.\mathrm{P}y}{\cos\beta} = \frac{\mathrm{R}\bar{z} - \Sigma.\mathrm{P}z}{\cos\gamma}; \qquad (144)$$

$$\therefore \quad \mathrm{R}\bar{x} - \Sigma.\mathrm{P}x = \mathrm{R}\bar{y} - \Sigma.\mathrm{P}y = \mathrm{R}\bar{z} - \Sigma\,\mathrm{P}z = 0; \qquad (145)$$

$$\therefore \quad \bar{x} = \frac{\Sigma.\mathrm{P}x}{\Sigma.\mathrm{P}}, \qquad \bar{y} = \frac{\Sigma.\mathrm{P}y}{\Sigma.\mathrm{P}}, \qquad \bar{z} = \frac{\Sigma.\mathrm{P}z}{\Sigma.\mathrm{P}}; \qquad (146)$$

the point $(\bar{x}, \bar{y}, \bar{z})$ is the point at which the resultant is applied in all these cases, and consequently is called *the centre of the parallel forces.*

The following are examples in which the centre of parallel forces is determined:

Ex. 1. Four parallel forces 2, 4, 6, 8 are applied at the angles of a square the length of the side of which is $2a$: find the centre of these parallel forces.

Let the plane of the square be the plane of (x, y), and let the origin be at the centre of the square. Let $(a, a, 0)$ be the point of application of 2, $(-a, a, 0)$ of 4, and so on; then

$$\Sigma.\mathrm{P} = \mathrm{R} = 20;$$
$$\Sigma.\mathrm{P}x = 0; \qquad \Sigma.\mathrm{P}y = -8a; \qquad \Sigma.\mathrm{P}z = 0:$$
$$\therefore \quad \bar{x} = 0; \qquad \bar{y} = -\frac{2a}{5}; \qquad \bar{z} = 0.$$

81.] The last case mentioned in Art. 70, viz. that in which R and G have both finite magnitudes, remains for discussion. In reference to an origin and a system of coordinate axes, both of which are arbitrarily chosen, the system of forces is reduced to a force of translation acting at the origin, and to a couple whose moment is G, the line of action of R and the rotation-axis of G being given by (101) and (106).

Whatever point is taken as the origin the magnitude of R is the same; all its lines of action are parallel, and its direction is the same.

But the value of G varies as the place of the origin varies, because L, M, N depend on the coordinates of the points of application of the forces; and if L_0, M_0, N_0 are the values of L, M, N at the new origin (x_0, y_0, z_0), then by Art. 76, the new axes being parallel to the former,

$$\left.\begin{aligned} \mathrm{L}_0 &= \mathrm{L} - \mathrm{Z}\,y_0 + \mathrm{Y}\,z_0, \\ \mathrm{M}_0 &= \mathrm{M} - \mathrm{X}\,z_0 + \mathrm{Z}x_0, \\ \mathrm{N}_0 &= \mathrm{N} - \mathrm{Y}x_0 + \mathrm{X}y_0; \end{aligned}\right\} \tag{147}$$

also if the axes are changed, see Art. 78,

$$\mathrm{LX} + \mathrm{MY} + \mathrm{NZ} = \mathrm{L'X'} + \mathrm{M'Y'} + \mathrm{N'Z'} = \mathrm{KR}; \tag{148}$$

and if ϕ is the angle between the action-line of R and the rotation-axis of G,

$$\mathrm{G}\cos\phi = \mathrm{K}; \tag{149}$$

so that the resolved part of every moment-axis along the line of action of R is constant. These are properties of G which have already been investigated.

And further let it be observed that of all axes passing through a given point, that corresponding to G is the one whose moment or moment-axis is the greatest; for the moment of the impressed couples with respect to a rotation-axis inclined at an angle θ to that of G is $\mathrm{G}\cos\theta$, as is plain from the law of resolution of such moment-axes; and as $\mathrm{G}\cos\theta$ is less than G, it follows that of all lines passing through a given point, the rotation-axis of the resultant couple is that with respect to which the moment-axis is the greatest. For this reason G is called *the complete* or *principal moment-axis* with reference to the point which is called *the moment-centre.* Hence also we infer that at a given moment-centre the moment-axis is the same for all axes which are inclined at the same angle to the axis of the principal moment; that is, all axes of equal moment with reference to a given moment-centre form a right circular cone which has the axis of principal moment for its axis of figure.

82.] Since $\text{G} \cos \phi = \text{K} =$ a constant, G has its least value when $\cos \phi$ has its greatest; that is, when $\phi = 0$, and when the rotation-axis coincides with the line of action of R.

Let (x_0, y_0, z_0) be the moment-centre at which this circumstance takes place; then

$$\frac{\text{L}_0}{\text{X}} = \frac{\text{M}_0}{\text{Y}} = \frac{\text{N}_0}{\text{Z}} = \frac{\text{G}_0}{\text{R}} = \frac{\text{K}}{\text{R}}; \tag{150}$$

and replacing L_0, M_0, N_0 by their values given in (128),

$$\frac{\text{L} - \text{Z}y_0 + \text{Y}z_0}{\text{X}} = \frac{\text{M} - \text{X}z_0 + \text{Z}x_0}{\text{Y}} = \frac{\text{N} - \text{Y}x_0 + \text{X}\text{Y}_0}{\text{Z}} = \frac{\text{K}}{\text{R}}; \tag{151}$$

whence we have

$$\frac{x_0 - \dfrac{\text{NY} - \text{MZ}}{\text{R}^2}}{\text{X}} = \frac{y_0 - \dfrac{\text{LZ} - \text{NX}}{\text{R}^2}}{\text{Y}} = \frac{z_0 - \dfrac{\text{MX} - \text{LY}}{\text{R}^2}}{\text{Z}}; \tag{152}$$

which are the equations to a straight line whose current coordinates are x_0, y_0, z_0; and as no other relation is given between x_0, y_0, z_0, that point may be anywhere on this line; and consequently this straight line is the locus of those moment-centres at which the rotation-axis of the principal moment coincides with the line of action of the resultant of translation. This line is called *the central axis* of the system (*Hauptdrehlinie*); and any plane perpendicular to it is called *a central plane.* If the system is reducible to a single force of translation, that force evidently acts along the central axis; and for this reason (134) and (152) are identical.

At all points of this line the principal moment is a minimum and is K; and K is called *the central principal moment:* and its rotation-axis coincides with the line of action of R. Consequently

The central axis is that line along which the system of forces produces a pressure of translation = R; and which is also the rotation-axis of the resultant couple whose moment is K. Thus the forces produce a shifting pressure along the central axis and a tendency to make the body rotate about the same line. This is indeed the most simple form in the nature of the case to which the system of forces can be reduced, and from this point of view the result is most important; but the complexity of the equations (152) often precludes us from making that use of them which we might, were they more simple, and the reduction to a single force of translation and to a couple whose moment-axis is G is employed in preference.

These results might have been arrived at from investigating the locus of those moment-centres at which the principal moment is a minimum, viz. when x_0, y_0, z_0 vary so that

$$G_0^2 = (L - Z y_0 + Y z_0)^2 + (M - X z_0 + Z x_0)^2 + (N - Y x_0 + X y_0)^2$$

is a minimum; and we should have the following results:

(1) With respect to moment-centres taken at any point in space, the moment of the rotation-axis coincident with the central axis is the least. Thus K is *the minimum maximorum moment.*

(2) If any point of the central axis is taken as the moment-centre, of all axes of rotation passing through that point, that coincident with the central axis has the greatest moment.

83.] The following is another mode of demonstrating the preceding results. In fig. 42, let O be the original moment-centre; OR the line of action of the force of translation acting at it; OG the moment-axis of the resultant principal couple at O: let GOR $= \phi$, so that

$$\cos\phi = \frac{LX + MY + NZ}{GR};$$

resolve OG into two parts, one along, and the other perpendicular to OR; then the part along OR is G $\cos\phi$, and that perpendicular to OR is G $\sin\phi$; the rotation-axis of G $\cos\phi$ is OR, and that of G $\sin\phi$ is a line in the plane containing OG and OR: at O draw OP perpendicular to this plane, and take OP of a length such that $R \times OP = G \sin\phi$; at P introduce two equal and opposite forces, each of which is equal to R, and whose line of action is parallel to that of R: then the couple whose arm is OP, and whose force is R, neutralizes the couple whose moment-axis is ON; and there remain (1) the force R acting at P, and in a line parallel to the original line of action of R, and (2) a couple whose moment-axis is G $\cos\phi$, and whose rotation-axis is along OR. Let the rotation-axis be transferred parallel to itself so as to pass through P, and we have finally a force of translation R acting along PR, and a couple whose rotation-axis is along the line of action of R, and whose moment-axis is $G \cos \phi$, which $= K$. Thus the line through P, and parallel to OR, is the central-axis; and its equation may thus be found. It passes through P, and its direction-cosines are proportional to X, Y, Z. Since $OP = \frac{G}{R} \sin \phi$, and OP is perpendicular to OR and to OG, the coordinates of P are

$$\frac{NY-MZ}{R^2}, \quad \frac{LZ-NX}{R^2}, \quad \frac{MX-LY}{R^2};$$

consequently the equations to PR are

$$\frac{x - \frac{NY-MZ}{R^2}}{X} = \frac{y - \frac{LZ-NX}{R^2}}{Y} = \frac{z - \frac{MX-LY}{R^2}}{Z};$$

which are the equations to the central axis.

As OP is perpendicular to both OG and PR, it is the shortest line between the rotation-axes of G and of K.

If $OP = r$, we have

$$Rr = G \sin \phi, \qquad K = G \cos \phi; \tag{153}$$

$$\therefore \quad R^2 r^2 + K^2 = G^2: \tag{154}$$

therefore G, the principal moment at a point, is the same at all points for which r is constant; that is, at all points equally distant from the central axis; and therefore the locus of all moment-centres, at which the principal moments are equal, is a circular cylindrical surface, of which the central axis is the axis of figure; and at all points along the same generating line of this cylinder, the rotation-axes of the principal moments are parallel, and all therefore lie in the plane touching the cylinder along the generating line.

But the directions of the rotation-axes change as we pass from one generating line to another; for since ϕ is the angle between the central axis and the rotation-axis of the principal moment corresponding to a moment-centre at a distance r from the central axis we have from (153)

$$\tan \phi = \frac{Rr}{K}; \tag{155}$$

and this is therefore constant for all points of the cylindrical surface mentioned above; and as the direction-cosines of the central axis are proportional to X, Y, Z, and those of the rotation-axis of the principal moment G to L, M, N, these lines in general do not meet: and therefore if a section is made of the cylindrical surface mentioned above by a plane perpendicular to the central axis, and the principal moment-axes are drawn for the moment-centres situated in this circular section, they will form a hyperboloid of revolution of one sheet, having the central axis for its axis of figure.

84.] These theorems however, and others of a like kind, may be investigated more easily by the following process:

Let a point in the central axis be taken as the origin, and let the central axis be the axis of z; so that the system of forces consists of a force of translation R acting along the z-axis, and a couple whose moment is K and whose rotation-axis is the z-axis also. At $(x_0, y_0, 0)$ let two equal and opposite forces, and each equal to R and acting parallel to the z-axis, be introduced; and let G_0 be the moment of the resultant couple, of which let L_0, M_0, N_0 be the axial components: then

$$L_0 = -R y_0, \qquad M_0 = R x_0, \qquad N_0 = K; \tag{156}$$

$$\therefore \quad G_0^2 = R^2 (x_0^2 + y_0^2) + K^2. \tag{157}$$

Let $x_0^2 + y_0^2 = r^2$, and let ϕ be the z-direction angle of the rotation-axis of G_0; then

$$N_0 = G_0 \cos\phi = K; \tag{158}$$

$$G_0 \sin\phi = (L_0^2 + M_0^2)^{\frac{1}{2}} = Rr; \tag{159}$$

$$x_0^2 + y_0^2 = \frac{K^2}{R^2} (\tan\phi)^2. \tag{160}$$

From these equations we have the following theorems:

(1) All moment-centres of equal principal moment are on the surface of a right circular cylinder, of which the central axis is the axis of revolution.

For from (157) we have

$$x_0^2 + y_0^2 = \frac{G_0^2 - K^2}{R^2}; \tag{161}$$

the right-hand member of which is constant, if G_0 is constant; and consequently all the moment-centres, at which G_0 is constant, lie on the surface of the right circular cylinder whose equation is (161).

Also the greater G_0 is, the greater is the radius of the cylinder, and the farther is the moment-centre from the central axis; and the least value of G_0 is K.

(2) At all points of equal principal moments, the rotation-axis is inclined at the same angle to the central axis.

This follows from (158), because $\cos\phi = \frac{\mathrm{K}}{\mathrm{G}_0}$; hence ϕ is constant when G_0 is constant, and the equation to the cylinder in (161) becomes

$$x_0^2 + y_0^2 = \frac{\mathrm{K}^2 (\tan\phi)^2}{\mathrm{R}^2}. \tag{162}$$

Also at all points in the same generating line of this cylinder, the principal rotation-axes are parallel, and lie in the plane which touches the cylinder along that generating line. Hence also the larger G_0 becomes, the smaller is $\cos\phi$, and if $\mathrm{G}_0 = \infty$, $\phi = 90°$; and as the tangent of the angle between the rotation-axis and the central axis is proportional to the distance of the moment-centre from the central axis, the rotation-axis is perpendicular to the central axis only when the moment-centre is at an infinite distance.

(3) The rotation-axes of the principal moments for the moment-centres lying in the circle given in (161) are in the surface of a hyperboloid of revolution of one sheet of which the central axis is the axis of figure.

By reason of (156) the equations of the rotation-axis corresponding to the moment-centre $(x_0, y_0, 0)$ are

$$\frac{x - x_0}{-\mathrm{R} y_0} = \frac{y - y_0}{\mathrm{R} x_0} = \frac{z}{\mathrm{K}}; \tag{163}$$

from which and (162), eliminating x_0 and y_0, we have

$$x^2 + y^2 - z^2 (\tan\phi)^2 = \frac{\mathrm{K}^2}{\mathrm{R}^2} (\tan\phi)^2; \tag{164}$$

which is the equation to a hyperboloid of revolution of one sheet, of which the z-axis, that is, the central axis, is the axis of figure.

This theorem is only a special one of a general class; viz. given the locus of the moment-centre to find the equation to the ruled surface generated by the corresponding rotation-axis of the principal moment. For from (163) we have

$$x_0 = \frac{(\mathrm{K}x + \mathrm{R}yz)\mathrm{K}}{\mathrm{K}^2 + \mathrm{R}^2 z^2}, \qquad y_0 = \frac{(\mathrm{K}y - \mathrm{R}xz)\mathrm{K}}{\mathrm{K}^2 + \mathrm{R}^2 z^2}; \tag{165}$$

consequently if the moment-centre moves along a given curve in the plane of (x, y) a relation is given between x_0 and y_0, and the substitution of the preceding values of x_0 and y_0 in that relation will give the equation of the ruled surface which is generated by the rotation-axis of the principal moment. The following theorem is an example of such a ruled surface:

(4) For all moment-centres lying in a straight line cutting the central axes at right angles, the corresponding rotation-axes of the principal moments lie on the surface of a hyperbolic paraboloid.

Let the straight line on which the moment-centre is be the axis of x, so that the moment-centre is $(x_0, 0, 0)$; consequently $\text{L}_0 = 0$, $\text{M}_0 = \text{R}x_0$, $\text{N}_0 = \text{K}$; and the equations to the rotation-axis of the principal moment are

$$x = x_0; \qquad \frac{y}{x_0 \text{R}} = \frac{z}{\text{K}}; \tag{166}$$

$$\therefore \quad \text{K}y = \text{R}xz; \tag{167}$$

which is the equation to a hyperbolic paraboloid.

Also generally if the moment-centre moves along a straight line which is perpendicular to, but does not cut, the central axis, the rotation-axis lies on a surface of the second degree.

(5) The line whose equations are (163) is evidently perpendicular to that which passes through the origin and (x_0, y_0); consequently this latter line is the shortest distance between the central axis and the principal rotation-axis corresponding to (x_0, y_0).

(6) The plane which contains the line of action of the resultant and the principal rotation-axis at a given moment-centre is perpendicular to the line drawn from that centre at right angles to the central axis.

85.] The preceding theorems supply means for investigating certain general properties of planes and lines with reference to moment-centres, and also criteria as to the reduction of systems of forces to a force of translation, and to a couple whose rotation-axis may coincide with a given line or be perpendicular to a given plane.

Whatever is the position of a plane, that plane is always a momental plane with reference to some point in itself which is the corresponding moment-centre: that is, the system of forces may always be reduced to a force of translation acting at a point in the plane, and to a couple the rotation-axis of which is normal to the plane.

If the plane is perpendicular to the central axis, it is a central plane, and the theorem is self-contained.

If the plane is not perpendicular to the central axis, at the point where the central axis intersects it, let a line be drawn in the plane perpendicular to the central axis; and along this line let a distance r be taken of such a length that if ϕ is the angle between the central axis, and the normal to the plane,

$$r = \frac{\text{K}}{\text{R}} \tan \phi; \tag{168}$$

then the point at the extremity of this line is the moment-centre; and the normal to the plane at it is the principal rotation-axis; and the line parallel to the central axis is the line of action of the resultant.

When the equation to the plane is given, the coordinates of its moment-centre may be found by the following process:

Let the equation to the plane be

$$\text{A}x + \text{B}y + \text{C}z = \text{D}; \tag{169}$$

and let the moment-centre in the plane be (x_0, y_0, z_0); then as the equations to the corresponding rotation-axis are

$$\frac{x-x_0}{-\text{R}y_0} = \frac{y-y_0}{\text{R}x_0} = \frac{z-z_0}{\text{K}},$$

and as this line is perpendicular to the given plane, we have

$$\frac{\text{A}}{-\text{R}y_0} = \frac{\text{B}}{\text{R}x_0} = \frac{\text{C}}{\text{K}};$$

$$\therefore \quad x_0 = \frac{\text{BK}}{\text{CR}}, \quad y_0 = -\frac{\text{AK}}{\text{CR}}, \quad z_0 = \frac{\text{D}}{\text{C}}; \tag{170}$$

which assign the moment-centre of the plane (169).

The value of z_0 shews that the moment-centre lies in the intersection of the given plane by a plane parallel to that of (x, y), and passing through the point at which the given plane cuts the central axis; and the line of intersection of these two planes is perpendicular to both the central axis and the principal rotation-axis. For a series of parallel planes, the values of x_0 and y_0 are constant; consequently all the moment-centres lie in a straight line parallel to the central axis.

Hence also if (x_0, y_0, z_0) is the moment-centre, the equation to the corresponding momental plane is

$$-\text{R}y_0 x + \text{R}x_0 y + \text{K}(z - z_0) = 0. \tag{171}$$

86.] And to consider this problem more generally, let the system be referred to an origin and coordinate-axes taken arbitrarily; then from the comparison of the direction-cosines of the normal of the plane (169), and of the axial components of the principal moment-axis given in (128), we have

$$\frac{L - Zy_0 + Yz_0}{A} = \frac{M - Xz_0 + Zx_0}{B} = \frac{N - Yx_0 + Xy_0}{C}$$

$$= \frac{LX + MY + NZ}{AX + BY + CZ}; \qquad (172)$$

$$\left.\begin{aligned} \therefore \quad x_0 &= \frac{DX + BN - CM}{AX + BY + CZ}, \\ y_0 &= \frac{DY + CL - AN}{AX + BY + CZ}, \\ z_0 &= \frac{DZ + AM - BL}{AX + BY + CZ}. \end{aligned}\right\} \qquad (173)$$

Hence the coordinates of the moment-centres of the three coordinate planes are,

$$\left.\begin{array}{llll} \text{Of the plane } (y, z), & x = 0, & y = -\frac{N}{X}, & z = \frac{M}{X}; \\ \quad - \; - \; - \quad (z, x), & x = \frac{N}{Y}, & y = 0, & z = -\frac{L}{Y}; \\ \quad - \; - \; - \quad (x, y), & x = -\frac{M}{Z}, & y = \frac{L}{Z}, & z = 0; \end{array}\right\} (174)$$

all which points evidently lie in the plane whose equation is

$$Lx + My + Nz = 0,$$

and which is the momental plane of the origin; and hence also we infer that the moment-centres of the three coordinate planes lie in a plane passing through the origin of coordinates.

Also if G_0 is the principal moment-axis with reference to the point (x_0, y_0, z_0) given in equations (173),

$$G_0 = \frac{(A^2 + B^2 + C^2)^{\frac{1}{2}}}{AX + BY + CZ} KR. \qquad (175)$$

Hence if G_x, G_y, G_z are the principal moment-axes of the planes of (y, z), (z, x), and (x, y) respectively,

$$G_x = \frac{KR}{X}; \qquad G_y = \frac{KR}{Y}; \qquad G_z = \frac{KR}{Z}; \qquad (176)$$

the moment-centres of these planes to which these moment-axes correspond are given in (174).

87.] In Article (85) it is demonstrated that if

$$Ax + By + Cz = D \qquad (177)$$

is the equation to a momental plane, $\left(\frac{BK}{CR}, -\frac{AK}{CR}, \frac{D}{C}\right)$ is its moment-centre; and also that, if (x_0, y_0, z_0) is a moment-centre,

$$-Ry_0 x + Rx_0 y + K(z - z_0) = 0 \qquad (178)$$

is its momental plane. Now from these relations problems of the following nature arise: (1) Given the locus of the moment-centres, find the envelope of the corresponding momental-planes; this will evidently be generally a developable surface, and the problem is the discovery of its equation; and (2) Given the law according to which momental planes are drawn, to find the locus of the corresponding moment-centres. The following are examples of these problems:

Ex. 1. To find the envelope of the momental planes, when the locus of the corresponding moment-centres is a plane.

Let (x_0, y_0, z_0) be the moment-centre; and let the plane in which it always is be

$$\text{A}x_0 + \text{B}y_0 + \text{C}z_0 = 0, \tag{179}$$

the origin, the position of which on the central axis is arbitrary, being taken at the point where the central axis intersects this plane. Consequently making x_0, y_0, z_0 to vary, and equating the ratios of the coefficients of the differentials of x_0, y_0, z_0 in (178) and (179), we have

$$\frac{\text{R}y}{\text{A}} = -\frac{\text{R}x}{\text{B}} = -\frac{\text{K}}{\text{C}};$$

$$\therefore \quad x = \frac{\text{BK}}{\text{CR}}, \qquad y = -\frac{\text{AK}}{\text{CR}}, \qquad z = 0; \tag{180}$$

which assign a point in the plane of (179), and which lies in the line of its intersection with the plane of (x, y); and this point is, as (170) shew, the moment-centre of the plane (179); consequently all the momental planes, corresponding to the moment-centres in (179), pass through the moment-centre of that plane, which is thus the envelope of them.

Let lines drawn in a plane from the moment-centre of the plane be called rays; then from the preceding result the following theorems arise:

If the moment-centre of a plane lies in the line of intersection of it with another plane, the moment-centre of the latter plane also lies in the same line of intersection.

The momental planes of all moment-centres lying in a ray intersect in that ray; or, in other words, a ray is the locus of the moment-centres of all planes passing through that ray.

The moment-centres of all planes which pass through one and the same point lie in a plane which is the momental plane of the point through which all the planes pass.

If the moment-centre is in the plane of (x, y), so that in (179)

$A = B = 0$, then from (180) the origin is the moment-centre, and the origin of rays; so that all the momental planes corresponding to moment-centres in the plane of (x, y) pass through the origin.

Since from (180) we have $Ax+By = 0$, and this is independent of C, all the moment-centres of the planes intersecting the plane of (x, y) in the line $Ax+By = 0$, lie in that line: and as this line passes through the origin which is the moment-centre of the plane of (x, y), it is a ray of that plane; consequently the ray is the locus of the moment-centres of all the planes passing through that ray.

Ex. 2. To find the envelope of the momental planes corresponding to moment-centres, of which the locus is a spherical surface; whose centre is on the central axis.

Let the equation to the sphere be

$$x_0^2+y_0^2+z_0^2 = a^2; \qquad (181)$$

then the envelope of the plane (178), when x_0, y_0, z_0 are subject to (181), is

$$a^2 R^2 (x^2+y^2)-K^2 z^2 = -K^2 a^2;$$

which is the equation to a hyperboloid of revolution of one sheet, the z-axis being the axis of figure.

Ex. 3. If the locus of the moment-centres is the ellipse $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, the envelope of the corresponding momental planes is the elliptic cone

$$R^2 (a^2 y^2+b^2 x^2)-K^2 z^2 = 0.$$

Ex. 4. To find the envelope of the momental planes, when the locus of the moment-centres is a straight line.

Let the line which is perpendicular to both the central-axis and the locus-line of the moment-centres be the axis of x, and let r be the perpendicular distance between those two lines; then the line is parallel to the plane of (y, z) and cuts the axis of x at a distance $= r$ from the origin. Let a be the angle at which it is inclined to the plane of (x, z); so that the equations to the locus of the moment-centre (x_0, y_0, z_0) are

$$\frac{x_0-r}{0} = \frac{y_0}{\sin a} = \frac{z_0}{\cos a}; \qquad (182)$$

then replacing x_0 and y_0 by these values in the equation of the momental plane, we have

$$-R z_0 x \tan a+R r y+K(z-z_0) = 0, \qquad (183)$$

whence, as z_0 varies, we have

$$\text{R}ry+\text{K}z = 0; \qquad \text{R}x\tan a+\text{K} = 0;$$

$$\therefore \quad x = -\frac{\text{K}\cot a}{\text{R}}, \qquad y = -\frac{\text{K}}{\text{R}r}z; \tag{184}$$

which express a straight line cutting the axis of x at right angles at a distance $= \frac{\text{K}\cot a}{\text{R}}$ on the negative side of the origin, and inclined at an angle $\tan^{-1}\left(-\frac{\text{K}}{\text{R}r}\right)$ to the plane of (y, z); and thus lying on the opposite side of the plane of (y, z) to that on which (182) is.

Consequently all the momental planes whose moment-centres are on (182) pass through the line (184), which is the envelope of them; and conversely, the moment-centres of all momental planes which pass through the same straight line lie in a straight line.

Now these two lines have many remarkable relations. If (184) is the locus of moment-centres, all the corresponding momental planes intersect along (182). For let (x_1, y_1) be a moment-centre on (184), and let $-\frac{\text{K}\cot a}{\text{R}} = r_1$, $-\frac{\text{K}}{\text{R}r} = \tan a_1$; so that the equations to (184) become

$$x = -\frac{\text{K}\cot a}{\text{R}} = r_1; \qquad y = -\frac{\text{K}}{\text{R}r}z = \tan a_1 z.$$

Consequently the equations to the line of intersection of the corresponding momental planes are

$$x = -\frac{\text{K}}{\text{R}}\cot a_1 = r; \qquad y = -\frac{\text{K}}{\text{R}r_1}z = \tan az,$$

which are the equations (182). Thus we have the following:

The momental planes of all moment-centres on (182) intersect in (184), and the momental planes of all moment-centres on (184) intersect in (182).

As these two lines have reciprocal relations, they are called *reciprocal lines, (gegenlinien.)* It is evident that to every line there is a reciprocal line.

Hence also it appears that the line, viz. the x-axis, which is perpendicular to both of them is also perpendicular to and intersects the central axis.

If r_0 and r_1 are, irrespectively of sign, the perpendicular distances between the central axis and the two reciprocal lines, and a_0 and a_1 are, also irrespectively of sign, the angles at which these lines are inclined to the central axis, we have the following relations:

$$r_1 = \frac{\mathrm{K}}{\mathrm{R}} \cot a_0 ; \qquad \tan a_1 = \frac{\mathrm{K}}{\mathrm{R}\, r_0} ; \tag{185}$$

$$\therefore \quad r_1 \tan a_0 = r_0 \tan a_1 = \frac{\mathrm{K}}{\mathrm{R}} . \tag{186}$$

If two reciprocal lines are coincident, this line is a ray of all planes passing through it. The analytical condition is

$$\mathrm{R} r \tan a + \mathrm{K} = 0.$$

If two reciprocal lines are perpendicular to each other, $a_0 + a_1 = 90^\circ$; $\quad \therefore \quad \mathrm{K}^2 + \mathrm{R}^2 r_0 r_1 = 0. \qquad (187)$

Ex. 5. Find the locus of the moment-centres of a series of planes, which intersect in one and the same straight line.

Let the equations of the line in which they intersect be

$$x - r_0 = 0 ; \qquad y - z \tan a = 0 ;$$

so that the equation to the planes which pass through this line is

$$\lambda (x - r_0) + y - z \tan a = 0,$$

where λ is an indeterminate quantity; then by (170) the coordinates of the moment-centre are

$$x = -\frac{\mathrm{K}}{\mathrm{R} \tan a}, \qquad y = \frac{\lambda \mathrm{K}}{\mathrm{R} \tan a}, \qquad z = -\frac{\lambda r_0}{\tan a} ;$$

$$\therefore \quad y = -\frac{\mathrm{K}}{\mathrm{R}\, r_0} z ; \tag{188}$$

which shew that all the moment-centres are in the line which is reciprocal to that in which the planes intersect.

Ex. 6. Find the locus of the moment-centres of all the planes which touch the sphere $x^2 + y^2 + z^2 = a^2$.

Let the equation of one of the tangent planes be

$$x \cos a + y \cos \beta + z \cos \gamma = a ;$$

so that by (170), if (x_0, y_0, z_0) is the moment-centre,

$$x_0 = \frac{\mathrm{K} \cos \beta}{\mathrm{R} \cos \gamma}, \qquad y_0 = -\frac{\mathrm{K} \cos a}{\mathrm{R} \cos \gamma}, \qquad z_0 = \frac{a}{\cos \gamma} ;$$

$$\therefore \quad \cos a = -\frac{\mathrm{R} a y_0}{\mathrm{K} z_0}, \quad \cos \beta = \frac{\mathrm{R} a x_0}{\mathrm{K} z_0}, \quad \cos \gamma = \frac{a}{z_0} ;$$

$$\therefore \quad \mathrm{R}^2 a^2 (x_0^2 + y_0^2) - \mathrm{K}^2 (z_0^2 - a^2) = 0 ; \tag{189}$$

which is the equation to a hyperboloid of revolution of two sheets, the axis of figure of which is the central axis.

88.] Although every point in space may be a moment-centre and have a momental plane and a principal rotation-axis passing through it, and although every plane may be a momental plane, and have its moment-centre in it, yet every straight line may not be a principal rotation-axis, and may not consequently have a moment-centre corresponding to itself in it. This result is evident from the properties of principal rotation-axes which are proved in Art. 84; for every principal rotation-axis touches a cylinder whose axis is the central axis, its corresponding moment-centre being the point of contact, and it is inclined to the central axis at an angle ϕ such that

$$\tan\phi = \frac{\mathrm{R}r}{\mathrm{K}}, \tag{190}$$

if r is the perpendicular distance between the given line and the central axis; and this is a relation between r and ϕ which all straight lines evidently do not satisfy.

The conditions however to be satisfied when a straight line is a principal rotation-axis, and also the coordinates of its moment-centre, may be ascertained in the following manner:

Let the equations to the straight line be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}; \tag{191}$$

and let (x_0, y_0, z_0) be the moment-centre on it. Then comparing (191) with (156) and (157), we have

$$\frac{l}{-\mathrm{R}y_0} = \frac{m}{\mathrm{R}x_0} = \frac{n}{\mathrm{K}} = \frac{(l^2+m^2)^{\frac{1}{2}}}{\mathrm{R}(x_0^2+y_0^2)^{\frac{1}{2}}} = \frac{1}{\mathrm{G}_0}; \tag{192}$$

$$\therefore \quad x_0 = \frac{m\mathrm{K}}{n\mathrm{R}}, \qquad y_0 = -\frac{l\mathrm{K}}{n\mathrm{R}}; \tag{193}$$

and from (191),

$$z_0 = c + \frac{m\mathrm{K}-na\mathrm{R}}{l\mathrm{R}} = c - \frac{l\mathrm{K}+nb\mathrm{R}}{m\mathrm{R}}; \tag{194}$$

which assign the moment-centre. Also from the two values of z_0, we have the condition

$$(l^2+m^2)\mathrm{K} = n\mathrm{R}(am-bl). \tag{195}$$

The geometrical meaning of this condition is that if ϕ is the angle at which the line is inclined to the central-axis, $\tan\phi = \frac{\mathrm{R}r}{\mathrm{K}}$; for from the first two members of (192) it appears that the line drawn from (x_0, y_0) at right angles to the central axis is also perpendicular to the given straight line; so that this line is the shortest distance between them; let it be equal to r; then

$$(\tan\phi)^2 = \frac{l^2+m^2}{n^2+m^2} = \frac{\text{R}(am-bl)}{n\text{K}}$$

$$= \frac{\text{R}^5(x_0{}^2+y_0{}^2)}{\text{K}^2};$$

$$\therefore \quad \tan\phi = \frac{\text{R}r}{\text{K}}.$$

Thus (193) and (194) assign the moment-centre; and if G_0 is the principal moment at it,

$$\text{G}_0{}^2 = \text{R}^2 r^2 + \text{K}^2. \tag{196}$$

If the origin and axes of coordinates are taken in the most general position, and the equations to the straight line are

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n},$$

then, if this line is a principal rotation-axis, whose moment-centre is (x_0, y_0, z_0),

$$\frac{l}{\text{L}-\text{Z}y_0+\text{Y}z_0} = \frac{m}{\text{M}-\text{X}z_0+\text{Z}x_0} = \frac{n}{\text{N}-\text{Y}x_0+\text{X}y_0}$$

$$= \frac{l\text{X}+m\text{Y}+n\text{Z}}{\text{KR}}; \tag{197}$$

whence x_0, y_0, z_0 may be determined; and the values are similar to those given in (173).

If one of the coordinate axes, say the axis of x, is a principal rotation-axis, $m = n = 0$; $y_0 = z_0 = 0$; consequently

$$x_0 = -\frac{\text{M}}{\text{Z}} = \frac{\text{N}}{\text{Y}};$$

and the condition, when this is the case, is

$$\text{MY}+\text{NZ} = 0, \tag{198}$$

and the moment-axis is L. A similar result is true of the other axes.

In further illustration of the preceding conditions, we can hereby shew that if two reciprocal lines are perpendicular to each other, each is then a principal rotation-axis, the moment-centres being on the axis of x in the configuration of Art. 87, Ex. 4. For in this case, by (186) and (187),

$$\left.\begin{aligned} \tan a_0 &= -\frac{\text{K}}{\text{R}r_1} = \frac{\text{R}r_0}{\text{K}}; \\ \tan a_1 &= -\frac{\text{K}}{\text{R}r_0} = \frac{\text{R}r_1}{\text{K}}; \end{aligned}\right\} \tag{199}$$

consequently both the reciprocal lines are principal rotation-axes.

Let G_0 and G_1 be the corresponding principal moment-axes; then

$$\text{K} = \text{G}_0 \cos \alpha_0 = \text{G}_1 \cos \alpha_1,$$
$$= \text{G}_1 \sin \alpha_0 ;$$

$$\therefore \quad \frac{1}{\text{G}_0^2} + \frac{1}{\text{G}_1^2} = \frac{1}{\text{K}^2}. \tag{200}$$

Since the product $r_0 \, r_1$ is a constant, by (187), when the reciprocal lines are perpendicular to each other, $r_0 + r_1$ is a minimum, when

$$r_0 = r_1 = \frac{\text{K}}{\text{R}} : \tag{201}$$

in which case $\alpha_0 = \alpha_1 = 45°$, and $\text{G}_0 = \text{G}_1 = \text{K}\, 2^{\frac{1}{2}}$; thus the two reciprocal lines are each inclined at 45° to the plane of (y, z).

Hereby also it may be shewn that the principal rotation-axes at $(r_0, 0, 0)$ and at $(-r_1, 0, 0)$ make equal angles with the reciprocal lines at these points.

SECTION 5.—*The reduction of a system of forces in space to two forces of translation acting along lines which are not in the same plane.*

89.] The reduction of a system of forces acting in space to two forces acting along lines which are not in the same plane, and consequently do not intersect one another, may be effected in various ways. Each of course demonstrates the possibility of the reduction. The following arise out of the processes of composition which have been employed in the preceding Articles.

Let us take the most general case of forces acting along lines in space.

Let P be the type-force, and (x, y, z) a point in its line of action, which we will suppose to be its point of application. Let A, B, C be three points taken arbitrarily and fixed; and let us assume that the point of application of P is not in the plane containing A, B, C. Let P at its point of application be equivalently replaced by three forces along lines passing through A, B, C respectively; and let all the forces be similarly resolved; then we shall have three groups of forces, corresponding to the points A, B, C respectively, each group consisting of forces whose lines of action have a common point. Let the forces of each group be compounded into a single force; so that the system is reduced to three forces acting each at an arbitrarily chosen point: let these forces be respectively Q, R, S acting at A, B, C respectively. Let D be a point in the line of intersection of the planes ABR, ACS; and let R be resolved into two forces, whose lines of action are BA and BD; and let S be resolved into two forces whose lines of action are CA and CD: thus the system is reduced to three forces whose lines of action pass through A and to two forces whose lines of action pass through D; let each of these groups be compounded into a single force; then we have finally two forces whose lines of action pass through A and D respectively, and evidently do not generally meet each other.

The magnitudes and lines of action of these two final resultants depend on the positions of A and D, and indeed of A, B, C; and as all these are arbitrary, so is the system of the two final resultants arbitrary; the extent to which the arbitrariness extends, that is, the determination of the conditions to which the elements of these two resultants must be subject, will be investigated hereafter: at all events the system is not unique, and the number of pairs of forces, which are equivalent to a system of forces in space, is indeterminate.

90.] For a second way of reduction, let the forces and their lines of action be referred to a system of rectangular coordinates. Let P, as heretofore, be the type-force, and by virtue of the principle of transmissibility let us assume it to act at the point where its line of action intersects the plane of (x, y). At that point let it be resolved into two forces the lines of action of which are in and perpendicular to the plane of (x, y) respectively. Then all the forces having been similarly resolved, we shall have (1) a group of forces the action-lines of which are all in the plane of (x, y), and which consequently generally admit of composition into a single force of translation; and (2) a group of forces all the action-lines of which are parallel to the axis of z, and which can be compounded into a single force of translation, the magnitude of which is equal to the sum of the magnitudes of the several components. Thus the system is reduced to two forces of translation, the lines of action of which do not generally meet; which, however, have the special property that the lines of action are perpendicular to each other.

However, as the origin and the coordinate-axes are arbitrary, and as the choice of the coordinate-axis along and perpendicular to which the forces are resolved is also arbitrary, so the system of the two resultants is arbitrary; and the number of ways in which a system of forces can be reduced to a pair of forces, whose action-lines are perpendicular to each other and do not meet, is indeterminate.

The magnitudes and lines of action of these two resultants may be determined in the following manner:

Let P be the type-force, and $(x, y, 0)$ its point of application: also let $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta$ be the direction-cosines of its line of action. Consequently if P at its point of application is resolved into components whose action-lines are in and perpendicular to the plane of (x, y), $P\sin\theta$ and $P\cos\theta$ are these components respectively; and they are applied at the point $(x, y, 0)$. Let all the forces be similarly resolved: and let R_1 and R_2 be the two resultants respectively in and perpendicular to the plane of (x, y). Then

$$R_2 = \Sigma P\cos\theta; \tag{202}$$

and if $(\bar{x}, \bar{y}, 0)$ is a point in its line of action

$$\bar{x}\,\Sigma P\cos\theta = \Sigma P x\cos\theta, \qquad \bar{y}\,\Sigma P\cos\theta = \Sigma P y\cos\theta; \tag{203}$$

and compounding the forces whose lines of action are in the plane of (x, y),

$$R_1^2 = (\Sigma P\sin\theta\cos\phi)^2 + (\Sigma P\sin\theta\sin\phi)^2; \tag{204}$$

and the equation to its line of action is, see (60), Art. 58.

$$x\,\Sigma P\sin\theta\sin\phi - y\,\Sigma P\sin\theta\cos\phi = \Sigma P\sin\theta(x\sin\phi - y\cos\phi). \tag{205}$$

Thus the magnitudes and lines of action of R_1 and R_2 are determined.

If the point $(\bar{x}, \bar{y})$ given in (203) lies in (205), the lines of action of R_1 and R_2 intersect, and as these may in that case be compounded into a single resultant, the system of forces is reducible to a single resultant. The substitution of (203) in (205) leads to the condition (130), Art. 77.

91.] Again, if all the forces are reduced, as in Art. 68, to a single force of translation acting at an arbitrarily chosen origin, and to a single couple, we may suppose one of the forces of the couple to act at the origin, the other acting along a determinate line parallel to the line of action of the former. Now the former force and the resultant of translation may be compounded into a single force acting at the arbitrarily chosen origin; and thus the system is reduced to two forces of translation acting along lines which do not meet.

If the arm of the resultant couple is turned in its own plane, the point of application of one of its forces, viz. of that at the origin, being unaltered, the resultant of that and of the original resultant of translation will vary, and consequently the system of pair of forces to which all the forces may be reduced is indeterminate.

The reduction, however, admits of the following simplification:

Let R be the resultant of translation at the origin, and let G be the moment of the resultant couple, and let all the other symbols be employed as in Art. 68 : let the arm of the couple be turned in its own plane until it is perpendicular to the line of action of R; let R′ and a be the force and the arm of the couple; both of these quantities being arbitrary, but subject to the condition $\text{R}'a = \text{G}$. Then, if ϕ is the angle between the line of action of R and the rotation-axis of G, so that

$$\cos\phi = \frac{\text{LX}+\text{MY}+\text{NZ}}{\text{RG}} = \frac{\text{K}}{\text{G}}, \tag{206}$$

$\frac{\pi}{2}-\phi$ is the angle between the action-lines of R and R′, these action-lines meeting at the origin. Let these forces be compounded into a single force R″; then

$$\text{R}''^2 = \text{R}^2+2\,\text{RR}'\sin\phi+\text{R}'^2; \tag{207}$$

and the system is reduced to the two forces R′ and R″, the lines of action of which do not meet, and the shortest distance between them being a which is perpendicular to both lines of action.

Also this reduction may be so arranged that the lines of action of the two forces shall be perpendicular to each other. Thus, as before, let the arm of the couple be perpendicular to the line of action of R; and let R be resolved into two parts $\text{R}\sin\phi$ and $\text{R}\cos\phi$ respectively in and perpendicular to the plane of the couple: so that there are, (1) three forces R′, −R′, $\text{R}\sin\phi$ in the plane of the couple, the lines of action of all of which are parallel and are perpendicular to the arm of the couple, and the resultant of which is $\text{R}\sin\phi$, which acts in the plane of the couple, at right angles to its arm, and at a distance r from the origin along the arm, such that $\text{R}r\sin\phi = \text{G}$; and (2) the force $\text{R}\cos\phi$ whose line of action is perpendicular to the plane of the couple.

Thus the system is reduced to the two forces $\text{R}\sin\phi$ and

$\mathrm{R}\cos\phi$ acting along lines perpendicular to each other which do not meet, and between which the shortest distance is r, where

$$r = \frac{\mathrm{G}}{\mathrm{R}\sin\phi}. \tag{208}$$

As the line of action of $\mathrm{R}\cos\phi$ passes through the origin and is perpendicular to the plane of the couple, its equations are

$$\frac{x}{\mathrm{L}} = \frac{y}{\mathrm{M}} = \frac{z}{\mathrm{N}}; \tag{209}$$

and as the line of action of $\mathrm{R}\sin\phi$ lies in the plane of the couple and passes at right angles through the extremity of r which is perpendicular to both the line of action of the original resultant of translation and to the rotation-axis of the couple, its equations are

$$\frac{x-\dfrac{\mathrm{NY}-\mathrm{MZ}}{\mathrm{R}\sin\phi}}{\mathrm{G}^2\mathrm{X}-\mathrm{LK}} = \frac{y-\dfrac{\mathrm{LZ}-\mathrm{NX}}{\mathrm{R}\sin\phi}}{\mathrm{G}^2\mathrm{Y}-\mathrm{MK}} = \frac{z-\dfrac{\mathrm{MX}-\mathrm{LY}}{\mathrm{R}\sin\phi}}{\mathrm{G}^2\mathrm{Z}-\mathrm{NK}}. \tag{210}$$

Thus the lines of action of the two forces are determined, and also the shortest distance between them.

As the equations to the line on which r lies are

$$\frac{x}{\mathrm{NY}-\mathrm{MZ}} = \frac{y}{\mathrm{LZ}-\mathrm{NX}} = \frac{z}{\mathrm{MX}-\mathrm{LY}}, \tag{211}$$

this line is perpendicular to the central axis whose equations are given in (152), and also intersects it. Consequently we have the following theorem:

If a system of forces is reduced to two forces of translation, which act along lines perpendicular to each other, the shortest distance between their lines of action intersects the central axis at right angles.

The sole indeterminateness which is involved in this mode of reduction arises from the arbitrary position of the origin. When that is fixed, all the quantities are assigned.

92.] Also if the system of forces is reduced to the force of translation R acting along the central axis, and to the couple K whose rotation-axis is the central axis, we may replace K by its two equal and opposite forces each of which is equal to $\frac{\mathrm{K}}{a}$, if a is the length of an arbitrary arm. Of these two forces let one act along a line passing through the central axis, and of course perpendicular to it; then it and R may be compounded into a single force R'', such that

$$\mathrm{R}''^2 = \mathrm{R}^2 + \frac{\mathrm{K}^2}{a^2}, \tag{212}$$

and there remains the other force of the principal central couple acting along a line, perpendicular indeed to the central axis but not meeting it, and not meeting the action-line of R''; and the shortest line between the action-line of these two resultants is a, which is such that, if R' is the force of K, $\mathrm{R}'a = \mathrm{K}$.

This reduction may also be effected more generally by the following process: Let us suppose the central axis to be the axis of z; and let R be replaced by two forces R_1 and R_2, the action-lines of which are parallel to the central axis, and which pass through two points Q_1 and Q_2 on the axis of x at distances r_1 and r_2 respectively from the origin, and on opposite sides of it; then we have

$$\mathrm{R} = \mathrm{R}_1 + \mathrm{R}_2; \qquad \mathrm{R}_1 r_1 = \mathrm{R}_2 r_2:$$

$$\therefore \quad \frac{\mathrm{R}_1}{r_2} = \frac{\mathrm{R}_2}{r_1} = \frac{\mathrm{R}_1+\mathrm{R}_2}{r_1+r_2} = \frac{\mathrm{R}}{r_1+r_2}. \tag{213}$$

Let the principal central couple be replaced by two equal forces R' acting in opposite directions along lines passing through Q_1 and Q_2 and parallel to the axis of y; then

$$\mathrm{K} = \mathrm{R}'(r_1+r_2). \tag{214}$$

Thus there are now four forces, viz. R_1 and R' at Q_1, and R_2 and $-\mathrm{R}'$ at Q_2; each pair acts in a plane perpendicular to the x-axis, and the action-lines of the forces in each pair are perpendicular to each other: let P_1 be the resultant of R_1 and R' which act at Q_1, and let P_2 be the resultant of R_2 and $-\mathrm{R}'$ which act at Q_2; then

$$\mathrm{P}_1^2 = \mathrm{R}_1^2 + \mathrm{R}'^2; \qquad \mathrm{P}_2^2 = \mathrm{R}_2^2 + \mathrm{R}'^2; \tag{215}$$

so that the system is now reduced to the two forces P_1 and P_2, the shortest distance between the action-lines of which is r_1+r_2.

As to the action-lines of P_1 and P_2; let θ_1 and θ_2 be the angles between them and the central axis; then

$$\mathrm{R}' = \mathrm{P}_1 \sin\theta_1 = \mathrm{P}_2 \sin\theta_2; \tag{216}$$

$$\mathrm{R}_1 = \mathrm{P}_1 \cos\theta_1; \qquad \mathrm{R}_2 = \mathrm{P}_2 \cos\theta_2; \tag{217}$$

consequently

$$\mathrm{P}_1 \cos\theta_1 + \mathrm{P}_2 \cos\theta_2 = \mathrm{R}; \tag{218}$$

$$\mathrm{P}_1 \sin\theta_1 = \mathrm{P}_2 \sin\theta_2 = \frac{\mathrm{K}}{r_1+r_2}; \tag{219}$$

$$\mathrm{P}_1 r_1 \cos\theta_1 = \mathrm{P}_2 r_2 \cos\theta_2; \tag{220}$$

$$\frac{\tan\theta_1}{r_1} = \frac{\tan\theta_2}{r_2} = \frac{\mathrm{K}}{\mathrm{R}\, r_1 r_2}; \tag{221}$$

so that if r_1 and r_2 are given, the forces and their incidents are completely determined.

93.] In reference to this system of two forces to which the general system has been reduced, the following theorems are noteworthy:

(1) On comparing (221) with (186) it appears that the action-lines of P_1 and P_2 are reciprocal lines; consequently as the position of a line is given when that of its reciprocal line is given, so if the action-line of one force is given that of the other force is also given.

(2) Let G_1 and G_2 be the principal moments at Q_1 and Q_2; then evidently,

$$G_1 \cos\theta_1 = G_2 \cos\theta_2 = K;$$

therefore from (220),
$$\frac{G_1}{r_1 P_1} = \frac{G_2}{r_2 P_2}; \tag{222}$$
which gives the ratio of the principal moments at Q_1 and Q_2.

(3) The volume of the tetrahedron of which the line-representatives of P_1 and P_2 are opposite edges is constant. For let V be the volume, then

$$\begin{aligned} V &= \frac{P_1 P_2}{6}(r_1+r_2)\sin(\theta_1+\theta_2) \\ &= \frac{r_1+r_2}{6}\{P_1 \sin\theta_1 P_2 \cos\theta_2 + P_1\cos\theta_1 P_2 \sin\theta_2\} \\ &= \frac{r_1+r_2}{6}\,\frac{KR}{r_1+r_2} \\ &= \frac{KR}{6}; \end{aligned} \tag{223}$$

which is constant; and consequently the volume of the tetrahedron is constant whatever is the position of the two forces which equivalently replace a system of forces.

If the volume of the tetrahedron vanishes, the two forces act in the same plane, and the system is reducible, either to a single force of translation, or to a couple: that is, either $K=0$, or $R=0$.

Hence also it is evident that if four forces are in equilibrium, the volume of the tetrahedron constructed on the line-representatives of any two is equal to that of the tetrahedron constructed on the line-representatives of the other two.

(4) If the action-lines of P_1 and P_2 are at right angles to each other, then $\theta_1+\theta_2 = 90°$; $\sin\theta_1 = \cos\theta_2$; $\sin\theta_2 = \cos\theta_1$; and consequently $\tan\theta_1 \tan\theta_2 = 1$:

$$r_1 r_2 = \frac{K^2}{R^2}; \qquad \tan\theta_1 = r_1\frac{R}{K}; \qquad \tan\theta_2 = r_2\frac{R}{K}; \tag{224}$$

whence if any one of the four quantities r_1, r_2, θ_1, θ_2 is given, all the others are given: as, however, the number of equations connecting the unknown quantities is less by one than the number of unknown quantities, the number of ways is infinite in which a system of forces may be reduced to two forces acting along lines at right angles to each other.

When any one of these quantities relating to one of the forces is assigned, then all the incidents of the other force are also assigned.

(5) The system of two forces is however unique, when the forces are equal and act along lines perpendicular to each other.

In this case $\text{P}_1 = \text{P}_2$; and consequently

$$\theta_1 = \theta_2 = 45°; \qquad r_1 = r_2 = \frac{\text{K}}{\text{R}}; \qquad \frac{\text{R}}{2^{\frac{1}{2}}} = \text{P}_1 = \text{P}_2;$$

and we have the following theorem :

A given system of forces acting on a rigid body may be replaced by two equal forces whose lines of action are perpendicular to each other, and each of which has a line of action inclined at 45° to the central axis; and the forces act perpendicularly at the ends of an arm which is bisected at right angles by that axis; the magnitude of each force is equal to $\frac{\text{R}}{2^{\frac{1}{2}}}$, and the length of the arm is $\frac{2\text{K}}{\text{R}}$.

This result may also be arrived at directly in the following manner :

Let R be resolved into two equal and parallel forces, each of which $= \frac{\text{R}}{2}$; and let them act at two points Q_1 and Q_2 on the axis of x which are equidistant from the central axis, and at a distance r from it on either side; also let the forces of K be $\frac{\text{R}}{2}$, and act at the points Q_1 and Q_2, so that $\text{R}r = \text{K}$. Then we have at Q_1 and at Q_2 two equal forces acting along lines which are perpendicular to each other; and the resultant at each point is equal to $\frac{\text{R}}{2^{\frac{1}{2}}}$, and acts along a line inclined to the central axis at an angle of 45°; but as these lines are on opposite sides of that axis, they are at right angles to each other.

This is the only unique system of a pair of forces to which a system may be reduced.

(6) The distance between the action-lines of the two forces which equivalently replace a system of forces is a minimum, when the forces are equal and their action-lines are perpendicular to each other.

SECTION 6.—*The equilibrium-axis of an equilibrium-system.*

94.] In this section I propose to investigate for an equilibrium-system of forces acting in space the conditions requisite that the system should also be an equilibrium-system, when the body receives the most general displacement, and the forces act at the same points of the body, along lines parallel to their former action-lines, and in the same direction as before displacement.

Whatever the displacement be, it may always be resolved into a displacement of translation and a displacement of rotation, the effects of which may be separately considered. Now the displacement of translation will be effected by transferring the point of the body which coincides with the origin in its original position to the point (x_0, y_0, z_0), and making all particles of the body describe equal and parallel paths: then if (x', y', z') is the place of the particle which was originally at (x, y, z),

$$x' = x + x_0, \qquad y' = y + y_0, \qquad z' = z + z_0. \tag{225}$$

As the systems, both displaced and original, are equilibrium-systems, and as the direction-angles of the action-lines of the forces are unchanged, we have the following conditions; viz.

$$\Sigma.\mathrm{P}\cos a = \Sigma.\mathrm{P}\cos\beta = \Sigma.\mathrm{P}\cos\gamma = 0; \tag{226}$$

$$\Sigma.\mathrm{P}(y\cos\gamma - z\cos\beta) = \Sigma.\mathrm{P}(z\cos a - x\cos\gamma)$$
$$= \Sigma.\mathrm{P}(x\cos\beta - y\cos a) = 0, \tag{227}$$

$$\Sigma.\mathrm{P}(y'\cos\gamma - z'\cos\beta) = \Sigma.\mathrm{P}(z'\cos a - x'\cos\gamma)$$
$$= \Sigma.\mathrm{P}(x'\cos\beta - y'\cos a) = 0; \tag{228}$$

and substituting from (225) in (228), (228) are identically satisfied by reason of (226) and (227); so that whatever is the displacement of translation an equilibrium-system continues an equilibrium-system.

Let the displacement of rotation be produced by making the body turn through an angle θ about an axis passing through the origin and of which the direction-angles are f, g, h: let (x, y, z) be the place of any particle of the body in its original position, and let this point after the rotation be $(x+\Delta x, y+\Delta y, z+\Delta z)$: let Δs be the distance between the two positions of this point, so that

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2; \tag{229}$$

and let p be the perpendicular distance from (x, y, z) to the axis of rotation; so that Δs is the chord of a circular arc, of radius p and angle θ, described by (x, y, z) revolving about the axis of rotation; and therefore

$$\Delta s = 2p \sin\frac{\theta}{2}. \tag{230}$$

As this point is in both its positions at the same distance from the origin, and also in the same plane perpendicular to the rotation-axis, we have

$$\Delta x(2x+\Delta x)+\Delta y(2y+\Delta y)+\Delta z(2z+\Delta z) = 0, \tag{231}$$

$$\Delta z\cos f+\Delta y\cos g+\Delta z\cos h = 0. \tag{232}$$

Also from (230),

$$(\Delta x)^2+(\Delta y)^2+(\Delta z)^2 = 4p^2\left(\sin\frac{\theta}{2}\right)^2; \text{ and} \tag{233}$$

$$(z\cos g-y\cos h)^2+(x\cos h-z\cos f)^2+(y\cos f-x\cos g)^2 = p^2. \tag{234}$$

Also as $z\cos g-y\cos h$, $x\cos h-z\cos f$, $y\cos f-x\cos g$ are proportional to the direction-cosines of the normal to the plane which contains the rotation-axis and (x, y, z), and Δx, Δy, Δz are proportional to the direction-cosines of the chord Δs, and as $\frac{\theta}{2}$ is the angle contained between these lines,

$$(z\cos g-y\cos h)\Delta x+(x\cos h-z\cos f)\Delta y+(y\cos f-x\cos g)\Delta z = p\Delta s\cos\frac{\theta}{2}.$$

Thus we have the three following linear equations in terms of Δx, Δy, Δz,

$$\begin{array}{llll}
(z\cos g-y\cos h)\Delta x+ & (x\cos h-z\cos f)\Delta y+ & (y\cos f-x\cos g)\Delta z = & p^2\sin\theta, \\
x\ \Delta x+ & y\ \Delta y+ & z\ \Delta z = & -2p^2\left(\sin\frac{\theta}{2}\right)^2, \\
\cos f\ \Delta x+ & \cos g\ \Delta y+ & \cos h\ \Delta z = & 0;
\end{array}$$

and from them we have

$$\left.\begin{aligned}
\Delta x &= \sin\theta\,(z\cos g-y\cos h) \\
&\quad +2\left(\sin\frac{\theta}{2}\right)^2\{\cos f(x\cos f+y\cos g+z\cos h)-x\}; \\
\Delta y &= \sin\theta\,(x\cos h-z\cos f) \\
&\quad +2\left(\sin\frac{\theta}{2}\right)^2\{\cos g(x\cos f+y\cos g+z\cos h)-y\}; \\
\Delta z &= \sin\theta\,(y\cos f-x\cos g) \\
&\quad +2\left(\sin\frac{\theta}{2}\right)^2\{\cos h\,(x\cos f+y\cos g+z\cos h)-z\}.
\end{aligned}\right\} \tag{235}$$

I may by the way observe, that if the angle through which the body is turned is infinitesimal, say $= d\theta$, then omitting powers of it higher than the first, and replacing Δx, Δy, Δz, which are also infinitesimal, by dx, dy, dz,

$$dx = (z\cos g - y\cos h)\,d\theta, \tag{236}$$

$$dy = (x\cos h - z\cos f)\,d\theta, \tag{237}$$

$$dz = (y\cos f - x\cos g)\,d\theta. \tag{238}$$

The signs of the terms in the right-hand members of these equations, which are ambiguous by reason of the system of squares in (234), have been taken in such a manner that if the x-axis were the axis of rotation, the positive direction of rotation would be from the y-axis to the z-axis. And the rotations about the other axes would have similar positive directions; so that the system is cyclical.

In (228) let x', y', z' be replaced by $x+\Delta x$, $y+\Delta y$, $z+\Delta z$ respectively; and let the following symbols be employed for the abbreviation of the results; viz.

$$\left.\begin{aligned} \Sigma.\mathrm{P}\,y\cos\gamma = \Sigma.\mathrm{P}\,z\cos\beta &= \mathrm{D}, \\ \Sigma.\mathrm{P}\,z\cos\alpha = \Sigma.\mathrm{P}\,x\cos\gamma &= \mathrm{E}, \\ \Sigma.\mathrm{P}\,x\cos\beta = \Sigma.\mathrm{P}\,y\cos\alpha &= \mathrm{F}; \end{aligned}\right\} \tag{239}$$

$$\left.\begin{aligned} \Sigma.\mathrm{P}(y\cos\beta + z\cos\gamma) &= \mathrm{U}, \\ \Sigma.\mathrm{P}\,(z\cos\gamma + x\cos\alpha) &= \mathrm{V}, \\ \Sigma.\mathrm{P}\,(x\cos\alpha + y\cos\beta) &= \mathrm{W}; \end{aligned}\right\} \tag{240}$$

the first three equalities following from (227); then we have the following equations, viz.

$$\cot\frac{\theta}{2}\,(-\mathrm{U}\cos f + \mathrm{F}\cos g + \mathrm{E}\cos h) - \cos h\,(\mathrm{F}\cos f - \mathrm{V}\cos g + \mathrm{D}\cos h) \\ + \cos g\,(\mathrm{E}\cos f + \mathrm{D}\cos g - \mathrm{W}\cos h) = 0; \tag{241}$$

$$\cot\frac{\theta}{2}\,(\mathrm{F}\cos f - \mathrm{V}\cos g + \mathrm{D}\cos h) - \cos f\,(\mathrm{E}\cos f + \mathrm{D}\cos g - \mathrm{W}\cos h) \\ + \cos h\,(-\mathrm{U}\cos f + \mathrm{F}\cos g + \mathrm{E}\cos h) = 0; \tag{242}$$

$$\cot\frac{\theta}{2}\,(\mathrm{E}\cos f + \mathrm{D}\cos g - \mathrm{W}\cos h) - \cos g\,(-\mathrm{U}\cos f + \mathrm{F}\cos g + \mathrm{E}\cos h) \\ + \cos h\,(\mathrm{F}\cos f - \mathrm{V}\cos g + \mathrm{D}\cos h) = 0; \tag{243}$$

but as equilibrium is to subsist for all angles through which the body is turned about the rotation-axis, θ is indeterminate; and consequently from these three equations the following result;

$$\left.\begin{aligned} -\mathrm{U}\cos f + \mathrm{F}\cos g + \mathrm{E}\cos h &= 0, \\ \mathrm{F}\cos f - \mathrm{V}\cos g + \mathrm{D}\cos h &= 0, \\ \mathrm{E}\cos f + \mathrm{D}\cos g - \mathrm{W}\cos h &= 0; \end{aligned}\right\} \tag{244}$$

and from these the direction-cosines of the rotation-axis are to be determined. As, however, they are more than sufficient for the purpose, a relation exists between them; and eliminating $\cos f$, $\cos g$, $\cos h$ we have

$$\mathrm{UVW}-\mathrm{D}^2\mathrm{U}-\mathrm{E}^2\mathrm{V}-\mathrm{F}^2\mathrm{W}-2\mathrm{DEF}=0; \qquad (245)$$

which expresses a relation between the forces, their action-lines, and their points of application, when an equilibrium-system is also an equilibrium-system after rotation through any angle about a certain axis. As this axis has important properties, it is convenient for it to have a distinctive name, and so it has been called *the equilibrium-axis*. Equation (245) is the condition that an equilibrium-system should have an equilibrium-axis. When that condition is satisfied, the direction-cosines of the equilibrium-axis are given by (244), and we have

$$\frac{(\cos f)^2}{\mathrm{D}^2-\mathrm{VW}}=\frac{(\cos g)^2}{\mathrm{E}^2-\mathrm{WU}}=\frac{(\cos h)^2}{\mathrm{F}^2-\mathrm{UV}}=\frac{1}{\mathrm{D}^2+\mathrm{E}^2+\mathrm{F}^2-(\mathrm{VW}+\mathrm{WU}+\mathrm{UV})}. \qquad (246)$$

As these equations give only the direction-cosines of the equilibrium-axis, all straight lines parallel to that thus assigned are also equilibrium-axes.

If $\mathrm{D}^2=\mathrm{VW}$, $\mathrm{E}^2=\mathrm{WU}$, $\mathrm{F}^2=\mathrm{UV}$, f, g, h are indeterminate, and the body is in equilibrium, whatever is the position of the axis about which it is turned.

If all the forces act in one plane, say in that of (x, y), then $\cos\gamma=0$, and consequently $\mathrm{D}=\mathrm{E}=0$, and from the last of (244), $\mathrm{W}=0$; that is,

$$\Sigma.\mathrm{P}(x\cos\alpha+y\cos\beta)=0,$$

which is the same condition as (82), Art. 63. Hence also $\cos f=\cos g=0$ and $\cos h=1$, so that the equilibrium-axis is perpendicular to the plane of the forces.

95.] The condition for the existence of an equilibrium-axis given in (245) will be more easily interpreted, if we take the most simple case. For this purpose let us assume the system of coordinate-axes to be so taken that the z-axis is the equilibrium-axis; then $\cos f=\cos g=0$; and consequently $\mathrm{D}=\mathrm{E}=0$, $\mathrm{W}=0$; that is,

$$\Sigma.\mathrm{P}y\cos\gamma=0,\quad \Sigma.\mathrm{P}x\cos\gamma=0,\quad \Sigma.\mathrm{P}(x\cos\alpha+y\cos\beta)=0; \qquad (247)$$

from the first two of which taken in combination with $\Sigma.\mathrm{P}\cos\gamma=0$, see (226), we infer that, if the forces are at their points of application resolved in directions parallel to the coordinate axes, those parallel to the axis of z are in equilibrium; and from the last, combined with the first two of (226) and the last of (227), we infer that the forces whose lines of action are parallel to the plane of (x, y) satisfy the conditions required for *a centre*, see Art. 63, and are therefore in equilibrium when the body is turned through any angle about the axis of z. Hence the meaning of the condition (245) is,

If the forces acting on a body are resolved along a certain straight line, and in planes perpendicular to that line; and if the forces parallel to the straight line are in equilibrium, and those in the planes perpendicular to the straight line are also in equilibrium, and satisfy the conditions required for a centre, then every line parallel to that line is an equilibrium-axis.

Also if the forces are such that the x- and y-axes are both equilibrium-axes: then from the equations (246)

$$\mathrm{D} = \mathrm{E} = \mathrm{F} = 0, \qquad \mathrm{U} = \mathrm{V} = 0;$$

and therefore $\cos h = 0$; and therefore any line parallel to the plane of (x, y) will also be an equilibrium-axis.

96.] To investigate generally the conditions requisite that any two lines inclined at any angle to each other should be equilibrium-axes; let the direction-angles of the two lines be f, g, h, f', g', h'; then from (244),

$$\left.\begin{aligned} -\mathrm{U}\cos f + \mathrm{F}\cos g + \mathrm{E}\cos h &= 0, \\ \mathrm{F}\cos f - \mathrm{V}\cos g + \mathrm{D}\cos h &= 0, \\ \mathrm{E}\cos f + \mathrm{D}\cos g - \mathrm{W}\cos h &= 0; \end{aligned}\right\} \tag{248}$$

$$\left.\begin{aligned} -\mathrm{U}\cos f' + \mathrm{F}\cos g' + \mathrm{E}\cos h' &= 0, \\ \mathrm{F}\cos f' - \mathrm{V}\cos g' + \mathrm{D}\cos h' &= 0, \\ \mathrm{E}\cos f' + \mathrm{D}\cos g' - \mathrm{W}\cos h' &= 0; \end{aligned}\right\} \tag{249}$$

whence we have the following relations:

$$\mathrm{DU} + \mathrm{EF} = 0, \qquad \mathrm{EV} + \mathrm{FD} = 0, \qquad \mathrm{FW} + \mathrm{DE} = 0; \tag{250}$$

and

$$\mathrm{D}^2 = \mathrm{VW}, \qquad \mathrm{E}^2 = \mathrm{WU}, \qquad \mathrm{F}^2 = \mathrm{UV}; \tag{251}$$

which are the conditions necessary that an equilibrium-system should admit of two equilibrium-axes not parallel to each other.

But by reason of (251), $\cos f$, $\cos g$, $\cos h$, as also $\cos f'$, $\cos g'$, $\cos h'$ are indeterminate; they are however subject to the following relation; if we substitute from (251) in either of (248), we have

$$\mathrm{U}^{\frac{1}{2}}\cos f + \mathrm{V}^{\frac{1}{2}}\cos g + \mathrm{W}^{\frac{1}{2}}\cos h = 0; \tag{252}$$

and if we substitute in either of (249), we have

$$\mathrm{U}^{\frac{1}{2}}\cos f' + \mathrm{V}^{\frac{1}{2}}\cos g' + \mathrm{W}^{\frac{1}{2}}\cos h' = 0; \tag{253}$$

which shews that both these lines are parallel to or lie in the plane whose equation is

$$\mathrm{U}^{\frac{1}{2}} x + \mathrm{V}^{\frac{1}{2}} y + \mathrm{W}^{\frac{1}{2}} z = 0; \tag{254}$$

but that the position of the lines in the plane is indeterminate.

Hence we infer that a body which is in equilibrium for two equilibrium-axes which meet and are not parallel to each other, is also in equilibrium for all axes parallel to the plane which contains these two equilibrium-axes. And hence

If a body has three equilibrium-axes which are not parallel to one and the same plane, so is any fourth axis an equilibrium-axis.

And as a body has an equilibrium-axis, if it is in equilibrium in two different and not parallel positions, so if it is in equilibrium in four different and not parallel positions, it is also in equilibrium in every fifth position.

And when this last case occurs, $\mathrm{D} = \mathrm{E} = \mathrm{F} = 0$, $\mathrm{U} = \mathrm{V} = \mathrm{W} = 0$; so that the position of the plane (254) becomes indeterminate.

97.] Although a system of forces acting on a rigid body and being in equilibrium admits of an equilibrium-axis, only when (245) is satisfied, and therefore not generally; yet if a system is in equilibrium, two new equal forces acting at certain definite points, along the same line of action and in opposite directions, may be introduced in such a manner that the system thus modified may have an equilibrium-axis in a given direction. The new forces, it will be observed, as introduced into the first position of the body, being equal and opposite, neutralize each other, and do not disturb equilibrium, and in the other positions form a couple which equilibrates with the impressed forces of the system in their new position.

Let, as in the preceding Articles, f, g, h be the direction-angles of the given equilibrium-axis; $-\mathrm{P}'$ and P' the two new forces, equal and opposite to each other; (x', y', z'), (x'', y'', z'') their points of application; l, m, n the direction-cosines of their common line of action; r the distance between their points of application; let $x''-x'$, $y''-y'$, $z''-z'$ be positive quantities; then

$$\frac{x''-x'}{l} = \frac{y''-y'}{m} = \frac{z''-z'}{n} = r;$$

and if the accented letters refer to the system when increased by the two new forces, and the unaccented letters to the original system,

$$\begin{aligned} \mathrm{D}' &= \mathrm{D} + y''\mathrm{P}'n - y'\mathrm{P}'n \\ &= \mathrm{D} + (y''-y')\mathrm{P}'n; \end{aligned}$$

$$\left.\begin{aligned} \therefore\quad & \mathrm{D}' = \mathrm{D} + \mathrm{P}' r\, mn, \\ \text{similarly}\quad & \mathrm{E}' = \mathrm{E} + \mathrm{P}' r\, nl, \\ & \mathrm{F}' = \mathrm{F} + \mathrm{P}' r\, lm; \end{aligned}\right\} \tag{255}$$

$$\mathrm{U}' = \mathrm{U} + \mathrm{P}' r (m^2 + n^2), \quad \mathrm{V}' = \mathrm{V} + \mathrm{P}' r (n^2 + l^2), \quad \mathrm{W}' = \mathrm{W} + \mathrm{P}' r (l^2 + m^2); \tag{256}$$

and substituting these values in the conditions (244), which are requisite for an equilibrium-axis, we have from the first of them

$$\begin{aligned} -\mathrm{U} \cos f + \mathrm{F} \cos g + \mathrm{E} \cos h & \\ &= \mathrm{P}' r \{(m^2 + n^2) \cos f - lm \cos g - ln \cos h\} \\ &= \mathrm{P}' r \{\cos f - l (l \cos f + m \cos g + n \cos h)\}. \end{aligned} \tag{257}$$

Let ϕ be the angle between the line of action of P' and the equilibrium-axis; then

$$\cos \phi = l \cos f + m \cos g + n \cos h; \tag{258}$$

and therefore we have

$$\left.\begin{aligned} -\mathrm{U} \cos f + \mathrm{F} \cos g + \mathrm{E} \cos h &= \mathrm{P}' r \{\cos f - l \cos \phi\} = u, \\ \mathrm{F} \cos f - \mathrm{V} \cos g + \mathrm{D} \cos h &= \mathrm{P}' r \{\cos g - m \cos \phi\} = v, \\ \mathrm{E} \cos f + \mathrm{D} \cos g - \mathrm{W} \cos h &= \mathrm{P}' r \{\cos h - n \cos \phi\} = w, \end{aligned}\right\} \tag{259}$$

employing u, v, w as abbreviating symbols for the left-hand members of the equations, which are known quantities.

Hence we have

$$\begin{aligned} u \cos f + v \cos g + w \cos h &= \mathrm{P}' r \{1 - (\cos \phi)^2\} \\ &= \mathrm{P}' r (\sin \phi)^2. \end{aligned}$$

$$\begin{aligned} \text{Also}\quad u^2 + v^2 + w^2 &= \mathrm{P}'^2 r^2 \{1 - (\cos \phi)^2\}, \\ &= \mathrm{P}'^2 r^2 (\sin \phi)^2; \end{aligned}$$

$$\therefore\quad \mathrm{P}' r = \frac{u^2 + v^2 + w^2}{u \cos f + v \cos g + w \cos h}, \tag{260}$$

$$(\sin \phi)^2 = \frac{(u \cos f + v \cos g + w \cos h)^2}{u^2 + v^2 + w^2}; \tag{261}$$

and therefore from (259) we are able to determine l, m, n; and thus the direction of the line of action of P' is completely determined. The intensity of P' and the position of its point of application are involved in only (260); and therefore we may take any two points on the line defined by (l, m, n) at a distance r apart, and at them apply two equal and opposite forces P' and $-\mathrm{P}'$ of such magnitude that $\mathrm{P}' r$ is equal to the right-hand member of (260).

From the preceding it appears that two equal forces, acting originally in opposite directions along the same line of action, will, when the body is turned about a certain axis, equilibrate with the forces of the system: but as the two forces in this displaced position form a couple, we infer that

If a rigid body, on which a system of forces in equilibrium acts, is turned about any axis, and if the forces act on the same points of the body as before and in the same directions, they are generally reducible to a couple; but in the particular case when the condition (245) is fulfilled, the moment of the couple vanishes.

98.] In Section 5 of the present Chapter it has been shewn in various ways that it is possible to reduce a system of forces acting on a rigid body to two forces, and that the two forces are generally indeterminate in all their elements; it was shewn, however, that the pair is unique and determinate, when the two forces were equal and acted along lines at right angles to each other. I propose now to shew that it is always possible to reduce a system of forces to two forces of translation, such that they with two other new forces shall be in equilibrium, and also shall have a given equilibrium-axis.

Let the two new forces be P' and P''; let $\alpha'\,\beta'\,\gamma'$, $\alpha''\,\beta''\,\gamma''$ be the direction-angles of their lines of action; (x', y', z'), (x'', y'', z'') their points of application; then for the condition of equilibrium of these two new forces, with the former forces of the system, we have

$$\left.\begin{aligned} \mathrm{P}'\cos\alpha' + \mathrm{P}''\cos\alpha'' + \mathrm{X} &= 0,\\ \mathrm{P}'\cos\beta' + \mathrm{P}''\cos\beta'' + \mathrm{Y} &= 0,\\ \mathrm{P}'\cos\gamma' + \mathrm{P}''\cos\gamma'' + \mathrm{Z} &= 0. \end{aligned}\right\} \qquad (262)$$

Also let

$$\begin{aligned} \Sigma.\mathrm{P}y\cos\gamma &= \mathrm{D}', & \Sigma.\mathrm{P}z\cos\beta &= \mathrm{D}'',\\ \Sigma.\mathrm{P}z\cos\alpha &= \mathrm{E}', & \Sigma.\mathrm{P}x\cos\gamma &= \mathrm{E}'',\\ \Sigma.\mathrm{P}x\cos\beta &= \mathrm{F}', & \Sigma.\mathrm{P}y\cos\alpha &= \mathrm{F}'', \end{aligned}$$

$$\Sigma.\mathrm{P}(y\cos\beta + z\cos\gamma) = \mathrm{U},\quad \Sigma.\mathrm{P}(z\cos\gamma + x\cos\alpha) = \mathrm{V},\quad \Sigma.\mathrm{P}(x\cos\alpha + y\cos\beta) = \mathrm{W};$$

then, as the three expressions for the moment-axes of the couples about the coordinate-axes are to vanish, we have

$$\left.\begin{aligned} &\mathrm{P}'y'\cos\gamma' + \mathrm{P}''y''\cos\gamma'' + \mathrm{D}'\\ &\qquad = \mathrm{P}'z'\cos\beta' + \mathrm{P}''z''\cos\beta'' + \mathrm{D}'' = \mathrm{D}\ (\text{say}),\\ &\mathrm{P}'z'\cos\alpha' + \mathrm{P}''z''\cos\alpha'' + \mathrm{E}'\\ &\qquad = \mathrm{P}'x'\cos\gamma' + \mathrm{P}''x''\cos\gamma'' + \mathrm{E}'' = \mathrm{E}\ (\text{say}),\\ &\mathrm{P}'x'\cos\beta' + \mathrm{P}''x''\cos\beta'' + \mathrm{F}'\\ &\qquad = \mathrm{P}'y'\cos\alpha' + \mathrm{P}''y''\cos\alpha'' + \mathrm{F}'' = \mathrm{F}\ (\text{say}). \end{aligned}\right\} \qquad (263)$$

Also let

$$\begin{aligned} \mathrm{P}'(y'\cos\beta' + z'\cos\gamma') + \mathrm{P}''(y''\cos\beta'' + z''\cos\gamma'') + \mathrm{U} &= \mathrm{U}', \\ \mathrm{P}'(z'\cos\gamma' + x'\cos\alpha') + \mathrm{P}''(z''\cos\gamma'' + x''\cos\alpha'') + \mathrm{V} &= \mathrm{V}', \\ \mathrm{P}'(x'\cos\alpha' + y'\cos\beta') + \mathrm{P}''(x''\cos\alpha'' + y''\cos\beta'') + \mathrm{W} &= \mathrm{W}'; \end{aligned}$$

and therefore, if the direction-angles of the given equilibrium-axis are f, g, h, the conditions required are, see (244),

$$\left.\begin{aligned} -\mathrm{U}'\cos f + \mathrm{F}\cos g + \mathrm{E}\cos h &= 0, \\ \mathrm{F}\cos f - \mathrm{V}'\cos g + \mathrm{D}\cos h &= 0, \\ \mathrm{E}\cos f + \mathrm{D}\cos g - \mathrm{W}'\cos h &= 0; \end{aligned}\right\} \qquad (264)$$

and these are all the conditions which are requisite for the existence of an equilibrium-axis: viz. the equations severally of (262), (263), and (264), and of which the whole number is nine; and they contain twelve undetermined quantities: viz. $\mathrm{P}'\cos\alpha'$, $\mathrm{P}'\cos\beta'$, ... $\mathrm{P}''\cos\gamma''$, x', y', ... z''; of these therefore nine may be eliminated, and there will remain a condition involving the other three: the elimination, however, is so long that I shall only state results. If we eliminate the forces P', P'', the direction-angles of their lines of action, and the coordinates of the point of application of one of them, say, x'', y'', z'', it will be found that the resulting equation is of the second degree in terms of x', y', z'; and will therefore represent a surface of the second order: and it will also be found that the point of application of the other force is also upon the same surface, and also that every point in the line joining the two points is on the same surface: the surface is therefore an hyperboloid of one sheet, the line joining the points of application of the forces being one of the generating straight lines of the surface; and the equilibrium-axis is the imaginary axis of the surface. And hence we conclude that into a system of forces which is not in equilibrium two forces may be introduced, so that the system thus modified may be in equilibrium and may also have an equilibrium-axis; and the points of application of these two forces may be at such points on the surface of a certain hyperboloid of one sheet, that the line joining them lies wholly in the surface; and when these points of application are given the lines of action of the forces are also determined.

Although I have applied to the theory of the equilibrium-axis only the geometrical changes of x, y and z, given in Art. 94, equations (235), yet they are of much wider application, and will hereafter be largely used.

SECTION 7.—*Stability and Instability of Equilibrium.*

99.] The investigations of the preceding section, as also those of Art. 63, are of great importance in determining a delicate question, viz. the character of equilibrium of an equilibrium-system. For if a body is at rest under the action of many forces, and receives a small displacement of the most general kind, but of such an infinitesimal amount that the forces, when applied at the same points as before, act in the same directions along lines parallel to, and infinitesimally distant from, their former lines of action; then the body in its new position will generally not be in equilibrium; and the acting forces may tend either to bring it back to its former position or to remove it farther from it; if the former is the character of the forces the equilibrium is said to be stable; and if the latter the equilibrium is said to be unstable. A heavy homogeneous sphere resting in a hollow bowl, a heavy oblate spheroid resting on a horizontal plane with its axis vertical, a heavy weight suspended as a pendulum and at rest, a loaded wheel with the load in the lowest possible position, are all cases of stable equilibrium. On the other hand, a loaded ball with its load as high as possible and resting on a horizontal plane, an egg balanced on the smaller end, a heavy beam resting on two inclined planes, a heavy ball balanced on the highest point of a sphere, are all instances of unstable equilibrium. If, however, the body in its displaced state is in a position of equilibrium, it may be so either for the displacement which it has actually undergone and for no other near to it, in which case the equilibrium is said to be neutral; or it may be in equilibrium for this and all other infinitesimal displacements, and then the equilibrium is said to be continuous. A heavy homogeneous cylinder having its ends perpendicular to the axis resting on a horizontal plane with its axis horizontal, and a heavy homogeneous circular cone having its base perpendicular to its axis resting with its slant side on a horizontal plane, are instances of neutral equilibrium; a heavy homogeneous sphere resting on a horizontal plane is an instance of continuous equilibrium.

100.] Now the most general displacement which a body can undergo always consists of a displacement of translation, and of a displacement of rotation about a determinate axis. In Art. 94 it has been shewn that if a body is at rest under the action of given forces, it is also at rest when it has undergone a displacement of translation, the paths described by every particle of the body being equal and parallel, the forces being applied at the same points as before, in the same direction, and along action-lines parallel to, and infinitesimally distant from, the former action-lines. Thus we have to consider only the effects of an infinitesimal displacement of rotation about a certain determinate axis. Let the direction-angles of the axis of rotation be f, g, h; and let $d\theta$ be the infinitesimal angle through which the body is turned about that axis; then the changes in the coordinates of the point (x, y, z), which are due to this infinitesimal displacement of rotation, are those which are given in Art. 94; and we have

$$\left.\begin{aligned} dx &= (z\cos g - y\cos h)\,d\theta, \\ dy &= (x\cos h - z\cos f)\,d\theta, \\ dz &= (y\cos f - x\cos g)\,d\theta. \end{aligned}\right\} \tag{265}$$

If, however, all the action-lines of the forces are in one plane, say, in the plane of (x, y), and the rotation-axis is perpendicular to that plane, then

$$dx = -y\,d\theta, \qquad dy = x\,d\theta. \tag{266}$$

In reference to equilibrium-axes it is evident that if a body in equilibrium under the action of certain forces has no equilibrium-axis, its equilibrium is either stable or unstable; if it has one or two equilibrium-axes which meet, its equilibrium is neutral, when the displacement of rotation takes place about one of them; and if the system of forces is such that every axis is an equilibrium-axis, then the equilibrium is continuous.

101.] In application of this theory I will first take the most simple case of a body held in equilibrium under the action of two forces only: these of course are equal to each other, and act along the same line, and in opposite directions: but these conditions may be satisfied in two ways: the forces may act to draw their points of application either nearer to, or farther from, each other. Let P_1, P_2, see fig. 66, be the two forces; A_1, A_2 their respective points of application. Let the body receive an infinitesimal displacement of rotation about an axis perpendicular to the line of action of the forces: so that the line $A_1 A_2$, which before the displacement was in the same line with the line of action of the forces, is now in one of the positions, relatively to them, indicated in the figures (α) and (β): (α) is evidently the state in which the forces applied at A_1 and A_2 tend to bring the points nearer to each other; and in which, now that the displacement has taken place, the action of the forces tends to remove the system farther and farther from its original position, and in which therefore the original equilibrium was unstable: (β) is the state in which the forces act to separate their points of application, and in which the forces act after the displacement to bring the body back to its original position; and in which therefore the equilibrium is stable. If the two forces act at the same point, equilibrium is continuous for every displacement of the body about an axis perpendicular to the line of action of the forces; and also because the point at which they act is their *centre*.

The following analytical investigation supplies a criterion of these several states of equilibrium. Let (x_1, y_1) (x_2, y_2) be the points of application of P_1 and of P_2 respectively; then the conditions of equilibrium of these two forces are

$$P_1 + P_2 = 0,$$

$$G = \sin a\, \Sigma.Px - \cos a\, \Sigma.Py = 0. \tag{267}$$

Let the body be turned about an axis perpendicular to the plane of (x, y) through an angle $d\theta$; then the forces, their points of application in the body, and directions being unchanged, and their lines of action being parallel and infinitesimally near to the former action-lines, G varies; and the change of it which is due to the displacement is the moment of the couple which acts on the body in its displaced state. Now the displacement involves a change of x and y, and we have

$$\begin{aligned} dG &= \sin a\, \Sigma.P\, dx - \cos a\, \Sigma.P\, dy, \\ &= -\{\sin a\, \Sigma.Py + \cos a\, \Sigma.Px\}\, d\theta; \end{aligned} \tag{268}$$

but according as $\frac{dG}{d\theta}$ is positive or negative, so does the couple brought into action by the displacement tend to remove the body further from, or to bring it back nearer to, the original place of equilibrium: that is, so is the equilibrium of the body unstable or stable. And consequently the equilibrium is stable or unstable, according as

$$\Sigma.Px \cos a + \Sigma.Py \sin a \tag{269}$$

is positive or negative.

And because a is the same for both the forces, and is also generally indeterminate, since the directions of the axes are arbitrary, the criterion (269) reduces itself to either $\Sigma.Px$ or $\Sigma.Py$, and thus the stability depends on the sign of either of these.

If (269) = 0, then, since $P_1+P_2=0$, $x_1=x_2=0$, $y_1=y_2=0$; that is, the forces are applied at the same point, viz. the origin, and the equilibrium is continuous.

The rotation has taken place about an axis perpendicular to the line of action of the forces. I would only further observe, that if it takes place about the line of action of the two forces, their points of application undergo no displacement, and no criterion of stability is obtained.

102.] The process of the preceding article is also generally applicable to the determination of the criterion of the stability and instability of forces all the action-lines of which are in the plane of (x, y). Let the forces and their several incidents be denoted by the same symbols as heretofore. Then for the equilibrium of the system we have

$$X = \Sigma.P\cos\alpha = 0, \qquad Y = \Sigma.P\sin\alpha = 0; \tag{270}$$

$$G = \Sigma.P(x\sin\alpha - y\cos\alpha). \tag{271}$$

Let the body on which the forces act undergo an infinitesimal displacement of rotation through $d\theta$ about an axis perpendicular to the plane of the forces; then

$$\begin{aligned} dG &= \Sigma.P(\sin\alpha\, dx - \cos\alpha\, dy) \\ &= -\Sigma.P(y\sin\alpha + x\cos\alpha)\, d\theta; \end{aligned} \tag{272}$$

and consequently the effect of the couple brought into action by the displacement is to remove the body further from, or to bring it back into, its former state, according as $\Sigma.P(x\cos\alpha + y\sin\alpha)$ is positive or negative; but this quantity is the radial moment, see Art. 63; consequently the equilibrium is stable or unstable according as the radial moment is positive or negative. If the radial moment vanishes, then the system has a centre, and an equilibrium-axis perpendicular to the plane of the forces, so that the body is in equilibrium in its displaced state, as also in its former state, and the equilibrium is neutral or continuous. Hence we have the following theorem:

Of a system of forces acting on a rigid body in a plane, and being in equilibrium, the equilibrium is stable, neutral, or unstable, according as $\Sigma.P(x\cos\alpha + y\sin\alpha)$, that is, the radial moment, is positive, zero, or negative.

The preceding criterion is true only for a displacement of the body about an axis perpendicular to the plane in which the forces act; for let us suppose four forces to act on a body in one plane and to be in equilibrium; and suppose them to be such that a pair of them is in equilibrium; and that therefore the other pair also equilibrates; let the body be turned about an axis coinciding with the line of action of the latter pair, the equilibrium of the other pair may evidently be either stable or unstable: and if the rotation takes place about the line of action of the former pair, the equilibrium of the latter pair may be either stable or unstable; and evidently there is no necessity that it should be of the same character as the other; hence in this case we are unable to determine *a priori* the axes of stable or of unstable equilibrium.

And the preceding test is applicable to the case of forces whose lines of action are parallel to a given plane when the displacement takes place about a line perpendicular to that plane.

103.] We can also hence derive another remarkable criterion of the stability and instability of an equilibrium-system. Let the radial moment, as in Art. 63, be denoted by H, so that

$$\mathrm{H} = \Sigma.\mathrm{P}\,(x\cos\alpha + y\sin\alpha); \tag{273}$$

$$\begin{aligned} \therefore\quad d\mathrm{H} &= \Sigma.\mathrm{P}\,(dx\cos\alpha + dy\sin\alpha), \\ &= \Sigma.\mathrm{P}\,(x\sin\alpha - y\cos\alpha)\,d\theta, \qquad (274) \\ &= \mathrm{G}\,d\theta = 0, \end{aligned}$$

since the system is in equilibrium and consequently $\mathrm{G} = 0$. Hence in an equilibrium-system the radial moment has a critical value, and is a maximum, a minimum, or a constant, zero being a particular value of the constant. To determine the character of this critical value, we differentiate again, and we have

$$\begin{aligned} \frac{d^2\mathrm{H}}{d\theta^2} = \frac{d\mathrm{G}}{d\theta} &= \Sigma.\mathrm{P}\left(\frac{dx}{d\theta}\sin\alpha - \frac{dy}{d\theta}\cos\alpha\right), \\ &= -\Sigma.\mathrm{P}\,(x\cos\alpha + y\sin\alpha), \\ &= -\mathrm{H}; \qquad (275) \end{aligned}$$

so that H has a maximum or minimum value according as it is positive or negative; but according as H, which is equal to $-\frac{d\mathrm{G}}{d\theta}$, is positive or negative so is the equilibrium stable or unstable; consequently we have the following criteria as to the character of equilibrium of a system of forces.

The equilibrium is stable or unstable according as H is a maximum or minimum; or according as H is positive or negative.

If $\mathrm{H} = 0$, the system has an equilibrium-axis, and the equilibrium is neutral.

If the action-lines of all the forces are parallel, let us take a line parallel to them for the axis of y; so that in this case $\alpha = 90°$, and

$$\text{H} = \Sigma.\text{P}\,y, \tag{276}$$

and equilibrium is stable or unstable according as this quantity is a maximum or a minimum. We shall hereafter have many applications of this equation.

104.] The following are examples in which the preceding criteria of stability are applied:

Ex. 1. When a heavy uniform beam rests on two inclined planes, is the equilibrium stable or unstable?

This is the case which is discussed in Ex. 2, Art. 60; and I will take the notation therein employed, and C for the origin, and the horizontal line through C for the axis of x. Then if $\text{CB} = r'$, $\text{CA} = r$, and as the forces are R, R′, W,

$$\begin{aligned}\text{H} &= \Sigma.\text{P}\,(x\cos\alpha + y\sin\alpha)\\ &= -\text{R}'r'\sin\beta\cos\beta - \text{R}r\sin\alpha\cos\alpha\\ &\quad + \text{R}'r'\sin\beta\cos\beta + \text{R}r\sin\alpha\cos\alpha - \text{W}(r'\sin\beta - a\sin\theta)\\ &= -\frac{\text{W}a}{\sin(\alpha+\beta)}\{2\sin\alpha\sin\beta\cos\theta + \sin(\alpha-\beta)\sin\theta\};\end{aligned}$$

consequently H is a negative quantity, and the equilibrium is unstable.

Also as the beam is at rest $\frac{d\text{H}}{d\theta} = 0$, and thus

$$\tan\theta = -\frac{\sin(\alpha-\beta)}{2\sin\alpha\sin\beta}.$$

Also $\frac{d^2\text{H}}{d\theta^2}$ is positive, so that the value of H is a minimum.

Ex. 2. If a heavy beam rests against a smooth wall, and has the other end fastened by a string to a given point in the wall, as in Ex. 3, Art. 60, what is the character of equilibrium?

Let us take the symbols which are given in Art. 60, and take C to be the origin, fig. 30, and the horizontal line drawn through it to be the x-axis, the y-axis being taken downwards. Then

$$\begin{aligned}\text{H} &= \Sigma.\text{P}\,(x\cos\alpha + y\sin\alpha)\\ &= \text{W}(x + a\cos\theta) - \text{T}\,b;\end{aligned}$$

and substituting in this equation the values given in Ex. 3, Art. 60, we have

$$\text{H} = -\frac{2a^2\text{W}3^{\frac{1}{2}}}{(b^2-4a^2)^{\frac{1}{2}}}.$$

Thus H is a negative quantity, and the equilibrium is unstable.

Ex. 3. What is the character of equilibrium in the problem of Ex. 6, Art. 60?

Let S be the origin; then

$$\begin{aligned} H &= -Rr\cos\frac{\theta}{2} + W\cos\theta\,(r-c) \\ &= -Wc\cos\theta \\ &= W(c-a), \end{aligned}$$

and this is positive or negative according as c is greater or less than a; hence the equilibrium is stable or unstable according as c is greater or less than a.

Ex. 4. Two heavy particles connected by a string support each other on the circumference of a circle in a vertical plane. Determine the nature of the equilibrium.

Let the weights of the particles be P and Q, and let the radii of the circle drawn to the points where P and Q rest make angles θ and ϕ with the vertical. Let the string subtend an angle $= \alpha$ at the centre, so that $\theta+\phi = \alpha$: then, if the origin is taken at the centre,

$$\begin{aligned} H &= -aP\cos\theta - aQ\cos\phi; \\ dH &= aP\sin\theta\, d\theta + aQ\sin\phi\, d\phi \\ &= a\{P\sin\theta - Q\sin\phi\}\, d\theta = 0, \\ \text{if} \quad & \frac{\sin\theta}{Q} = \frac{\sin\phi}{P}; \end{aligned}$$

consequently H is negative, and the equilibrium is unstable.

105.] In the case of a rigid body in equilibrium under the action of many forces acting along lines of action in space, we have to consider only the effects of a displacement of rotation, as to the kind of equilibrium which the body is in.

Let the direction-angles of the axis of rotation be f, g, h; and let the moment-axes of the impressed couples along the three coordinate axes be L, M, N; then, if G is the moment of the couple tending to turn the body about the rotation-axis, by reason of the law of resolution of couples,

$$\begin{aligned} G &= L\cos f + M\cos g + N\cos h \\ &= \cos f\,\Sigma.P\,(y\cos\gamma - z\cos\beta) + \ldots + \ldots; \qquad (277) \end{aligned}$$

$$\begin{aligned} \frac{dG}{d\theta} &= \cos f\,\Sigma.P\Big(\cos\gamma\frac{dy}{d\theta} - \cos\beta\frac{dz}{d\theta}\Big) + \ldots + \ldots \\ &= -(\cos f)^2\,\Sigma.P(y\cos\beta + z\cos\gamma) + \cos g\cos h\,\Sigma.P\,(y\cos\gamma + z\cos\beta) \\ &\quad -(\cos g)^2\,\Sigma.P(z\cos\gamma + x\cos\alpha) + \cos h\cos f\,\Sigma.P\,(z\cos\alpha + x\cos\gamma) \\ &\quad -(\cos h)^2\,\Sigma.P(x\cos\alpha + y\cos\beta) + \cos f\cos g\,\Sigma.P(x\cos\beta + y\cos\alpha); \end{aligned}$$

and employing the abbreviating notation of Art. 94,

$$\frac{d\mathrm{G}}{d\theta} = -\mathrm{U}(\cos f)^2 - \mathrm{V}(\cos g)^2 - \mathrm{W}(\cos h)^2 + 2\mathrm{D}\cos g\cos h + 2\mathrm{E}\cos h\cos f + 2\mathrm{F}\cos f\cos g; \tag{278}$$

and since the effect of G due to a small variation of θ is to bring back the system to its former position, or to remove it farther therefrom, according as $\frac{d\mathrm{G}}{d\theta}$ is negative or positive, so is the equilibrium stable or unstable according as the right-hand member of (278) is negative or positive.

For convenience of reference let us denote this quantity by S; so that

$$\mathrm{S} = -\mathrm{U}(\cos f)^2 - \mathrm{V}(\cos g)^2 - \mathrm{W}(\cos h)^2 + 2\mathrm{D}\cos g\cos h + 2\mathrm{E}\cos h\cos f + 2\mathrm{F}\cos g;$$

then equilibrium is stable or unstable, according as S is negative or positive: and the sign evidently depends, not only on the impressed forces and their incidents, but also on the direction-angles of the rotation-axis; and therefore an equilibrium-system may be stable for one rotation-axis, unstable for another, and neutral for a third; that is, in the third case the system may have an equilibrium-axis, and S may be equal to zero.

For suppose that S is arranged in the form

$$\{-\mathrm{U}\cos f + \mathrm{F}\cos g + \mathrm{E}\cos h\}\cos f + \{\mathrm{F}\cos f - \mathrm{V}\cos g + \mathrm{D}\cos h\}\cos g + \{\mathrm{E}\cos f + \mathrm{D}\cos g - \mathrm{W}\cos h\}\cos h,$$

and that we have also

$$\left.\begin{aligned} -\mathrm{U}\cos f + \mathrm{F}\cos g + \mathrm{E}\cos h &= 0, \\ \mathrm{F}\cos f - \mathrm{V}\cos g + \mathrm{D}\cos h &= 0, \\ \mathrm{E}\cos f + \mathrm{D}\cos g - \mathrm{W}\cos h &= 0; \end{aligned}\right\}$$

so that $\mathrm{UVW} - \mathrm{D}^2\mathrm{U} - \mathrm{E}^2\mathrm{V} - \mathrm{F}^2\mathrm{W} - 2\mathrm{DEF} = 0;$

then this is the condition requisite for the existence of an equilibrium-axis; and in this case $\mathrm{S} = 0$, and the equilibrium is neutral.

If also, according to Art. 96, equations (251),

$$\mathrm{D}^2 = \mathrm{VW}, \quad \mathrm{E}^2 = \mathrm{WU}, \quad \mathrm{F}^2 = \mathrm{UV},$$

and if the axis about which the rotation takes place is parallel to the plane whose equation is

$$\mathrm{U}\ x + \mathrm{V}^{\frac{1}{2}}y + \mathrm{W}^{\frac{1}{2}}z = 0,$$

then equilibrium is neutral for all such axes; and is continuous, if the change of axis is from any one line to any other line lying in the plane.

And if in addition, $$\left.\begin{array}{l} D = E = F = 0, \\ U = V = W = 0, \end{array}\right\}$$
so that any axis about which the body is turned is an equilibrium-axis, then the equilibrium is continuous for all axes.

I may also observe that, if the directions of action of all the forces are reversed, the signs of U, V, W, D, E, F are changed, and therefore the sign of S is changed; and thus the nature of the equilibrium is changed: in the case, however, of neutral equilibrium no alteration takes place.

106.] And S admits of the following geometrical interpretation: on the straight line drawn through the origin, and whose direction-angles are f, g, h, let a point (x, y, z) be taken: then x, y, z are proportional to $\cos f$, $\cos g$, $\cos h$, and S becomes proportional to
$$-Ux^2 - Vy^2 - Wz^2 + 2Dyz + 2Ezx + 2Fxy, \qquad (279)$$
which, when equated to zero, is the equation to a cone of the second degree; and therefore for all lines passing through the origin, and lying within this cone, and employed as rotation-axes, the above expression has a different sign to that which it has for all lines lying outside of the cone; and for all lines on the surface of the cone it vanishes; so that for all the generating lines of the cone, equilibrium is neutral; and the cone divides space into two parts such, that for all axes within its surface, the equilibrium is the opposite to that which it is for axes outside the surface.

I may, however, observe that, if lines are drawn through the vertex of the cone, and if these are called *interior* or *exterior* lines according as from points on them real tangent planes cannot, or can, be drawn to the cone; then will interior lines be axes of stable, and exterior lines axes of unstable, equilibrium, if
$$UVW - D^2U - E^2V - F^2W - 2DEF = V \text{ (say)}$$
is positive; and if V is negative, the converse is the case.

If $V = 0$, we have the following circumstances. If we reduce the expression (279) so as to deprive it of the terms containing the products of the variables, we obtain the discriminating cubic, of which the constant term is V; and therefore if $V = 0$, one of the roots of this cubic is zero, and the reduced equation becomes of the form $$U'x^2 \pm V'y^2 = 0,$$

which, if the upper sign is taken, represents the axis of z; and, if the lower sign is taken, two planes perpendicular to the plane of (x, y). In the former of these two cases the axis of z is an axis of neutral equilibrium, and other lines are axes either all of unstable, or all of stable, equilibrium: in the latter case, any line in either of the planes is an axis of neutral equilibrium, and the other lines are either all axes of stable, or all of unstable, equilibrium.

One or two special forms of (277) require notice: if the z-axis is the rotation-axis, the condition requires that

$$\mathrm{W} = \Sigma.\mathrm{P}\,(x\cos\alpha + y\cos\beta)$$

should be positive for stable, and negative for unstable, equilibrium: which is the same result as that of Art. 102.

And if all the forces are parallel to the axis of z, so that $\cos\alpha = \cos\beta = 0$, $\cos\gamma = 1$, then

$$\frac{d\mathrm{G}}{d\theta} = -(\sin h)^2\,\Sigma.\mathrm{P}\,z + \cos h\,\{\cos f\,\Sigma.\mathrm{P}\,x + \cos g\,\Sigma.\mathrm{P}\,y\}; \qquad (280)$$

and if the axis about which the infinitesimal rotation takes place is at right angles to the lines of action of the forces, then $h = 90°$, and we have

$$\frac{d\mathrm{G}}{d\theta} = -\Sigma.\mathrm{P}\,z; \qquad (281)$$

and therefore equilibrium is stable or unstable according as $\Sigma.\mathrm{P}\,z$ is positive or negative.

Now on referring to Art. 80, (146), it appears that if $(\bar{x}, \bar{y}, \bar{z})$ is the centre of a system of parallel forces, $\bar{z}\,\Sigma.\mathrm{P} = \Sigma.\mathrm{P}\,z$; consequently the equilibrium is stable or unstable according as $\bar{z}$ is positive or negative. In the following Chapter we shall have many illustrations of this theorem.

107.] The condition for the stability of equilibrium of a system of forces acting in space may be expressed in a form similar to that of Art. 103 by the following process:

Let the infinitesimal rotation take place about an axis whose direction-angles are f, g, h; so that, as the moment-axes of the couples, whose rotation-axes are the coordinate-axes, are L, M, N, for equilibrium we have

$$\mathrm{L}\cos f + \mathrm{M}\cos g + \mathrm{N}\cos h = 0;$$

and thus, replacing L, M, N by their values, and introducing $d\theta$,

$$\Sigma.\mathrm{P}\,\{\cos\alpha\,(z\cos g - y\cos h) + \cos\beta\,(x\cos h - z\cos f) \\ + \cos\gamma\,(y\cos f - x\cos\gamma)\}\,d\theta = 0;$$

and by means of (265),

$$\Sigma.\mathrm{P}.\,\{\cos\alpha\,dx + \cos\beta\,dy + \cos\gamma\,dz\} = 0 = d\mathrm{H}\ (\text{say});$$

therefore by integration

$$\text{H} = \Sigma.\text{P}\,(x\cos\alpha + y\cos\beta + z\cos\gamma); \qquad (282)$$

and therefore H is a maximum, a minimum, or a constant. And since, see equation (277),

$$\frac{d\text{H}}{d\theta} = \text{G} = 0;$$

$$\therefore \quad \frac{d^2\text{H}}{d\theta^2} = \frac{d\text{G}}{d\theta}$$

$$= \text{S}, \qquad (283)$$

see equation (278); therefore $\text{H} = \Sigma.\text{P}\,(x\cos\alpha + y\cos\beta + z\cos\gamma)$ is a maximum or minimum, according as S is negative or positive, that is, according as equilibrium is stable or unstable.

Now S, as given in (278), admits of being put into the form,

$$\text{S} = \Sigma.\text{P}\,\{(x\cos f + y\cos g + z\cos h)(\cos\alpha\cos f + \cos\beta\cos g + \cos\gamma\cos h)\}$$
$$-\Sigma.\text{P}\,(x\cos\alpha + y\cos\beta + z\cos\gamma); \qquad (284)$$

and as for a given rotation-axis $x\cos f + y\cos g + z\cos h$ is the projection on the axis of rotation of the distance from the origin of the point of application of the force P, and

$$\text{P}\,(\cos\alpha\cos f + \cos\beta\cos g + \cos\gamma\cos h)$$

is the resolved part of P, along the rotation-axis; and as both these quantities are constant for a given-rotation-axis, and independent of the rotation; the value of S can only change by means of the last term in the right-hand member of (284): but this term is H; hence equilibrium is stable or unstable according as H is greater than or less than

$$\Sigma.\text{P}\,\{(x\cos f + y\cos g + z\cos h)(\cos\alpha\cos f + \cos\beta\cos g + \cos\gamma\cos h)\};$$

and if $\text{S} = 0$, equilibrium is either neutral or continuous.

In Art. 60, the forces have been resolved along, and perpendicular to, the radius vector of the point of application; and

$$\Sigma.\text{P}\,(x\cos\alpha + y\sin\alpha)$$

has been called the radial moment of the system, because it is the product of the radius vector of the point of application, and of the radial component. Similarly in space, if we resolve P along the radius vector of its point of application, and call U its radial component,

$$\text{U} = \frac{\text{P}\,(x\cos\alpha + y\cos\beta + z\cos\gamma)}{r},$$

where r is the radius vector of the point of application of P: therefore

$$\text{H} = \Sigma.\text{P}\,(x\cos\alpha + y\cos\beta + z\cos\gamma) \text{ [illegible]} \qquad (285)$$

and H is called *the radial moment of the system.* Hence we have the following theorem:

The equilibrium of a system of forces is stable or unstable according as the radial moment is a maximum or a minimum.

The radial moment also possesses the following two other properties. Let us suppose the body or system of particles on which the forces act to receive a small displacement, and all the forces to act at their points of application, along lines of action parallel to the former ones, and in the same directions. Then if the motion of the body is constrained in translation along a given line, and ds is the space described along that line, $\frac{dH}{ds}$ is the sum of the components of the forces estimated along that line; and if the motion is one of rotation about a given axis, and θ is the amplitude of rotation, then $\frac{dH}{d\theta}$ is, in any position, the moment-axis of the couple arising out of the system of forces about that axis.

SECTION 8.—*The principle of Virtual Velocities.*

108.] Let a body, or a system of material particles on which an equilibrium-system of forces acts, receive the most general infinitesimal geometrical displacement that is possible, so that the forces may act at the same points as before the displacement, along lines parallel to, and infinitesimally distant from, the original action-lines, and in the same directions. Let ξ, η, ζ be the infinitesimal distances along the coordinate-axes through which the body is displaced, and let f, g, h be the direction-angles of the rotation-axis about which the body is turned through the angle $d\theta$. Then all these quantities being arbitrary, the total displacement is of the most general kind.

Let us employ the symbol δ to signify this most general displacement; so that d signifies a particular form of it, viz. that in which the change of value is restricted to given conditions. Then $\delta x, \delta y, \delta z$ being the variations of x, y, z, which are the coordinates of any point in the original system, due to these displacements,

$$\left.\begin{aligned} \delta x &= \xi + (z \cos g - y \cos h)\, d\theta, \\ \delta y &= \eta + (x \cos h - z \cos f)\, d\theta, \\ \delta z &= \zeta + (y \cos f - x \cos g)\, d\theta. \end{aligned}\right\} \qquad (286)$$

As the system of forces is in equilibrium, we have the following six conditions:

$$\Sigma.\mathrm{P}\cos\alpha = 0, \qquad \Sigma.\mathrm{P}\cos\beta = 0, \qquad \Sigma.\mathrm{P}\cos\gamma = 0,$$

$$\Sigma.\mathrm{P}(y\cos\gamma - z\cos\beta) = 0,$$

$$\Sigma.\mathrm{P}(z\cos\alpha - x\cos\gamma) = 0,$$

$$\Sigma.\mathrm{P}(x\cos\beta - y\cos\alpha) = 0;$$

let these be severally multiplied by ξ, η, ζ, $\cos f\,d\theta$, $\cos g\,d\theta$, $\cos h\,d\theta$, and added; then we have

$$\Sigma.\mathrm{P}\{\xi\cos\alpha + \eta\cos\beta + \zeta\cos\gamma$$
$$(z\cos g - y\cos h)\cos\alpha\,d\theta + (x\cos h - z\cos f)\cos\beta\,d\theta + (y\cos f - x\cos g)\cos f\,d\theta\} = 0;$$

and by reason of (286) this becomes

$$\Sigma.\mathrm{P}(\cos\alpha\,\delta x + \cos\beta\,\delta y + \cos\gamma\,\delta z) = 0. \qquad (287)$$

Now as δx, δy, δz are the projections on the coordinate-axes of the displacement of (x, y, z), which is the point of application of of P, and as α, β, γ are the direction-angles of the action-line of P, $\cos\alpha\,\delta x + \cos\beta\,\delta y + \cos\gamma\,\delta z$ is the projection of the displacement along the action-line of P. Let this projected displacement $= \delta p$; then (287) becomes

$$\Sigma.\mathrm{P}\,\delta p = 0. \qquad (288)$$

This equation expresses a theorem which is known as the *Principle of Virtual Velocities*, and which may be enuntiated as follows:

If a system of forces acting on a rigid body, or on a system of material particles which are at relative rest, is in equilibrium, and the body receives an infinitesimal displacement of the most general kind possible, whereby the points of application of the forces are displaced; but the forces act along lines parallel to, and infinitesimally distant from, their former lines of action; then the sum of the products of each force and the projection on its line of action of the displacement of its point of application, is equal to zero.

The projection on the line of action of a force of the infinitesimal displacement of its point of application is called *the virtual velocity* of the force: and as that projection may take place along the line either in the direction of the force or in the opposite direction, so it is in these alternative cases to be affected with a different sign. I shall take the virtual velocity to be positive when the projection on the action-line of P is in the direction in which the force acts. Thus in fig. 140, let AP be the line of action of P, ere the displacement takes place: let the system be infinitesimally displaced, so that the point of application of the force is shifted from A to A′; AA′ being of infinitesimal length; let us suppose the line of action of the force after the displacement to be parallel to its line of action before the displacement, so that A′P′ is parallel to AP. From A′ let a perpendicular A′M be drawn to the original line of action of the force, so that AM is the orthogonal projection of AA′ on that line: AM is called the virtual velocity of the force P; and is the infinitesimal distance, over which the point of application of P moves, in its own line of action. If, as in the first figure of fig. 140, AM lies along AP in the direction in which P acts, the virtual velocity is taken to be positive: and if it lies in the direction of AP produced backwards, as in the second figure, then it is taken to be negative.

Hence, if the displacement of the point of application takes place along the line of action of P, the whole displacement becomes the virtual velocity: and is positive or negative according as it takes place in the direction towards which P acts, or in the opposite direction.

Hence also, if the point of application of the force is displaced in a line which is perpendicular to the line of action of the force, the virtual velocity of the force is zero.

The quantity $P\delta p$ is frequently called the *virtual moment* of the force P in any assigned displacement. The importance and meaning of this quantity in a Dynamical respect will be seen hereafter.

This principle of virtual velocities is of the greatest importance. It includes all Statics under the single equation (288), for as δp in its most general form involves six arbitrary quantities which correspond to the six possible degrees of freedom, so it comprehends six conditions, which are the six equations of equilibrium, and which may be deduced from it by a process the reverse of the preceding. It also includes all Dynamics, as we shall see hereafter; and we shall also see that the equation of it may be deduced from Dynamical principles, and may be independent of the parallelogram of forces, by means of which we have now proved it.

This principle has been made by Lagrange the foundation of that great work of his on Mechanics, Mecanique Analytique.

Also, if every force at its point of application is resolved into three forces of which the action-lines are parallel to the axes of x, y, z respectively, and if we call X, Y, Z the axial components of the force P, then the equation of virtual velocities takes the form

$$\Sigma.(\mathrm{X}\,\delta x + \mathrm{Y}\,\delta y + \mathrm{Z}\,\delta z) = 0. \qquad (289)$$

In connection with the theory of stability of equilibrium and of the radial moment, which have been discussed in the preceding section, it will be observed that as $\mathrm{H} = \Sigma.\mathrm{P}(x\cos\alpha + y\cos\beta + z\cos\gamma)$, so the principle of virtual velocities as given in (287) expresses, that consistently with the most general variations of x, y, z, $d\mathrm{H} = 0$; and that consequently in an equilibrium-system the radial moment has a critical value. This is indeed no more than what is expressed by (288).

109.] The following are various problems which are solved by the principle of virtual velocities.

Ex. 1. Three forces P, Q, R act in given lines at the point A, and are in equilibrium: it is required to determine the relation between them.

Let the angles severally between the lines of action of Q and R, of R and P, of P and Q, be α, β, γ: let the point of application of the forces be shifted from A to A$'$, see fig. 141; and from A$'$ let perpendiculars A$'m$, A$'n$, A$'p$ be drawn to the lines of action of P, Q, R respectively; then Am, An, Ap are the virtual velocities of P, Q, R respectively: so that (288) becomes

$$\mathrm{P} \times \mathrm{A}m + \mathrm{Q} \times \mathrm{A}n - \mathrm{R} \times \mathrm{A}p = 0.$$

Let $\mathrm{AA}' = \delta s$; $\mathrm{A'AP} = \theta$; $\mathrm{QAR} = \alpha$, $\mathrm{RAP} = \beta$, $\mathrm{PAQ} = \gamma$: so that this equation becomes

$$\mathrm{P}\,\delta s\cos\theta + \mathrm{Q}\,\delta s\cos(\theta+\gamma) + \mathrm{R}\,\delta s\cos(\beta-\theta) = 0;$$

$$\therefore \quad \mathrm{P} + \mathrm{Q}\cos\gamma + \mathrm{R}\cos\beta + (\mathrm{R}\sin\beta - \mathrm{Q}\sin\gamma)\tan\theta = 0;$$

and as the line along which A is displaced is indeterminate, θ is indeterminate, and therefore

$$\mathrm{P} + \mathrm{Q}\cos\gamma + \mathrm{R}\cos\beta = 0,$$

$$\mathrm{R}\sin\beta - \mathrm{Q}\sin\gamma = 0:$$

from the latter we have

$$\frac{\mathrm{P}}{\sin\alpha} = \frac{\mathrm{Q}}{\sin\beta} = \frac{\mathrm{R}}{\sin\gamma};$$

the first term of the equality being inferred by reason of the symmetry. Also we have

$$\mathrm{R}\cos\beta = -\mathrm{P} - \mathrm{Q}\cos\gamma, \qquad \mathrm{R}\sin\beta = \mathrm{Q}\sin\gamma;$$

whence, squaring and adding,

$$R^2 = P^2 + 2PQ \cos\gamma + Q^2:$$

these are respectively the mathematical expressions of the triangle and of the parallelogram of forces.

Ex. 2. To determine the conditions of equilibrium of the straight lever.

Let ACB be the lever, fig. 142, which turns about a horizontal axis through C: let the forces P and Q act at the ends A and B along lines of action which are inclined to ACB at angles α and β respectively: let $AC = a$, $CB = b$.

Let the lever be turned about the horizontal axis through an infinitesimal angle $d\theta$, so that $AA' = a\,d\theta$, $BB' = b\,d\theta$: then the projections of these quantities on the lines of action of P and Q respectively are $a\,d\theta \sin\alpha$, $b\,d\theta \sin\beta$; and as the virtual velocity of Q is negative, (288) becomes

$$Pa\,d\theta \sin\alpha - Qb\,d\theta \sin\beta = 0;$$

$$\therefore \quad Pa \sin\alpha = Qb \sin\beta:$$

which is the ordinary equation of moments about C.

Ex. 3. To determine the conditions of equilibrium of the wheel and axle.

Let a = the radius of the wheel on which P acts: b = the radius of the axle on which W acts: and let the system be turned through a small angle $d\theta$, so that P (say) descends through a vertical distance $a\,d\theta$, and W ascends through a vertical space $b\,d\theta$: then (288) becomes

$$-a\,d\theta\,P + b\,d\theta\,W = 0; \qquad \therefore \quad Pa = Wb.$$

Ex. 4. To find the conditions of equilibrium in the screw.

In this mechanical power, as it is called, I shall assume that there is no friction. Let h be the vertical distance between two successive winds of the thread: let l be the length of the lever, measured from the axis of the screw, at the end of which P acts: let W be the weight on the screw. Then as W descends through a vertical distance equal to h, the point of application of P moves round the circumference of a circle whose radius is b: so that h and $2\pi b$ are evidently proportional to the virtual velocities of W and P; and equation (288) becomes

$$-2\pi b P + Wh = 0;$$

$$\therefore \quad P = \frac{h}{2\pi b} W.$$

Ex. 5. To determine the condition of equilibrium of a heavy body resting on an inclined plane under the action of given forces.

In applying the principle of virtual velocities to problems wherein some of the forces are pressures against lines or surfaces, the reactions will not enter into the equation, if the displacement of the point of application of the reaction is perpendicular to its line of action, because in that case the virtual velocity vanishes. Hence also if one surface rolls on another, and the resulting displacement is the arbitrary displacement out of which the virtual velocity arises, the mutual reaction of the surface does not appear in the equation of virtual velocities. Several instances of this circumstance will be given in this and the following examples.

In this example let us take the symbols, &c., of Ex. 1, Art. 26, fig. 12. Let Q be shifted over a distance δs up the plane; then the virtual velocity of P is $\delta s \cos\beta$, that of $W = -\delta s \sin\alpha$, and that of $R = 0$; so that

$$P\delta s\cos\beta - W\delta s\sin\alpha = 0;$$

$$\therefore \quad P\cos\beta - W\sin\alpha = 0.$$

Ex. 6. Solve by virtual velocities the problem given in Ex. 1, Art. 60.

Let the system as described in fig. 28 be shifted so that A and B may still be in contact with the horizontal and vertical planes respectively; and let $\alpha = BAC$ be diminished by $\delta\alpha$; then the virtual velocity of $T = \delta.2a\cos\alpha = -2a\sin\alpha\,\delta\alpha$, and that of $W = \delta.a\sin\alpha = a\cos\alpha\,\delta\alpha$; and those of the reactions vanish: so that

$$-T2a\sin\alpha\,\delta\alpha + Wa\cos\alpha\,\delta\alpha = 0;$$

$$\therefore \quad 2T\sin\alpha - W\cos\alpha = 0.$$

Ex. 7. In the problem given in Ex. 3, Art. 60, fig. 30, let the beam be shifted so that A is still in contact with the wall; then the principle of virtual velocities gives

$$W\delta.(b\cos\phi - a\cos\theta) = 0;$$

$$\therefore \quad b\sin\phi\,\delta\phi - a\sin\theta\,d\theta = 0.$$

But $$b\sin\phi = 2a\sin\theta;$$

$$\therefore \quad b\cos\phi\,\delta\phi = 2a\cos\theta\,\delta\theta:$$

$$\therefore \quad \tan\theta = 2\tan\phi;$$

which leads to the results given in Ex. 3, Art. 60.

Ex. 8. Find the form of the curve in a vertical plane, such that a heavy rod resting on its concave side, and on a peg at a given point, say the origin, may be at rest in all positions.

Let the place of the peg be the origin, and let the rod be inclined to the vertical at the angle θ; let r be the radius vector of the curve which coincides with the rod, and let $2a$ be the length of the rod. Then by the principle of virtual velocities,

$$\text{w}\,\delta.(r-a)\cos\theta = 0;$$

$$\therefore\quad (r-a)\cos\theta = \text{a constant} = k, \text{ say}:$$

$$\therefore\quad r = a + k\sec\theta;$$

which is the equation to the conchoid of Nicomedes.

Ex. 9. In Ex. 3, Art. 37, prove that (79) is the equation of virtual velocities; and that in case (4), (84) is also the equation of virtual velocities.

Ex. 10. A particle is attracted by two centres of force which vary inversely as the square of the distance; find the form of the surface on all points of which the particle will be at rest.

Let μ and μ' be the absolute attractive forces, and let r and r' be the distances of the particle from the centres; then by the principle of virtual velocities we have

$$\frac{\mu\,dr}{r^2} + \frac{\mu'dr'}{r'^2} = 0;$$

$$\therefore\quad \frac{\mu}{r} + \frac{\mu'}{r'} = \text{a constant};$$

which condition expresses the form of the surface.

110.] A remarkable theorem discovered by Gauss, and published for the first time, so far as I know, in the fourth volume of Crelle's Journal, may be deduced immediately from the equation of virtual velocities.

For a system of forces in equilibrium we have

$$\Sigma.\text{P}\,\{\cos\alpha\,dx + \cos\beta\,dy + \cos\gamma\,dz\} = 0. \tag{290}$$

Let the forces be replaced by line-representatives, and let (x, y, z) be the point of application of the type-force P, and (ξ, η, ζ) the other extremity of the representative; then replacing $\text{P}\cos\alpha$, $\text{P}\cos\beta$, $\text{P}\cos\gamma$ respectively by $\xi-x$, $\eta-y$, $\zeta-z$, (290) becomes

$$\Sigma.\{(\xi-x)dx + (\eta-y)dy + (\zeta-z)dz\} = 0 = -\frac{\text{D}\Omega}{2}\text{ (say)}; \tag{291}$$

and if the displacement of the system is such that the extremity (ξ, η, ζ) of the line-representative of the type-force is fixed, while the other extremity (x, y, z) receives an infinitesimal displacement, then integrating (291) we have

$$\Sigma.\{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2\} = \Omega; \qquad (292)$$

and thus Ω, which is the sum of the squares of the line-representatives of the forces, is a maximum, a minimum, or a constant.

Hence we have the following theorem :

If there are n points, at invariable distances apart, the system of which is however moveable, and also if there is a system of n points wholly fixed, each of which corresponds to a point of the former system, then if the sum of the squares of the distances between each of the moveable points and its corresponding fixed point has a critical value, the system of forces represented as to intensity and line of action by these distances, and acting severally at the moveable points, is in equilibrium; and the equilibrium is stable or unstable, according as the sum of the squares of the distances is a minimum or a maximum, and is neutral if it is constant.

Also differentiating again (291) we have

$$\begin{aligned} \mathrm{D}^2\Omega = {} & 2\,\Sigma.(dx^2+dy^2+dz^2) \\ & -2\,\Sigma.\{(\xi-x)d^2x+(\eta-y)d^2y+(\zeta-z)d^2z\}; \qquad (293) \end{aligned}$$

and if the displacement, to which the variations of the coordinates of the points of application of the forces are due, is such that $d^2x = d^2y = d^2z = 0$, then $\mathrm{D}^2\Omega$ is necessarily positive, and Ω is a minimum; also if $\Sigma.\{(\xi-x)d^2x+(\eta-y)d^2y+(\zeta-z)d^2z\}$ is negative, that is, when equilibrium is stable, Ω is a minimum.

The line-representatives of the forces, however, can always be taken so small that $\xi-x$, $\eta-y$, $\zeta-z$ shall be infinitesimal; whereby the second part of (293) being infinitesimal, and of the third order, must be neglected; and as the first part is positive, Ω is a minimum; that is, the sum of the squares of the line-representatives is a minimum.

To this subject, however, we shall return hereafter, and in a more general way. And in respect of the preceding it is also to be observed that, in the displaced position of the body on which the forces act, $\frac{\xi-x}{\mathrm{P}\cos a} = \ldots = \ldots = \frac{r}{\mathrm{P}}$ are supposed to act along lines parallel to their original lines of action; whereas, in the most general case, the new lines of action would be functions of the original points of application.

SECTION 9.—*Constrained Equilibrium.*

111.] The material body or system of material particles, which receives the pressures considered in the preceding Articles, has been supposed to be free from all constraint; we must now investigate the modifications required in the general results when the system is subject to certain given constraints.

Firstly, suppose one point of the body to be fixed; let this be taken for the origin: it is evident that, because it is fixed, it will bear any pressure of translation acting on it, and that the body will not move owing to that pressure; but the effects of a pressure of rotation about a rotation-axis passing through that point are not affected by the fixedness of the point; the impressed forces therefore must be so related that, see Art. 70, $\mathrm{G} = 0$; and therefore that,

$$\mathrm{L} = 0, \qquad \mathrm{M} = 0, \qquad \mathrm{N} = 0; \tag{294}$$

which three conditions are requisite, so that a body, of which one point is fixed, should be at rest. These three conditions, it will be observed, satisfy equation (130), and therefore indicate that the impressed pressures may be compounded into a single force of translation: that, viz. which passes through the fixed point.

And the pressure on the fixed point, and the direction of its line of action, may thus be found: let R be the pressure, and a, b, c the direction-angles of its line of action; let the impressed forces be $\mathrm{P}_1, \mathrm{P}_2, \ldots \mathrm{P}_n$, and the direction-angles of their lines of action $\alpha_1, \beta_1, \gamma_1$, &c.; then

$$\left.\begin{aligned} \mathrm{R}\cos a &= \Sigma.\mathrm{P}\cos\alpha, \\ \mathrm{R}\cos b &= \Sigma.\mathrm{P}\cos\beta, \\ \mathrm{R}\cos c &= \Sigma.\mathrm{P}\cos\gamma; \end{aligned}\right\} \tag{295}$$

$$\therefore \quad \mathrm{R}^2 = (\Sigma.\mathrm{P}\cos\alpha)^2 + (\Sigma.\mathrm{P}\cos\beta)^2 + (\Sigma.\mathrm{P}\cos\gamma)^2; \tag{296}$$

and therefore by (295) a, b, c are known.

112.] Secondly, let us suppose two points of the body to be fixed; and let the axis of z pass through the two points, and the origin be at the middle point of the line joining them; and let the z-ordinates to the points be $+z_1$ and $-z_1$; then it is manifest that the body cannot have any motion of translation, and can have motion of rotation about the axis of z only. The impressed forces therefore must be so related that the rotation-pressure about the axis of z should be equal to zero; therefore the necessary condition is

$$\text{N} = 0. \tag{297}$$

And the pressures on the two points may be determined in the following manner: let them be represented by R_1 and R_2, and let the direction-angles of their lines of action be a_1, b_1, c_1; a_2, b_2, c_2; then

$$\left.\begin{aligned} \text{R}_1 \cos a_1 + \text{R}_2 \cos a_2 &= \Sigma.\text{P} \cos a, \\ \text{R}_1 \cos b_1 + \text{R}_2 \cos b_2 &= \Sigma.\text{P} \cos \beta, \\ \text{R}_1 \cos c_1 + \text{R}_2 \cos c_2 &= \Sigma.\text{P} \cos \gamma; \end{aligned}\right\} \tag{298}$$

$$\left.\begin{aligned} \text{L} + \text{R}_1 z_1 \cos b_1 - \text{R}_2 z_1 \cos b_2 &= 0, \\ \text{M} - \text{R}_1 z_1 \cos a_1 + \text{R}_2 z_1 \cos a_2 &= 0. \end{aligned}\right\} \tag{299}$$

From the first two of (298), and from (299), we have

$$\text{R}_1 \cos b_1 = \frac{z_1 \Sigma.\text{P} \cos \beta - \text{L}}{2 z_1};$$

$$\text{R}_2 \cos b_2 = \frac{z_1 \Sigma.\text{P} \cos \beta + \text{L}}{2 z_1};$$

$$\text{R}_1 \cos a_1 = \frac{z_1 \Sigma.\text{P} \cos a + \text{M}}{2 z_1};$$

$$\text{R}_2 \cos a_2 = \frac{z_1 \Sigma.\text{P} \cos a - \text{M}}{2 z_1};$$

and thus the pressures on the fixed points, which are parallel to the axes of x and y, are determined: but the pressures along the axis of z are involved in only the third equation of (298), which shews that the sum of the pressures is equal to $\Sigma.\text{P} \cos \gamma$, and therefore that each pressure is indeterminate: now this is, at first sight, a startling fact, and has been urged heretofore as an argument against the truth of our mechanical results and principles; because it is said that, when a body is supported in the manner assumed in the problem, say a gate or a door on its two hinges, the vertical pressures are determinate and may be experimentally determined at both hinges; our mechanical formulæ therefore ought to yield a corresponding result. In any actual case the pressures without doubt are determinate, and may be determined by mechanical means: but then the bodies which are the subjects of the experiments are more or less compressible and extensible: they are not rigid; and therefore do not satisfy the conditions required in the preceding theory, however nearly they may approach to them; thus if to a door, being in a *horizontal* position, two 'eyes' are attached, which correspond to two hooks fixed in a vertical doorpost, and if the distance between the eyes when the door is horizontal is equal to that between the hooks in the vertical doorpost; then doubtless, if the body were perfectly rigid and inextensible, and were attached by the eyes to the hooks, either one or the other hook would be sufficient to bear the vertical pressure; and we should be unable to determine whether one or the other carried the whole weight, and whether it was distributed between them, and in what proportion; yet as such a door is extensible, both hooks would bear a part of the weight, and the respective proportions will depend on the extensibility and the elasticity of the material. Thus if the distance between the eyes is greater than that between the hooks, the pressure will for the most part be on the lower hook, although the compression of the material due to its weight may cause the eyes so to approach each other, that some of the pressure may be brought upon the upper hook; and a similar effect may occur at the lower hook, when the distance between the hooks is greater than that between the eyes. Thus it appears that the determinateness of the pressures is due to the extensibility, compressibility and elasticity of the material which is in nature the subject of the experiment; and the truth of the result which is arrived at in (298) for a rigid body is not affected: for in nature we have nothing of perfect rigidity. We shall see a further example of indeterminateness of the same kind in dynamics.

Again, suppose the circumstances of constraint to be such, that the body is capable of *sliding* along, as well as of turning about, the axis passing through the two fixed points; then the points will be able to bear the pressures arising from the forces which are resolved at right angles to the axis, and parallel to the axes of x and y; but will not offer any resistance to those along the axis of z: if therefore equilibrium exists, the forces must satisfy the conditions,

$$\Sigma . \mathrm{P} \cos \gamma = 0, \qquad \mathrm{N} = 0.$$

113.] And lastly, if three or more points of the body are fixed, and if all these are not in the same straight line, it is evident that the body is fixed; and therefore whatever are the impressed forces as to intensity, point of application, line of action, and direction, the body is in equilibrium, if we suppose the fixed points of it to be capable of bearing the pressures which are due to the impressed forces.

And it is evident by the following reasoning that, if these points are fixed, the body is also fixed. For suppose the body to consist of n particles; then each of these particles is at rest, if the forces, including the tensions, mutual reactions, &c., acting on it satisfy the three conditions (69), Art. 34: and therefore if all are at rest, $3n$ conditions are required. Now if three points of a body are fixed, the mutual distances of them are also fixed, and hereby we have three conditions; also as the body is rigid, the distances of each of the remaining $n-3$ particles from each of the three fixed points are given, and thus we have $3n-9$ conditions; and as the equations of equilibrium of a rigid body are six, we have six more conditions: and thus altogether we have, as before, $3n$ equations. If the three fixed points are in one and the same straight line, one of the conditions is lost, and the number is insufficient for equilibrium.

114.] Another form in which a body under the action of impressed forces may be in constraint is, when it rests with points of it on a plane, or against any surface.

Let us consider first the more simple case of a smooth plane: and let us suppose the plane to be that of (x, y), and n points of the body to rest on it; let these be (x_1, y_1), (x_2, y_2), ... (x_n, y_n); and let the pressures at these points be R_1, R_2, ... R_n; the lines of action of which are parallel to the axis of z: thus the equations of equilibrium become

$$\Sigma.\text{P} \cos \alpha = 0, \qquad \Sigma.\text{P} \cos \beta = 0, \qquad \Sigma.\text{P} \cos \gamma - \Sigma.\text{R} = 0; \qquad (300)$$

$$\text{L} - \Sigma.\text{R} y = 0, \qquad \text{M} + \Sigma.\text{R} x = 0, \qquad \text{N} = 0. \qquad (301)$$

Here are six equations, of which only three involve the pressures against the plane and the coordinates of their points of action; there are always therefore three independent conditions to be fulfilled by the impressed forces.

Now if only one point of the body is in contact with the plane, the pressure at that point will be given by the third equation, and the impressed forces must be such as to fulfil the other five.

If two points are in contact, the pressures at them may be determined by either two of the third, fourth, and fifth equations, and the forces must satisfy the remaining four conditions.

But if three points are in contact, the pressures at them may be determined by means of the three equations which involve the pressures, and the other three equations must be satisfied by the impressed forces.

If more than three points are in contact, the pressures are indeterminate, because there is not a sufficient number of equations for their determination.

In all cases the pressure which the plane has to bear is given by the third equation of (300); and for the existence of equilibrium, if the body only presses against the plane, it is necessary that the $\Sigma . \text{P} \cos \gamma$ should act *towards*, and not *from*, the plane; it is also necessary that the line of action of this pressure should pierce the plane of (x, y) at some point within the area determined by straight lines joining the points of contact of the body and the plane: otherwise the rotation-pressure of the z-force will cause the body to turn about one of the bounding lines of this area.

And of the indeterminateness of the several pressures, which act at the points of contact, when more than three points are in contact with the plane, an explanation similar to that of Art. 112 may be given. Suppose a heavy body to rest on a horizontal table, and to be in contact with it at many points; the sum of all the pressures is doubtless equal to the weight of the body; but if the points of contact are more than three, each pressure, so far as the preceding theory enables us to determine it, is indeterminate; and so it would be in fact, if the table were accurately plane, and it and the body were perfectly rigid; but such a table and such a body do not exist: and so our results when applied to flexible and compressible matter are not true. If however we knew the laws of flexibility and elasticity, and could thus bring into calculation all the conditions of the problem, the result would be determinate and true; and thus it seems that the non-applicability of the mechanical principles is only apparent, and is due to the omission of certain conditions which the true solution of the problem requires.

115.] Again, suppose the body to be in contact with surfaces whose equations are $\text{F}_1 = 0$, $\text{F}_2 = 0, \ldots \text{F}_n = 0$; and the mutual pressures between the body and the several surfaces to be R_1, $\text{R}_2, \ldots \text{R}_n$; the direction-angles of the lines of action of these to be a_1, b_1, c_1; a_2, b_2, c_2; $\ldots a_n, b_n, c_n$; and the points of contact to be (x_1, y_1, z_1), $(x_2, y_2, z_2), \ldots (x_n, y_n, z_n)$; then employing the ordinary notation, see Art. 36,

$$\cos a = \frac{\text{U}}{\text{Q}}, \qquad \cos b = \frac{\text{V}}{\text{Q}}, \qquad \cos c = \frac{\text{W}}{\text{Q}};$$

and the equations of equilibrium become

$$\left.\begin{aligned} \Sigma.\mathrm{P}\cos\alpha+\Sigma.\mathrm{R}\cos a &= 0, \\ \Sigma.\mathrm{P}\cos\beta+\Sigma.\mathrm{R}\cos b &= 0, \\ \Sigma.\mathrm{P}\cos\gamma+\Sigma.\mathrm{R}\cos c &= 0; \end{aligned}\right\} \tag{302}$$

$$\left.\begin{aligned} \mathrm{L}+\Sigma.\mathrm{R}(y\cos c-z\cos b) &= 0, \\ \mathrm{M}+\Sigma.\mathrm{R}(z\cos a-x\cos c) &= 0, \\ \mathrm{N}+\Sigma.\mathrm{R}(x\cos b-y\cos a) &= 0. \end{aligned}\right\} \tag{303}$$

To which equations, as to the number of points in contact between the body and the surfaces, the remarks of the last three Articles are applicable.

One point however requires further elucidation: suppose that the surface of the body on which the forces act meets n given and fixed points; then the equations (302) and (303) contain n undetermined pressures which act at these points. Now as the equations are six in number, if $n = 6$, the six pressures at the points may be determined; and the directions of their lines of action will be along the normals to the surface of the body at the points; if n is greater than 6, $n-6$ of the pressures may be indeterminate, and when they receive given values, the other 6 will be known: and when n is less than 6, the pressures at the given points may be eliminated from the preceding equations, and the remaining $6-n$ conditions must be fulfilled by the impressed forces acting on the body. And hence we infer that generally a body under the action of given forces is in equilibrium and fixed, if the bounding surface of it passes through six given and fixed points*; and that the mobility of it is not taken away, if the surface has to pass through fixed points of which the number is less than six.

116.] And hereby I am led to another subject: viz. to the investigation of the conditions requisite that many bodies subject to given pressures, and in contact with, or under mutual action from, each other, should be in equilibrium.

Let the number of bodies be n; let $\mathrm{P}_1, \mathrm{P}_2, \ldots \mathrm{P}_n$ be the types of the forces which act on the first, second, ... nth body respectively; let R be the general type of the reacting pressures at the points of contact, and a, b, c the direction-angles of its line of action, and (x, y, z) the point of its application; $\mathrm{L}_1, \mathrm{M}_1, \mathrm{N}_1$; $\mathrm{L}_2, \mathrm{M}_2, \mathrm{N}_2$; ... the moment-axes of the component couples which act on the several bodies; then the conditions of equilibrium for the several bodies are

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* For various other properties of this kind let me refer the reader to Möbius, Lehrbuch der Statik, Zweiten Theil, Erstes Kapitel; Leipzig, 1837.

$$\left.\begin{aligned}\Sigma.\text{P}_1\cos\alpha_1+\Sigma.\text{R}_1\cos a_1&=0,\\ \Sigma.\text{P}_1\cos\beta_1+\Sigma\,\text{R}_1\cos b_1&=0,\\ \Sigma.\text{P}_1\cos\gamma_1+\Sigma.\text{R}_1\cos c_1&=0;\end{aligned}\right\}\qquad(304)$$

$$\left.\begin{aligned}\text{L}_1+\Sigma.\text{R}_1(y_1\cos c_1-z_1\cos b_1)&=0,\\ \text{M}_1+\Sigma.\text{R}_1(z_1\cos a_1-x_1\cos c_1)&=0,\\ \text{N}_1+\Sigma.\text{R}_1(x_1\cos b_1-y_1\cos a_1)&=0;\end{aligned}\right\}\qquad(305)$$

. . . .

$$\left.\begin{aligned}\Sigma.\text{P}_n\cos\alpha_n+\Sigma.\text{R}_n\cos a_n&=0,\\ \Sigma.\text{P}_n\cos\beta_n+\Sigma.\text{R}_n\cos b_n&=0,\\ \Sigma.\text{P}_n\cos\gamma_n+\Sigma.\text{R}_n\cos c_n&=0;\end{aligned}\right\}\qquad(306)$$

$$\left.\begin{aligned}\text{L}_n+\Sigma.\text{R}_n(y_n\cos c_n-z_n\cos b_n)&=0,\\ \text{M}_n+\Sigma.\text{R}_n(z_n\cos a_n-x_n\cos c_n)&=0,\\ \text{N}_n+\Sigma.\text{R}_n(x_n\cos b_n-y_n\cos a_n)&=0.\end{aligned}\right\}\qquad(307)$$

Now if, of all these groups of equations, all the first of the first sets are added, $\Sigma.\text{R}\cos a$ will disappear, because, the reactions of the several bodies being equal and opposite, the same quantity will appear twice, and with different signs; so that we shall finally obtain $\Sigma.\text{P}\cos\alpha = 0$; similarly, by adding all the second equations of the first set in each group, and by adding all the third equations of the first set, we shall have

$$\Sigma.\text{P}\cos\beta = 0, \qquad \Sigma.\text{P}\cos\gamma = 0.$$

In the same way, by adding the several equations of the second sets of the groups, we shall obtain equations free from the R's, and shall have ultimately

$$\text{L} = 0, \qquad \text{M} = 0, \qquad \text{N} = 0;$$

and thus the equations of condition necessary for the equilibrium of a system of rigid bodies are of the same form and of the same number as those required for the equilibrium of a single rigid body.

117.] Examples illustrative of the preceding Articles.

Ex. 1. A heavy uniform beam is fixed by a hinge to a given inclined plane: between the beam and the plane a heavy sphere is in equilibrium; determine its position and the several pressures.

Let fig. 37 represent a vertical section of the system made by the plane of the paper: $\text{POB} = \alpha$; $\text{POQ} = 2\theta$; $\text{OG} = \text{GA} = a$; CP

$= \mathrm{CQ} = c$; $W =$ the weight of the beam; $w =$ the weight of the sphere; $R =$ reaction existing between the beam and the sphere; $R' =$ the pressure of the sphere on the inclined plane. And let us consider separately the conditions of equilibrium of the sphere and of the beam.

For the equilibrium of the sphere, resolving the forces along the plane, we have

$$w \sin\alpha = R \sin 2\theta.$$

For the equilibrium of the beam, taking moments about O, we have

$$W a \cos(\alpha + 2\theta) = R \times \mathrm{OQ},$$
$$= R c \cot\theta;$$
$$\therefore \quad W a \cos(\alpha + 2\theta) \sin 2\theta = w c \sin\alpha \cot\theta:$$

whence may θ be determined; and thence R; and since

$$R' = w \cos\alpha + R \cos 2\theta,$$

R' may also be found.

Ex. 2. Two heavy beams OA and OA′ of equal lengths are connected, fig. 38, at O by a hinge, and at AA′ by a string of given length; between them a heavy sphere is placed, and the string remains horizontal; determine the tension of the string and the pressure against the beams.

Let length of each beam be $2a$, weight of each beam $= W$; $2c =$ length of string; $T =$ the tension of the string; $b =$ the radius of the sphere; $w =$ the weight of the sphere; $\alpha =$ the angle $\mathrm{AOB} = \sin^{-1}\dfrac{c}{2a}$; then for the equilibrium of either of the beams, taking the moments of the forces about O, we have

$$T\, 2a \cos\alpha = W a \sin\alpha + R b \cot\alpha;$$

and for the equilibrium of the sphere, taking vertical forces, we have

$$w = 2R \sin\alpha;$$
$$\therefore \quad T = \frac{W}{2}\tan\alpha + \frac{b}{a}\,\frac{w}{4}(\operatorname{cosec}\alpha)^2.$$

SECTION 10.—*On Friction.*

118.] All the surfaces, which we have imagined to be in contact in the preceding Articles, are supposed to be smooth, and, as such, to offer no resistance to the motion of the points in contact with them in directions perpendicular to the normal at the points; and therefore the reaction arising from the contact acts along the common normal line only. In nature, however, we have no surfaces perfectly smooth; the constitution of all bodies is such, that on their bounding surfaces are small elevations and depressions, arising, as it seems, from their constituent molecules not being continuous and in perfect contact: so that if the surfaces of two bodies are pressed against each other, the elevations of one fit, at least in a measure, into the depressions of the other, and the surfaces interpenetrate each other; and the mutual penetration is of course greater, if the pressing force is greater; much of this *roughness* may be removed by polishing, and the effect of much of it may be destroyed by lubrication: all however cannot be, and there still remains a resistance due to it, when force is applied so as to cause one body to move or to have a tendency to move on another with which it is in contact. This resistance is called *friction*, and is of two kinds; either of sliding or of rolling: the first is that of a heavy body dragged on a plane or other surface; of an axle turning in a fixed box; of a vertical shaft turning on a horizontal plate, or of a millstone turning upon another concentric stone about a vertical axis. Friction of the second kind is that of a wheel rolling along a plane; the resistance however of which seems to arise from the necessity of the wheel overcoming small obstacles which are successively in its path. It is of friction of the first kind only that I shall at present state the laws and give examples; and first as to its line of action: it is manifestly along that tangent line of the surfaces at the point of contact which is the line of the tendency to motion; and its direction is opposite to that of the line of motion. Suppose therefore many forces to act on a material particle which is in contact with a rough surface; and, the lines of action of the forces being unaltered, their magnitudes to change, so that motion is on the point of taking place (1) in one direction, and (2) in an opposite direction: the line of action of friction is in both cases the same; but the *direction* of it in the former case is contrary to that of it in the latter. Also the magnitudes of the forces may evidently vary within certain limits, and the particle may still be at rest. Examples of the determination of these limits are given in the following Article.

In our ignorance of the constitution of bodies, and of their molecular action, the laws of friction must be deduced from experiment; and therefore I shall enunciate those only which are necessary for our purpose, and refer the reader to the Treatise by M. Morin*, wherein he will find the subject investigated in all its completeness.

I. Friction is proportional to the normal pressure, when the materials of the surfaces in contact are the same.

II. Friction is independent of the extent of the surfaces in contact.

III. Friction is independent of the velocity of motion.

As to law I; suppose R to be the normal pressure between two surfaces, and F to be the friction, then $F = \mu R$, where μ is a constant quantity for the same materials and is the value of F when $R = 1$; μ is called *the coefficient of friction.* And this law, it may be observed, appears to arise out of the preceding theory of friction; because the greater is the pressure, the greater is the interpenetration of the molecules at the surface of the bodies, and the greater is the resistance to be overcome, when motion is just about to take place.

As to law II; it signifies that if the pressure remains the same, and the surface in contact increases, the total resistance is still the same, whilst the pressure on each element and the friction corresponding to that element are diminished in the inverse ratio of the area of the surfaces in contact.

The treatise of M. Morin will be found to contain a complete account of the modes of determining μ for different substances; but the following manner of considering the subject is sufficiently simple, and sufficiently general for our purpose.

Let a given heavy body rest with a plane face of a finite area on a horizontal plane; and let the plane be turned about a horizontal line in it, so that it becomes inclined to the horizontal plane, that is, becomes tilted: the body will begin to slide when the inclination has reached a certain limit; and this inclination will manifestly depend on the friction which exists between the body and the plane, and may be determined as follows. See fig. 39.

Let W be the weight of the body; μ = the coefficient of friction; α = the angle between the inclined and the horizontal planes just as motion is beginning to take place; R = the pressure on the plane; so that

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* Nouvelles Expériences sur le frottement faites a Metz, imprimées par ordre de l'Académie des Sciences; 3 vols. in 4to. 1832–1835.

$$F = \mu R; \tag{308}$$

and resolving along, and perpendicular to, the plane,

$$F = W \sin\alpha, \qquad R = W\cos\alpha;$$

$$\therefore \quad \tan\alpha = \mu, \qquad \alpha = \tan^{-1}\mu: \tag{309}$$

α is called *the angle of friction*, and *the angle of repose*. The body will rest on the plane when the angle of inclination is less than the angle of friction, and will slide, if the angle of inclination exceeds that angle.

119.] Various problems involving friction.

Ex. 1. A small ring under the action of known pressures is capable of sliding on a rough curved material line in space; it is required to determine the limits of the forces, so that the ring may be at rest.

Let the resolved parts of the impressed forces along the coordinate axes be X, Y, Z, of which let the resultant be R; so that if x, y, z are the coordinates to the position of the ring on the curve, the whole impressed force along the tangent, which we will call T, is

$$T = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}. \tag{310}$$

Let N = the normal pressure: then

$$N^2 + T^2 = R^2,$$

$$= X^2 + Y^2 + Z^2;$$

$$\therefore \quad N^2 = X^2 + Y^2 + Z^2 - \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)^2.$$

Now in order that motion should not take place,

$$T^2 < \mu^2 N^2 < \mu^2 (R^2 - T^2);$$

$$\therefore \quad \frac{T^2}{R^2} < \frac{\mu^2}{1+\mu^2} < (\sin\alpha)^2, \text{ see equation (309)};$$

$$\therefore \quad \left(\frac{X\,dx + Y\,dy + Z\,dz}{R\,ds}\right)^2 < (\sin\alpha)^2; \tag{311}$$

$$\text{and if} \quad \frac{X\,dx + Y\,dy + Z\,dz}{R\,ds} = \pm\sin\alpha, \tag{312}$$

the particle will begin to slide; the ± sign assigning the limits within which the forces are to be confined.

Ex. 2. As an example, let us take the helix whose equations are

$$x = a\cos\phi, \qquad y = a\sin\phi, \qquad z = ka\phi;$$

and let the force which acts on the ring be its own weight, and $= w$, and have its line of action parallel to the axis of z: then $\text{Z} = \text{R} = w$: and

$$\frac{dz}{ds} = \frac{k}{(1+k^2)^{\frac{1}{2}}} = \pm \sin\alpha; \qquad \therefore\quad k = \pm \tan\alpha;$$

that is, the angle of inclination of the thread of the helix to the horizontal plane is equal to the angle of friction.

Ex. 3. To determine the limits of the pressures, so that a particle under the action of them may be at rest on a given rough surface.

Let $\text{F}(x, y, z) = 0$ be the equation to the surface: then employing the ordinary symbols, if $\text{N} =$ the normal pressure, $\text{T} =$ the tangential force, and $\text{R} =$ the resultant of the acting forces, of which the resolved parts along the coordinate axes are $\text{X}, \text{Y}, \text{Z}$,

$$\text{N} = \frac{\text{XU}+\text{YV}+\text{ZW}}{\text{Q}}, \qquad \text{T}^2 = \text{R}^2 - \text{N}^2;$$

therefore that the particle should be at rest

$$\text{T}^2 < \mu^2\text{N}^2, \qquad \text{R}^2 - \text{N}^2 < \mu^2\text{N}^2, \qquad \frac{\text{R}^2}{\text{N}^2} < 1 + \mu^2;$$

$$\therefore\quad \frac{\text{Q}^2\text{R}^2}{(\text{XU}+\text{YV}+\text{ZW})^2} < 1 + \mu^2 < (\sec\alpha)^2; \tag{313}$$

and therefore if

$$\frac{\text{QR}}{\text{XU}+\text{YV}+\text{ZW}} = \pm \sec\alpha, \tag{314}$$

the particle will just begin to move; the $\pm$ sign assigns the limits of the impressed pressures. As an example let us take the following:

Ex. 4. An ellipsoid has its least axis in a vertical direction; determine on the surface the curve, on all points within which a heavy material particle being placed shall remain at rest.

In this case $\quad \text{X} = 0, \quad \text{Y} = 0, \quad \text{Z} = \text{R};$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

$$\begin{aligned} \text{Q}^2 &= \text{U}^2 + \text{V}^2 + \text{W}^2 \\ &= 4\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right); \end{aligned}$$

therefore (314) becomes

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = (\sec\alpha)^2 \frac{z^2}{c^4};$$

$$\therefore\quad \frac{x^2}{a^4} + \frac{y^2}{b^4} - (\tan\alpha)^2 \frac{z^2}{c^4} = 0;$$

which is the equation to a cone, whose vertex is at the centre of the ellipsoid; and the line of intersection of which with the ellipsoid is the required bounding curve.

Ex. 5. A heavy particle rests on a rough inclined plane, and is acted on by a given force in a vertical plane which is perpendicular to the inclined plane; determine the limits of the force, and the angle at which the least force capable of drawing the particle up the plane must act.

Let fig. 40 represent a vertical section of the inclined plane, and containing the force P; let the inclination of the plane to the horizontal plane be i; and let θ be the angle between the inclined plane and the line of action of P; μ = coefficient of friction: and let us first suppose the tendency to motion to be down the plane, so that friction is a force acting up the plane: then resolving along, and perpendicular to, the plane,

$$F + P\cos\theta = W\sin i, \qquad R + P\sin\theta = W\cos i, \qquad F = \mu R;$$

$$\therefore \quad P = W\frac{\sin i - \mu\cos i}{\cos\theta - \mu\sin\theta}. \tag{315}$$

And if P is increased so that motion up the plane is just beginning, F acts in an opposite direction, and therefore the sign of μ must be changed, and we have

$$P = W\frac{\sin i + \mu\cos i}{\cos\theta + \mu\sin\theta}. \tag{316}$$

Now to determine θ in this latter case, so that P shall be the least,

$$\frac{dP}{d\theta} = W(\sin i + \mu\cos i)\frac{\sin\theta - \mu\cos\theta}{(\cos\theta + \mu\sin\theta)^2} = 0,$$

$$\text{if} \quad \tan\theta = \mu;$$

that is, if θ is equal to the angle of friction. Hence we infer that

A given power acts to the greatest advantage in dragging a weight up a hill, if the angle at which its line of action is inclined to the hill is equal to the angle of friction of the hill. And, similarly, a power acts to the greatest advantage in dragging a weight along a horizontal plane, if its line of action is inclined to the plane at the angle of friction of the plane. Hereby also may we determine the angle at which the 'traces' of a drawing horse should be inclined to the plane of traction. The preceding results are those which are *a priori* to be expected, because some part of the power ought to be expended in *lifting* the weight from the plane, so that friction may be diminished.

Ex. 6. Also let us consider the case of a rough cylindrical axis, on which given forces act and produce a pressure of rotation, capable of turning within a rough hollow coaxal cylinder.

Let fig. 41 be a section perpendicular to the axis of the cylinder; the smaller and interior circle being a section of the cylindrical axis, and the larger circle of the hollow cylinder; let C be the point of contact of the two cylinders, and at which of course the resultant of all the impressed forces acts: let this force = P, and let θ be the angle between the lines of action of R and P: then

$$R = P\cos\theta, \qquad F = P\sin\theta,$$
$$F = \mu R; \qquad \therefore \quad \tan\theta = \mu;$$

therefore θ is equal to the angle of friction. If therefore the angle between R and P is less than the angle of friction, the cylinder will continue at rest; and if it is greater, it will move.

Ex. 7. A heavy circular shaft rests in a vertical position, with its end, which is a circular section, on a horizontal plate; determine the resistance due to friction which is to be overcome, when the shaft begins to revolve about a vertical axis.

Let a be the radius of the circular section of the shaft; and let the plane of (r, θ) be the horizontal one of contact between the end of the shaft and the plate; and let the centre of the circular area of contact be the pole; now the vertical pressure on each element of this area manifestly varies as the area; and therefore, if $r\,dr\,d\theta$ is the area-element and k is the coefficient of variation, since, by law III, friction is independent of the velocity of motion,

$$\text{the pressure on the element} = kr\,dr\,d\theta;$$
$$\therefore \quad \text{the friction of the element} = \mu kr\,dr\,d\theta;$$

the moment of friction about the vertical axis through the centre

$$= \mu kr^2\,dr\,d\theta;$$

$\therefore$ the moment of friction of the circular end

$$= \int_0^{2\pi}\int_0^a \mu kr^2\,dr\,d\theta$$
$$= \frac{2\mu k\pi a^3}{3}.$$

Now if W = the weight of the shaft; since k is the pressure on an unit of area,

$$W = \pi k a^2;$$

$\therefore$ the moment of the friction of the circular end $= \dfrac{2\mu \mathrm{w} a}{3}$,

and consequently varies as the radius. Hence arises the advantage of reducing to the smallest possible dimensions the area of the base of a vertical shaft revolving with its end resting on a horizontal bed.

Similarly may the friction of the upper millstone moving on the nether one be calculated.

Ex. 8. If the shaft is a square prism of the weight w, and rotates about an axis in the centre of the shaft, then the moment of friction varies as the side of the square section of the shaft.

Ex. 9. If the shaft is composed of two circular cylinders placed side by side, and rotates about the line of contact of the two cylinders, then

the moment of the friction of the surface

in contact with the horizontal plane $= \dfrac{32\mu a \mathrm{w}}{9\pi}$.

Ex. 10. A heavy straight rod rests on a rough horizontal plane, and at one end of the rod, in a line perpendicular to its length and in the plane, a force pulls the rod, the magnitude of which is just sufficient to move the rod in the plane. Shew that the point about which the rod begins to turn is at a distance $= a\sqrt{2}$ from the other end of the rod, if the length of the rod is $2a$.

CHAPTER IV.

ON GRAVITY, AND CENTRE OF GRAVITY.

SECTION 1.—*Elementary considerations on mass, gravity, and weight.*

120.] Into the investigations of this and of subsequent Chapters there will enter certain elementary conceptions of matter beyond those which have hitherto been stated. In Chapter II. matter was defined as the subject of force; occupying space, and consequently possessing form: capable of infinite divisibility, and thus resoluble into particles; capable of rigidity, in which state the particles are in relative rest; and transmitting force in the line of action of the force only, so that the external forces acting on the matter are of infinitesimal magnitude in comparison of the internal forces which act on the several particles and keep them in relative rest; for the relative equilibrium is not affected by the action of the forces which act on the matter from without. Now we require other properties of matter.

Matter is *impenetrable;* that is, two particles of matter cannot occupy the same place at the same time.

Matter is *porous;* that is, although matter is composed of particles or molecules or atoms, yet these are not packed in close and continued contact; but there are intervals or interstices, which do not contain the matter of the body, whatever that is by which they are occupied.

According to the greater or less degree of closeness with which the particles are packed, so is matter more or less dense; and density is predicated of it in respect of this quality. If the density of matter is constant throughout a given body, the body is said to be homogeneous; but if the density changes, either continuously or discontinuously, the body is said to be heterogeneous; in the more general case the density varies continuously, and at a given point is a function of the coordinates of the point. Thus the earth is not homogeneous; the density of it increases as we pass from the surface to the centre; it is doubtless composed of concentric shells, each of which has surfaces of the form of an oblate spheroid and is homogeneous; and the density of which is a function of the axes of the shell. The average density of a heterogeneous body is called its *mean* density. The mean density of the earth is about five times that of distilled water.

121.] As the quantity of matter contained in a body is a function of the volume of the body and of the density of the matter, it is necessary to have means of measuring the same with precision.

Quantity of matter is called *mass*; so that the mass of a body is the quantity of matter contained in the body.

Density is the quantity of matter contained in an unit-volume; the absolute density or the closeness with which the particles are packed being uniform throughout that unit-volume. This definition is directly applicable if a body is homogeneous; but if it is heterogeneous, and the density varies from point to point, the density at any point is the quantity of matter contained in an unit-volume, throughout which the density is the same as that at the point. Density is commonly denoted by the symbol ρ, which is constant in homogeneous bodies, and in heterogeneous bodies is a function of the coordinates.

Thus if v is the volume of a homogeneous body of which ρ is the density,

$$\text{the mass} = \rho \mathrm{v}; \tag{1}$$

and if the body is heterogeneous, and is referred to a system of rectangular coordinate axes; and if ρ is the density at (x, y, z), then

$$\text{the mass} = \int \rho \, d\mathrm{v}; \tag{2}$$

$d\mathrm{v}$ being an element of the volume, ρ being a function of the coordinates of the place of $d\mathrm{v}$, and the sign of integration denoting the process of summation, whether that involves one, two, or three integrations, according to the dimensions of the body, and the integrations extending through the space occupied by the body.

Density is usually measured by means of comparison with some substance the density of which is assumed to be the unit-density. This latter substance is commonly taken to be distilled water at the temperature 39°.4 Fahrenheit, and under a barometric pressure of 2116.4 lbs. on the square foot; so that by means of this comparison ρ is a number; and the value of it for any given substance is called *the specific density* of that substance. Thus for platinum, $\rho = 21.5$, and this means that, bulk for bulk, and under the stated conditions, platinum contains 21.5 times more matter than distilled water.

The following are examples in which mass is determined, when the law of varying density is given.

Ex. 1. To find the mass of a straight wire or rod, the density of which varies directly as the distance from one end.

Let the end of the rod be taken as the origin, and let a be the length of it; and let the distance of any point of it from that end $= x$; let $\omega =$ the area of a transverse section of it; then $d\mathrm{v} = \omega\, dx$; and $\rho = kx$; therefore

$$\begin{aligned}\text{the mass of the rod} &= \int_0^a k\omega x\, dx \\ &= \frac{k\omega a^2}{2}.\end{aligned}$$

Ex. 2. To find the mass of a circular plate of uniform thickness, the density of which varies as the distance from the centre.

Let τ be the thickness of the plate and a its radius: let the centre of the plate be the origin, and let it be referred to polar coordinates; so that $d\mathrm{v} = \tau r\, dr\, d\theta$: let $\rho = kr$; then

$$\begin{aligned}\text{the mass of the plate} &= \int_0^{2\pi}\int_0^a k\tau r^2\, dr\, d\theta \\ &= \frac{2\pi k\tau a^3}{3}.\end{aligned}$$

If the density is constant, and the thickness varies directly as the distance from the centre; then $\tau = kr$, and we have

$$\begin{aligned}\text{the mass of the plate} &= \int_0^{2\pi}\int_0^a \rho k r^2\, dr\, d\theta \\ &= \frac{2\pi\rho k a^3}{3}.\end{aligned}$$

Ex. 3. The mass of a sphere, the density of which varies inversely as the distance from the centre $= 2\pi\rho a^3$, where ρ is the density of the outside stratum.

Ex. 4. The mass of an ellipsoid composed of shells the principal sections of which are similar ellipses, and the density of which varies as the semi-axis major of the largest principal section of each shell, is equal to $\pi\rho a^2 bc$, where ρ is the density of the outside stratum.

Ex. 5. To determine the bounding curve of a thin ribbon of uniform thickness and density, such that the breadth of it corresponding to each ordinate may be proportional to the mass of the ribbon beyond it.

Let the curve be that delineated in fig. 63. Let the axis of x be vertical, and that of y horizontal. OM $= x$, MP $= y$, OA $= a$. Let τ be the constant thickness of the ribbon, ρ its density; then taking the part of the ribbon on the positive side of the axis of x, the mass of it below MP

$$= \int_{y=0}^{y=y} \tau \rho y \, dx;$$

therefore by the data $\int_{y=0}^{y=y} \tau \rho y \, dx = \mu y;$

$$\therefore \quad y\,dx = k\,dy, \qquad dx = k\frac{dy}{y};$$

$$\therefore \quad y = a^{\frac{x}{k}};$$

the equation to the logarithmic curve. Similarly, if OA$'=a'$, for the curve on the other side we shall have

$$y' = a'^{\frac{x}{k}}.$$

122.] The letter m is usually employed to denote mass, and M to denote the sum of many masses, and consequently the mass of a body, so that M $= \Sigma . m$. Now when many particles occupying points in space are the subjects of our inquiry, there is a certain point in reference to their masses and to their positions which is frequently of great importance towards the simplification of the investigation. Let there be n particles whose masses are respectively $m_1, m_2, \ldots m_n$, and let the places of them be (x_1, y_1, z_1), (x_2, y_2, z_2), ... (x_n, y_n, z_n). If these particles are all equal, and each is equal to the unit-particle, the *mean*, or average, of their distances from a given plane is

$$\frac{p_1 + p_2 + \ldots + p_n}{n},$$

if $p_1, p_2, \ldots$ are the distances of the particles severally from the plane. But if the mass of a particle is m, that particle contains m unit-particles, so that in the preceding formula m of the p's become identical; and thus if all the particles are of masses different or not as the case may be, the formula becomes

$$\frac{m_1 p_1 + m_2 p_2 + \ldots + m_n p_n}{m_1 + m_2 + \ldots + m_n};$$

which we denote by $\frac{\Sigma.mp}{\Sigma.m}$. Hence if $\bar{x}, \bar{y}, \bar{z}$ are the mean distances of the places of the several particles from the planes of (y, z), (z, x), (x, y) respectively,

$$\bar{x} = \frac{\Sigma.mx}{\Sigma.m}, \qquad \bar{y} = \frac{\Sigma.my}{\Sigma.m}, \qquad \bar{z} = \frac{\Sigma.mz}{\Sigma.m}. \tag{3}$$

The point $(\bar{x}, \bar{y}, \bar{z})$ thus defined, and thus determined, is called *the centre of mass*, or *mass-centre*, of the system of particles, and is a definite point in every system; for whatever are the values of the numerators in the preceding expressions, the denominator is a positive quantity, and cannot vanish, so that the expressions cannot take an indeterminate form.

If the system of masses is a body, and is continuous, and the density at any point is ρ, then

$$\bar{x} = \frac{\int \rho\, x\, dv}{\int \rho\, dv}, \qquad \bar{y} = \frac{\int \rho\, y\, dv}{\int \rho\, dv}, \qquad \bar{z} = \frac{\int \rho\, z\, dv}{\int \rho\, dv}; \tag{4}$$

so that the centre of mass of any system of particles is that point whose distance from any plane is equal to the sum of the products of each mass into its distance from that plane divided by the sum of the masses.

Hence, if the centre of mass of a system of material particles is taken as the origin,

$$\Sigma.mx = \Sigma.my = \Sigma.mz = 0; \tag{5}$$

and if the system of particles is a continuous body

$$\int \rho x\, dv = \int \rho y\, dv = \int \rho z\, dv = 0. \tag{6}$$

And here I might proceed to consider the various forms which (3) and (4) take according to the continuous or other distribution of matter, and according to the bounding forms of bodies, and to apply them largely to special cases, and there would be a theoretical advantage in such a method, as it would preserve the generality of the expressions, and this point is of great importance in many subsequent investigations. But as the preceding expressions have been almost universally considered and applied from another point of view, and as there is no practical inconvenience in following that course, I will take it; the number of applications of (3) and (4) will not thereby be lessened; and these remarks will prevent the student from limiting his view of the subject to the restricted aspect which this latter conception of it presents to him.

123.] Of all terrestrial, and indeed of all cosmical matter, as far as our knowledge extends, every particle attracts towards itself every other particle; and all would come into close contact, did not some forces act to hinder them. This property is inherent in cosmical matter, but we know neither the cause of it nor its mode of operation. It is called *gravity*, and its action-line is the line which joins the two particles, and its intensity varies inversely as the square of the distance between the particles, so that if the distance is increased, say, twofold, the attraction is diminished, and is only one-fourth of what it was before. We shall enter on the inquiry into these and kindred subjects hereafter. By reason of this power of attraction the earth attracts all other matter towards itself, and we shall shew hereafter that the resultant attractive force of all the particles of the earth on a particle outside of it varies approximately inversely as the square of the distance of the particle from the earth's centre.

Now of bodies which are the subject of investigation to us, and are near to the earth's surface, the dimensions in all directions are usually so small in comparison of the distance of the body from the centre of the earth that we may, without sensible error, suppose the earth to exert an equal force on all particles of the body which are of equal mass; and as gravity is a force which penetrates matter, and acts with equal effect, whether the particle on which it acts is within a body, or on its bounding surface, or separate, so the effect of it on a body varies as the mass of the body; the amount of this attraction of the earth on a body is called its *weight;* and is thus measured. Let the mass-element of the body be m, and let g be the weight of an unit-mass; that is, g is the amount of the earth's attraction on an unit-mass at the place; then

$$\text{the weight of } m = mg;$$

and if the mass of the body is M,

$$\text{the weight of the body} = \mathrm{M}g:$$

M having been determined by the processes indicated in Art. 121. So that of a body of the most general form, and heterogeneous in structure,

$$\text{the weight} = \iiint \rho g \, dx \, dy \, dz. \tag{7}$$

If the volume of a homogeneous body is v and its density is ρ, then its weight $= \rho g \mathrm{v}$; consequently if $\mathrm{v} = 1$,

$$\text{the weight of an unit-volume} = \rho g; \tag{8}$$

this weight is sometimes called *the specific gravity*, but sometimes and more correctly called *the specific weight* of a substance. It is evidently the product of the specific density, and the weight of the unit-mass at the place.

124.] I have been obliged to limit g to the weight of an unit-mass *at a given place:* for although mass is the same wherever the body may be, yet the weight of it varies from place to place; gravity is not the same at all places of the earth's surface: it increases as we go from the equator, where it has its least value, towards the poles, where it has its greatest value: and this increase is according to the following law given by Clairaut. Let G and g be gravity at the equator, and a place whose latitude is λ, respectively; then

$$g = G\{1+.005133\,(\sin\lambda)^2\}.$$

This increase is due to two causes: (1) the statical attraction of the earth, and (2) the dynamical action of centrifugal force: to the consideration of both these causes we shall return hereafter. And it also changes, as we pass further from the centre of the earth: for bodies external to the earth's gravity decreases in the ratio of the inverse square of the distance from the centre of the earth; also as we pass from the surface of the earth towards the centre, as e. g. down a mine, its intensity decreases, and varies directly as the distance from the centre of the earth. A proof of these propositions will be given hereafter. Gravity also varies according to the nature of the materials of the earth in the neighbourhood of the place where it is considered: its value on an island is different to that on a continent: it is also affected by neighbouring mountains, and in line of action as well as in intensity.

The line of action of it is *vertical,* that is, is perpendicular to the surface of still water. Now although the earth is not quite spherical, so that all verticals do not meet at the centre; yet its radius, about 4000 miles, is so large, compared with the dimensions of any bodies which we shall at present consider to be subject to gravity, that all vertical lines corresponding to molecules of the same body may be reckoned parallel; and therefore all the particles of material bodies may be considered to be acted on by forces whose lines of action are parallel.

Another point also requires some remarks. In these Articles different concrete units are involved. Now the symbols ρ, dv, g are symbols of numbers; and therefore their product is a number; but the quantity which we commence with is volume-element, and that which we end with is weight-element: it remains therefore to seek the source whence this change arises; it is true, as it is convenient, that $d\mathrm{v}$ expresses the *number* of the volume-units, ρ the *number* of mass-units in a volume-unit, and g the *number* of earth's attraction-units in a mass-unit: but how does the result of all this imply weight? In the first place, the process 'multiplication' must be used in a sense wider than its numerical one, so as to include within its subjects of operation quantities of different kinds; and so that the product may be of a kind different to that of either of the multiplicands: and thus the product of two concrete units is a concrete unit of a different kind; the product of the volume-unit and of the density-unit is mass-unit; and the product of the mass-unit and of the earth's attraction-unit is weight-unit; the change of concrete unit therefore arises from the product of the different concrete units; and weight-unit is the product of three different concrete units. The units are of course arbitrary, and therefore we choose those which are most convenient; and thus we take a cubic inch to be the volume-unit; the density of distilled water, at a certain temperature and under certain atmospheric pressure, to be the density unit; and the earth's attraction at a given place on a mass-unit to be the gravity-unit; and by means of these we obtain the weight of a cubic inch of distilled water at a certain place, and compare all other weights with it.

125.] Thus by reason of the earth's attraction every mass-element of the body becomes the source and point of application of a force which varies as the mass of the element; and the action-lines of all these forces are vertical and parallel. Consequently they are subject to the laws of composition of such forces which are investigated in Arts. 79, 80. The resultant is equal to the sum of the components; that is, the weight of the body or system of particles is equal to the sum of the weights of the component particles. Its action-line is vertical. It has also a definite point of application the coordinates of which are assigned by (146) Art. 80. This point is called *the centre of gravity*, being the centre of the parallel forces; and if it is fixed the body will rest in all positions, and every line passing through it is an equilibrium-axis, the equilibrium of the body thus supported being continuous.

Firstly, let the system consist of many material particles separate from each other; let their masses be $m_1, m_2, \dots m_n$, and let the positions of them be $(x_1, y_1, z_1), \dots (x_n, y_n, z_n)$; let the centre of gravity be $(\bar{x}, \bar{y}, \bar{z})$; then as the weights are $m_1 g, m_2 g, \dots m_n g$,

$$\text{R} = \Sigma . mg = g\, \Sigma . m; \tag{9}$$

$$\left.\begin{array}{ll} \bar{x}\, \Sigma . mg = \Sigma . mgx; & \therefore \quad \bar{x}\, \Sigma . m = \Sigma . mx; \\ \bar{y}\, \Sigma . mg = \Sigma . mgy; & \qquad \bar{y}\, \Sigma . m = \Sigma . my; \\ \bar{z}\, \Sigma . mg = \Sigma . mgz; & \qquad \bar{z}\, \Sigma . m = \Sigma . mz; \end{array}\right\} \tag{10}$$

whereby both the resultant and the position of its point of application are known. And from the form of these equations it follows that, in the investigation of the centre of gravity of a system of material particles or bodies, we may, if it is convenient, divide the system into groups, and calculate separately the centre of gravity of each group; and by a similar process deduce from them the centre of gravity of the whole system.

Secondly, let us take the case of many material particles aggregated into a continuous body, so that the symbol of summation becomes that of integration; and let the coordinates to the type volume-element of the body be x, y, z: then the type-force is $\rho g\, d\text{v}$; let $(\bar{x}, \bar{y}, \bar{z})$ be the centre of gravity; then from (146) Art. 80,

$$\left.\begin{array}{l} \bar{x}\int \rho g\, d\text{v} = \int \rho g x\, d\text{v}, \\ \bar{y}\int \rho g\, d\text{v} = \int \rho g y\, d\text{v}, \\ \bar{z}\int \rho g\, d\text{v} = \int \rho g z\, d\text{v}; \end{array}\right\} \tag{11}$$

$\int$ is used on both sides of the equations as a general symbol of summation; and is to be replaced by the symbols of single, double, or triple integration according to the different values of $d\text{v}$, and the integration is to extend through the space occupied by the body.

In reference to these values it is to be observed that the centre of gravity is the point of application of the resultant of all the weights of the several component particles of a body, which resultant is equal to the sum of the separate weights; it is therefore that point at which, if the weight of the whole body acts, an effect is produced the same as that of all the particles of the body taken in combination; or, in other and equivalent words, *the centre of gravity is that point at which, if the body is collected into a material particle, the circumstances of pressure are the same as those of the body in its actual state.*

There are of course many cases where the centre of gravity is known at once, by reason of the symmetry of the figure; thus the centre of gravity of a straight wire or rod, of the same density and thickness throughout, is at the middle point of the rod; the centre of gravity of a circular wire of the same density and thickness throughout is at the centre of the circle: that of a circular or of an elliptical plate of constant thickness and density is at the centre: that of a homogeneous sphere and of a homogeneous ellipsoid is at the centre: and in a similar manner we shall frequently conclude from the symmetry of the figure, that the centre of gravity of a body is in a particular line which can be at once determined.

126.] Since g in (10) and (11) denotes a constant quantity, it may be divided out from both sides of the equations; and if this is done, the results are then identical with (3) and (4), Art. 122; and thus it appears that the centre of gravity always coincides with centre of mass. These points however arise from two different and distinct conceptions; the latter depends on the constitution of the body only, and its position is geometrically derivable from that constitution without any relation to external circumstances; it is independent of the place of the body and of any forces acting on it. The former, on the other hand, involves the conception of the earth's attraction, assumes that the action-lines of the force of gravity which acts on each particle are parallel for all particles, and that these forces are proportional to the masses of the particles. These assumptions are only approximately true; and consequently the point is more truly conceived of as the centre of mass than as the centre of gravity. Although in deference to usage I shall call the point the centre of gravity, yet the place of it will always be determined by the formulae which were investigated by means of its conception as a centre of mass; and I may say that the most important applications of it involve the conception of centre of mass and not that of centre of gravity.

It is also to be observed that as gravity is not the same at different places on the earth, the weight of a given mass is not the same at all places. Mass however is the same at all places; and consequently a certain mass and not a certain weight must be taken as the measure of comparison of other masses. Thus standards of weight, as they are called, are masses and not weights. As the weight however at a given place varies as the mass, two masses may at a given place be compared by means of their weights at that place. Two masses are equal if their weights at the same place are equal, and thus one mass is n times another if the weight of the former is n times that of the latter. Weights are easily compared by means of the balance and its varied forms. Thus these instruments indirectly compare masses: and herein their great value consists; and hence arises the necessity of their perfection. In commerce too, no less than in experimental physics, the comparison of mass and not the comparison of weight is required. Mass is absolute; weight is relative. We shall return to the subject of the comparison of masses at a future stage of the treatise.

SECTION 2.—*The centre of gravity of material lines or wires, straight and curved.*

127.] Let us first consider the centre of gravity of a curved material line or wire, of which the thickness is infinitesimal in comparison of the length.

Let ω = the area of a transverse section of the wire, and ds = a length-element, so that $dv = \omega ds$; let ρ be the density at the point (x, y), and g = the earth's attraction; and let $(\bar{x}, \bar{y}, \bar{z})$ be the coordinates of the centre of gravity; then

$$\left.\begin{aligned} \bar{x}\int \rho g\omega\, ds &= \int \rho g\omega x\, ds, \\ \bar{y}\int \rho g\omega\, ds &= \int \rho g\omega y\, ds, \\ \bar{z}\int \rho g\omega\, ds &= \int \rho g\omega z\, ds. \end{aligned}\right\} \qquad (12)$$

The integrals are of course definite, and the limits are fixed by the conditions of the problem. If the curve of the wire lies approximately wholly in one plane, we may take that to be the plane of (x, y), or of (r, θ), and in that case, the first two of (12) are sufficient to determine the centre of gravity, since $\bar{z} = 0$. If the curve of the wire is of double curvature all three equations are required.

It will be found that in many cases the centre of gravity of a material line is outside of the line; and it is necessary therefore that it should be rigidly connected with it if the wire or rod is to have physical support; but such connection is not necessary for the centre of mass.

128.] **Ex. 1.** To find the centre of gravity of a wire of uniform thickness and density, bent into the form of a quadrant of a circle.

Let the radius of the circle be a; fig. 43; then as $\rho\omega$ and g are constant, they may be divided out, and (12) become

$$\bar{x}\int ds = \int x\, ds, \qquad \bar{y}\int ds = \int y\, ds;$$

$$\text{also} \qquad x^2 + y^2 = a^2;$$

$$\therefore \quad \frac{dx}{y} = \frac{dy}{-x} = \frac{ds}{a};$$

$$\therefore \quad \bar{x}\int_0^a \frac{a\,dx}{(a^2-x^2)^{\frac{1}{2}}} = \int_0^a \frac{ax\,dx}{(a^2-x^2)^{\frac{1}{2}}};$$

$$\bar{x}\left[\sin^{-1}\frac{x}{a}\right]_0^a = \left[-(a^2-x^2)^{\frac{1}{2}}\right]_0^a;$$

$$\bar{x}\frac{\pi}{2} = a; \qquad \bar{x} = \frac{2a}{\pi};$$

$$\bar{y}\int_0^a \frac{a\,dx}{(a^2-y^2)^{\frac{1}{2}}} = \int_0^a a\,dx, \qquad \bar{y}\left[\sin^{-1}\frac{x}{a}\right]_0^a = \left[x\right]_0^a;$$

$$\therefore \quad \bar{y} = \frac{2a}{\pi}.$$

Or thus by means of polar coordinates; $r = a$;

$$\therefore \quad \bar{x}\int_0^{\frac{\pi}{2}} d\theta = \int_0^{\frac{\pi}{2}} a\cos\theta\, d\theta, \qquad \therefore \quad \bar{x} = \frac{2a}{\pi};$$

$$\bar{y}\int_0^{\frac{\pi}{2}} d\theta = \int_0^{\frac{\pi}{2}} a\sin\theta\, d\theta, \qquad \therefore \quad \bar{y} = \frac{2a}{\pi}.$$

Ex. 2. To find the centre of gravity of a wire of constant thickness and density, and bent into the form of a complete cycloid.

Let the starting point of the cycloid be the origin, and let the equation to the curve be

$$x = a\,\text{versin}^{-1}\frac{y}{a} - (2ay-y^2)^{\frac{1}{2}};$$

$$\therefore \quad \frac{dx}{y^{\frac{1}{2}}} = \frac{dy}{(2a-y)^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}};$$

it is evident that the centre of gravity will be in the line perpendicular to the base at its point of bisection; therefore $\bar{x} = \pi a$; and as ρ, g, ω are constant,

$$\bar{y}\int_0^{2a} \frac{dy}{(2a-y)^{\frac{1}{2}}} = \int_0^{2a} \frac{y\,dy}{(2a-y)^{\frac{1}{2}}}; \qquad \therefore \quad \bar{y} = \frac{4a}{3}.$$

For a wire in the form of a semicycloid, $\bar{x} = \frac{4a}{3}$, $\bar{y} = \frac{4a}{3}$.

Ex. 3. To find the centre of gravity of a wire of constant thickness and density, bent into the form of an arc of a circle.

Let the radius of the circle be a; and let the line passing through the middle point (the vertex) of the circular arc and the centre of the circle be the axis of x; then as the arc, fig. 44, is symmetrical with respect to this line, $\bar{y} = 0$. Let the arc BOB′ $= 2s$, and let the chord BB′ $= 2c$, OD $= b$; then

$$y^2 = 2ax - x^2;$$

$$\frac{dy}{a-x} = \frac{dx}{y} = \frac{ds}{a};$$

$$\text{and} \quad \bar{x}\int_0^b \frac{dx}{(2ax-x^2)^{\frac{1}{2}}} = \int_0^b \frac{x\,dx}{(2ax-x^2)^{\frac{1}{2}}};$$

$$\therefore \quad \bar{x} = a - \frac{ac}{s}.$$

Ex. 4. To find the centre of gravity of a wire in the form of a half of one loop of a lemniscate.

Let the equation be $r^2 = a^2 \cos 2\theta$; and let l be the length of the half loop; then

$$\frac{dr}{-a^2 \sin 2\theta} = \frac{r\,d\theta}{a^2 \cos 2\theta} = \frac{ds}{a^2};$$

$$\therefore \quad \bar{x}\,l = \int_0^{\frac{\pi}{4}} r \cos\theta\,ds = \frac{a^2}{2^{\frac{1}{2}}};$$

$$\bar{y}\,l = \int_0^{\frac{\pi}{4}} r \sin\theta\,ds = a^2\,\frac{2^{\frac{1}{2}}-1}{2^{\frac{1}{2}}}.$$

Ex. 5. To find the centre of gravity of a straight rod, the thickness of which varies directly as the distance from one end.

Let the end of the rod whence the variation of the thickness is reckoned be taken as the origin, and the line as the axis of x: then $\omega = \kappa x$; let $a =$ the length of the rod; and we have

$$\bar{x}\int_0^a \rho g \kappa x\,dx = \int_0^a \rho g \kappa x^2\,dx; \qquad \bar{x} = \frac{2a}{3}.$$

Ex. 6. To find the centre of gravity of a straight rod, the density of which varies as the nth power of the distance of each point from a given point in the line of the rod produced.

Let O be the point from which the variation of the density takes place; fig. 45; OA $= a$, OB $= b$, OP $= x$, PQ $= dx$; $\rho = \kappa x^n$; then

$$\bar{x}\int_a^b \kappa\omega g x^n\,dx = \int_a^b \kappa\omega g x^{n+1}\,dx;$$

$$\therefore \quad \bar{x} = \frac{n+1}{n+2}\,\frac{b^{n+2}-a^{n+2}}{b^{n+1}-a^{n+1}}.$$

If $n = -2$, then

$$\bar{x}\int_a^b \kappa\omega g\frac{dx}{x^2} = \int_a^b \kappa\omega g\frac{dx}{x}; \qquad \bar{x} = \frac{ab}{b-a}\log\frac{b}{a}.$$

Ex. 7. To find the centre of gravity of a wire bent into the form of a cycloid, the thickness of which varies directly as the distance from the middle point of the wire.

The middle point of the wire is the highest point of the cycloid; let it be taken as the origin; and let the axis of the cycloid be the axis of x; then $\bar{y} = 0$; let the length of the wire be $8a$; then, see Integral Calculus, Art. 155, Ex. 3, the radius of the base-circle is a; and the equation to the cycloid is

$$s^2 = 8ax;$$

and since $\rho = \kappa s$, we have

$$\bar{x}\int g\omega\kappa s\,ds = \int g\omega\kappa s x\,ds, \qquad \bar{x}\int_0^{4a} s\,ds = \frac{1}{8a}\int_0^{4a} s^3\,ds;$$

$$\bar{x} = a.$$

Ex. 8. Find the curve whose extreme points are (x_0, y_0), (x, y), such that $m\bar{x} = x - x_0$, $n\bar{y} = y - y_0$.

129.] If the wire is in space, having all its elements either in or not in one plane, we must determine all the coordinates of the mass-centre which are given in (12).

Ex. 1. A wire of constant thickness and density is bent into the form of a helix; find its centre of gravity.

Let $a =$ the radius of the base-cylinder; and let the wire commence at the axis of x, that is, at the point $(a, 0, 0)$, see fig. 125, Differential Calculus; and let its end be at (x, y, z); then

$$x = a\cos\phi, \qquad y = a\sin\phi, \qquad z = ka\phi;$$

$$ds = (1+k^2)^{\frac{1}{2}}\,a\,d\phi;$$

$$\bar{x}\int_0^{\phi} \rho g\omega(1+k^2)^{\frac{1}{2}}\,a\,d\phi = \int_0^{\phi} \rho g\omega(1+k^2)^{\frac{1}{2}}\,a^2\cos\phi\,d\phi;$$

$$\bar{x}\phi = a\sin\phi; \qquad \therefore\quad \bar{x} = ka\frac{y}{z};$$

$$\bar{y}\phi = a(1-\cos\phi); \qquad \bar{y} = ka\frac{a-x}{z};$$

$$\bar{z}\phi = \frac{ka\phi^2}{2}; \qquad \bar{z} = \frac{z}{2}.$$

Ex. 2. To find the centre of gravity of the perimeter of a triangle in space, the three sides of which are thin rods of constant thickness and density.

Let the lengths of the sides be l_1, l_2, l_3; and the angular points be $(x_1, y_1, z_1) \dots (x_3, y_3, z_3)$: $\rho =$ the constant density, $\omega =$ the area of a transverse section of the rods: then the centres of gravity of l_1, l_2, l_3 are manifestly at the points

$$\frac{x_2+x_3}{2}, \quad \frac{y_2+y_3}{2}, \quad \frac{z_2+z_3}{2},$$
$$\cdots\cdots\cdots$$
$$\frac{x_1+x_2}{2}, \quad \frac{y_1+y_2}{2}, \quad \frac{z_1+z_2}{2};$$

and therefore by the formulæ (10),

$$\bar{x}(l_1+l_2+l_3) = \frac{1}{2}\{l_1(x_2+x_3)+l_2(x_3+x_1)+l_3(x_1+x_2)\},$$

$$\bar{y}(l_1+l_2+l_3) = \frac{1}{2}\{l_1(y_2+y_3)+l_2(y_3+y_1)+l_3(y_1+y_2)\},$$

$$\bar{z}(l_1+l_2+l_3) = \frac{1}{2}\{l_1(z_2+z_3)+l_2(z_3+z_1)+l_3(z_1+z_2)\}.$$

By a similar process the centre of gravity of the perimeter of a polygon formed by heavy rods in space may be determined.

130.] The determination of the centres of gravity of material lines or wires also suggests the following problem, which is solved by the Calculus of Variations:

To find the equation to the curve into which a thin heavy rod or string of uniform thickness and density and of given length is to be bent, so that its ends being fixed at two given points, the centre of gravity may be in the lowest possible position.

Let the axis of z be parallel to the direction of gravity; and let $2c$ be the length of the rod; and (x_1, y_1, z_1) and (x_0, y_0, z_0) the ends of the line; then

$$\int_0^1 ds = 2c, \tag{13}$$

$$\bar{z}\, 2c = \int_0^1 z\, ds; \tag{14}$$

and $\bar{z}$ will be a maximum or a minimum according as the plane of (x, y) is above or below the centre of gravity of the suspended wire; in either case, $\delta\bar{z} = 0$; therefore from (14),

$$2c\delta\bar{z} = 0 = \int_0^1 \delta.z\, ds$$

$$2c\delta\bar{z} = 0 = \int_0^1 (z\delta.ds + ds\,\delta z)$$
$$= \int_0^1 \left\{ z\left(\frac{dx}{ds}\, d.\delta x + \frac{dy}{ds}\, d.\delta y + \frac{dz}{ds}\, d.dz\right) + ds\, dz \right\},$$

$$2c\delta\bar{z} = 0 = \left[z\left(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\right)\right]_0^1$$

$$-\int_0^1 \left\{d.z\frac{dx}{ds}\delta x + d.z\frac{dy}{ds}\delta y + \left(d.z\frac{dz}{ds} - ds\right)\delta z\right\}. \quad (15)$$

Of this quantity the first part vanishes by reason of the limits being fixed; also from (13),

$$\delta.2c = 0 = \delta\int_0^1 ds$$

$$= \int_0^1 \left\{\frac{dx}{ds}d.\delta x + \frac{dy}{ds}d.\delta y + \frac{dz}{ds}d.\delta z\right\}$$

$$= \left[\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\right]_0^1$$

$$-\int_0^1 \left\{d.\frac{dx}{ds}\delta x + d.\frac{dy}{ds}\delta y + d.\frac{dz}{ds}\delta z\right\};$$

and of this quantity the first part vanishes by reason of the limits being fixed; and as the second part is to consist with the second part of (15), we have

$$\frac{d.z\dfrac{dx}{ds}}{d.\dfrac{dx}{ds}} = \frac{d.z\dfrac{dy}{ds}}{d.\dfrac{dy}{ds}} = \frac{d.z\dfrac{dz}{ds} - ds}{d.\dfrac{dz}{ds}} = \lambda \text{ (say)}; \quad (16)$$

from the first two members of which equality we have

$$\frac{d.\dfrac{dx}{ds}}{\dfrac{dx}{ds}} = \frac{d.\dfrac{dy}{ds}}{\dfrac{dy}{ds}}; \qquad \therefore \quad \frac{dx}{x_1 - x_0} = \frac{dy}{y_1 - y_0};$$

the constants being introduced consistently with the curve passing through (x_1, y_1, z_1) and (x_0, y_0, z_0);

$$\therefore \quad \frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0};$$

whence it follows that the curve is a plane curve, and is in a plane perpendicular to that of (x, y). Let the plane of (x, z) be taken so as to contain the curve; then $y = 0$; and from the first of (16) we have

$$d.z\frac{dx}{ds} = \lambda d.\frac{dx}{ds};$$

$$\therefore \quad zd.\frac{dx}{ds} + dz\frac{dx}{ds} = \lambda d.\frac{dx}{ds}, \qquad \frac{dz}{z - \lambda} + \frac{d.\dfrac{dx}{ds}}{\dfrac{dx}{ds}} = 0;$$

$$\therefore \quad \log(z-\lambda)+\log\frac{dx}{ds} = \log a; \qquad \text{or,} \quad (z-\lambda)\frac{dx}{ds} = a,$$

where a is an arbitrary constant of integration; and since $ds^2 = dx^2+dz^2$,

$$\frac{dx}{a} = \frac{dz}{\{(z-\lambda)^2-a^2\}^{\frac{1}{2}}};$$

$$\therefore \quad z-\lambda = \frac{b}{2}\left\{e^{\frac{x-b}{a}}+e^{-\frac{x-b}{a}}\right\}; \tag{17}$$

where b is another arbitrary constant of integration; and the three undetermined constants a, b, λ may be determined by the conditions of the curve passing through two given points, and of the length of the curve between those points being given. The equation (17) is that of the catenary, the properties of which will be investigated hereafter; and the result is important, inasmuch as it shews that the curve in which a perfectly flexible and inextensible heavy string will hang when suspended from two fixed points is also that of which the centre of gravity has the lowest possible position.

The form of the problem as stated in equation (14) shews that it is identical with the determination of the form of the curve of given length, which passes through two given points, and revolving about a line in the same plane with the two points generates a surface whose area is a maximum. This problem is solved in Art. 326, Vol. II.

131.] The formulæ given in (12) lead also to the following theorem. If the wire or line is of constant thickness and density, and is infinitesimally then, then

$$\bar{y}\int ds = \int y\,ds;$$

$$\therefore \quad 2\pi\bar{y}\times s = \int 2\pi y\,ds. \tag{18}$$

Now if the plane curve whose length is s revolves about the axis of x, and generates thereby a thin shell (or surface) of revolution, the right-hand member of (18) is the area of the surface generated; see Art. 232, Vol. II; and the left-hand member of (18) is the product of the length of the generating line and of the path described during an entire revolution by the centre of gravity of it; hence we conclude that,

If a plane curve lies wholly on one side of a line in its own plane, and revolving about that line generates thereby a surface of revolution, the area of the surface is equal to the (geometrical) product of the length of the revolving line, and of the path described by its centre of gravity.

This theorem is one of those known by the name of the Theorems of Pappus or of Guldinus; it is a geometrical relation existing between a curve, the surface which it generates by revolving about a line in its own plane, and the distance of its centre of gravity from that line; the curve must not intersect the axis of x; for if it does, y will change its sign; and (18) may be an inexact expression; the generating curve may however be a closed figure. Also as (18) expresses the equality of the two sides of the equation for a whole revolution, so will a similar theorem be true for any part of a revolution. Two or three examples are subjoined.

Ex. 1. A circle of radius a, revolves about an axis in its own plane at a distance c from its centre; it is required to find the area of the surface of the ring thereby generated.

The circumference of the generating curve is $2\pi a$; and as the centre of gravity of it is at its centre, the path described by the centre of gravity during a complete revolution is $2\pi c$;

$$\therefore \quad \text{the area of the surface of the ring} = 4\pi^2 ac.$$

Ex. 2. A right-angled triangle revolves about its hypothenuse, and its sides thereby describe a surface; it is required to find the area of the surface described.

Let a, b be the sides of the triangle, and h the length of the perpendicular from the right angle to the hypothenuse, so that

$$\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2};$$

then the area of the surface $= \pi(a+b)h$

$$= \frac{\pi(a+b)ab}{(a^2+b^2)^{\frac{1}{2}}}.$$

Also if the area of a surface is known, and the length of the generating line is known, the distance of the centre of gravity of the line from the axis of revolution may be determined. Thus, the surface of a sphere of radius $a = 4\pi a^2$, the length of a semicircle $= \pi a$; therefore from (18),

$$2\pi\bar{y} \times \pi a = 4\pi a^2;$$

$$\therefore \quad \bar{y} = \frac{2a}{\pi}.$$

SECTION 3.—*Centre of gravity of thin plates and curved shells, bounded by lines straight or curved.*

132.] In the next place let us consider a plane plate of infinitesimal thickness, bounded by curved or straight lines, and refer it to rectangular coordinates. Let the plane of the plate be that of (x, y) and let the coordinates of any element in the plane surface of the plate be x, y; so that the area of the element E is $dx\,dy$; see fig. 46. Let the thickness of the plate at E $= \tau$; then

$$d\mathrm{v} = \tau\,dx\,dy;$$

and the first two of equations (11) become

$$\left.\begin{aligned}\bar{x}\iint\rho g\tau\,dy\,dx &= \iint\rho g\tau x\,dy\,dx,\\ \bar{y}\iint\rho g\tau\,dy\,dx &= \iint\rho g\tau y\,dy\,dx;\end{aligned}\right\} \qquad (19)$$

the integrations extending over the area assigned by the problem.

Ex. 1. It is required to find the centre of gravity of a thin plate of uniform thickness and density, bounded by a parabola, its axis, and an ordinate; fig. 46.

Let OA $= a$, AB $= b$; $\tau =$ the thickness of the plate, $\rho =$ the density: then the equation to the parabola is $ay^2 = b^2x$; let $a\mathrm{Y}^2 = b^2x$; so that we have

$$\bar{y}\int_0^a\int_0^{\mathrm{Y}} dy\,dx = \int_0^a\int_0^{\mathrm{Y}} x\,dy\,dx,$$

$$\bar{x}\int_0^a x^{\frac{1}{2}}\,dx = \int_0^a x^{\frac{3}{2}}\,dx;$$

$$\therefore\quad \bar{x} = \frac{3}{5}a:$$

$$\bar{y}\int_0^a\int_0^{\mathrm{Y}} dy\,dx = \int_0^a\int_0^{\mathrm{Y}} y\,dy\,dx;$$

$$\therefore\quad \bar{y} = \frac{3b}{8}.$$

Ex. 2. To find the centre of gravity of a thin plate of uniform thickness and density in the form of an elliptic quadrant.

Let $\qquad \mathrm{Y} = \frac{b}{a}(a^2-x^2)^{\frac{1}{2}};$

then $\qquad \bar{x}\int_0^a\int_0^{\mathrm{Y}} dy\,dx = \int_0^a\int_0^{\mathrm{Y}} x\,dy\,dx,$

$$\bar{x}\int_0^a (a^2-x^2)^{\frac{1}{2}}\,dx = \int_0^a x\,(a^2-x^2)^{\frac{1}{2}}\,dx\,;$$

$$\therefore\quad \bar{x} = \frac{4a}{3\pi}:$$

$$\bar{y}\int_0^a\int_0^{\mathrm{Y}} dy\,dx = \int_0^a\int_0^{\mathrm{Y}} y\,dy\,dx\,;$$

$$\therefore\quad \bar{y} = \frac{4b}{3\pi}.$$

Hence for a thin plate in the form of a quadrant, the position of the mass-centre in reference to the centre of the circle is given by

$$\bar{x} = \bar{y} = \frac{4a}{3\pi}.$$

Ex. 3. To find the centre of gravity of a thin triangular plate of constant thickness and density.

Let τ be the thickness of the plate, and $\rho =$ the density. Take the angle O, fig. 47, for the origin, and the sides OA, OB for the coordinate axes; $\mathrm{OA} = a$, $\mathrm{OB} = b$, so that the equation to AB is

$$\frac{x}{a}+\frac{y}{b} = 1.$$

Let the angle at $\mathrm{O} = \omega$; then the area of the surface at E $= dx\,dy\sin\omega$; $d\mathrm{V} = \tau\,dx\,dy\sin\omega$. Then if

$$\mathrm{Y} = \frac{b}{a}(a-x),$$

the equations of moments about the axes are

$$\bar{x}\sin\omega\int_0^a\int_0^{\mathrm{Y}} dy\,dx\sin\omega = \int_0^a\int_0^{\mathrm{Y}} x\,(\sin\omega)^2\,dy\,dx,$$

$$\bar{y}\sin\omega\int_0^a\int_0^{\mathrm{Y}} dy\,dx\sin\omega = \int_0^a\int_0^{\mathrm{Y}} y\,(\sin\omega)^2\,dy\,dx\,;$$

$$\therefore\quad \bar{x} = \frac{a}{3},\qquad \bar{y} = \frac{b}{3};$$

the centre of gravity therefore is situated on the line passing through O and bisecting AB, at a distance from O equal to two-thirds of the bisecting line; and as the result is independent of the particular angle, it is equally true for all the angles; and therefore the centre of gravity of a triangular thin plate is at the point of intersection of the three bisectors of the sides drawn from the opposite angles. This is also manifest from the following reason: let OAB be a triangular plate, fig. 48; and let OC be drawn from O to C, the middle point of the opposite side AB;

let us imagine the plate to be divided into a series of thin slices by lines parallel to AB; then the centre of gravity of each of these slices will be at its middle point, that is, at its intersection with OC. Imagine therefore each slice to be condensed into its centre of gravity; there is then a series of particles of increasing weight arranged along the line OC, the law of increase being that of the distance directly, because PP′ varies as OM; if therefore OM $= x$, and OC $= h$, we have from (19)

$$\bar{x}\int_0^h x\,dx = \int_0^h x^2\,dx;$$

$$\therefore \quad \bar{x} = \frac{2}{3}h.$$

Hereby also we conclude that if the coordinates to the angles of a triangular plate in space are x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3; then

$$\bar{x} = \frac{x_1+x_2+x_3}{3},$$

$$\bar{y} = \frac{y_1+y_2+y_3}{3},$$

$$\bar{z} = \frac{z_1+z_2+z_3}{3}.$$

Ex. 4. If a thin plate is in the form of a complete cycloid, the distance of the centre of gravity from the vertex is $\frac{7a}{6}$.

Ex. 5. Of a thin plate bounded by a cissoid and its asymptote, the distance of the centre of gravity from the cusp is five-sixths of the diameter of the base-circle.

Ex. 6. The centre of gravity of a thin plate bounded by the witch of Agnesi is at a distance from the asymptote equal to the eighth part of the diameter of the base circle.

Ex. 7. To find the centre of gravity of a cycloidal plate, the thickness of which varies as the nth power of the distance from the base, and of which the density is constant.

In this case taking the starting point as the origin, and the base as the axis of x,

$$x = a\,\text{versin}^{-1}\frac{y}{a} - (2ay-y^2)^{\frac{1}{2}};$$

$$\therefore \quad dx = \frac{y\,dy}{(2ay-y^2)^{\frac{1}{2}}}.$$

Let $\tau = hy^n =$ thickness, $\rho =$ density; it is plain that $\bar{x} = \pi a$;

and $$\bar{y}\int_0^{2\pi a}\int_0^y y^n\,dy\,dx = \int_0^{2\pi a}\int_0^y y^{n+1}\,dy\,dx;$$

$$\therefore\quad \bar{y}\int_0^{2\pi a}\frac{y^{n+1}}{n+1}\,dx = \int_0^{2\pi a}\frac{y^{n+2}}{n+2}\,dx,$$

$$\bar{y}\int_0^{2a}\frac{y^{n+2}\,dy}{(2ay-y^2)^{\frac{1}{2}}} = \frac{n+1}{n+2}\int_0^{2a}\frac{y^{n+3}\,dy}{(2ay-y^2)^{\frac{1}{2}}}$$

$$= \frac{n+1}{n+2}\,\frac{2n+5}{n+3}\,a\int_0^{2a}\frac{y^{n+2}\,dy}{(2ay-y^2)^{\frac{1}{2}}};$$

$$\therefore\quad \bar{y} = \frac{n+1}{n+2}\,\frac{2n+5}{n+3}\,a.$$

Ex. 8. Find the centre of gravity of a thin plate contained by an ellipse, and the chord joining the extremities of the two principal axes.

Ex. 9. Find the centre of gravity of a thin plate contained by a parabola and a straight line through the vertex.

Ex. 10. If $x = m\bar{x}$, where x is the abscissa to the bounding ordinate of a thin plate contained between the axis of x, the origin and the bounding curve, the equation to the bounding curve is $$\left(\frac{x}{a}\right)^{2-m} = \left(\frac{y}{b}\right)^{m-1}.$$

133.] If the plane surface of the plate is referred to polar coordinates, and rectangular coordinates are retained for the centre of gravity, then the area of the surface-element of the plate is $r\,dr\,d\theta$, and $x = r\cos\theta$, $y = r\sin\theta$, so that the equations (19) become

$$\left.\begin{aligned}\bar{x}\iint\rho g\tau\, r\,dr\,d\theta &= \iint\rho g\tau\, r^2\cos\theta\,dr\,d\theta,\\ \bar{y}\iint\rho g\tau\, r\,dr\,d\theta &= \iint\rho g\tau\, r^2\sin\theta\,dr\,d\theta.\end{aligned}\right\}\qquad(20)$$

Ex. 1. To find the centre of gravity of a plate in the form of a sector of a circle, the thickness of which varies directly as the distance from the centre of the circle.

Let a = radius of circle, 2α = the angle which the sector subtends at the centre; and let the axis of x be the line bisecting the angle 2α, so that $\bar{y} = 0$; then $\tau = kr$, and we have

$$\bar{x}\int_{-\alpha}^{\alpha}\int_0^a r^2\,dr\,d\theta = \int_{-\alpha}^{\alpha}\int_0^a r^3\cos\theta\,dr\,d\theta;$$

$$\therefore\quad \bar{x} = \frac{3a}{4}\,\frac{\sin\alpha}{\alpha}.$$

Ex. 2. To find the centre of gravity of a thin plate of uniform thickness and density in the form of the loop of the lemniscata.

The equation to the bounding curve is

$$r^2 = a^2 \cos 2\theta;$$

and as the loop is symmetrical with respect to the axis of x, $\bar{y} = 0$. Let $\mathrm{r} = a(\cos 2\theta)^{\frac{1}{2}}$; then from (20),

$$\bar{x}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\mathrm{r}} r\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\mathrm{r}} r^2\cos\theta\,dr\,d\theta,$$

$$\bar{x}\frac{a^2}{2} = \frac{a^3}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos 2\theta)^{\frac{3}{2}}\cos\theta\,d\theta$$

$$= \frac{a^3}{3}2^{\frac{3}{2}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left\{\frac{1}{2}-(\sin\theta)^2\right\}^{\frac{3}{2}}d.\sin\theta;$$

let $k^2 = \frac{1}{2}$, and $\sin\theta = x$; then

$$\bar{x} = \frac{2^{\frac{5}{2}}a}{3}\int_{-k}^{k}(k^2-x^2)^{\frac{3}{2}}dx,$$

$$= \frac{2^{\frac{5}{2}}a}{3}\frac{3}{8}\frac{\pi}{4}$$

$$= \frac{\pi a}{2^{\frac{5}{2}}}.$$

Ex. 3. The centre of gravity of a thin plate bounded by the curve whose equation is $r = a(1+\cos\theta)$ is at a distance from the origin equal to $\frac{5a}{6}$.

Ex. 4. A thin plate in the form of a circular sector is generated by the motion of one of its bounding radii; if a is the radius, prove that the locus of the centre of gravity is

$$r = \frac{2a}{3}\frac{\sin\theta}{\theta}.$$

134.] Centre of gravity of a thin shell of revolution.

Let the axis of revolution be the axis of x; and let the equation to the curve, by the revolution of which the exterior surface of the shell is generated, be $y = f(x)$: let $\tau =$ the thickness of the shell; $\rho =$ the density; $g =$ the earth's attraction; and imagine the shell, see fig. 49, to be divided into a series of circular rings or annuli of breadth dx by means of planes perpendicular to the axis of revolution, and at an infinitesimal distance apart: then, if ds is a length-element of the generating curve, the volume of any one of these rings corresponding to a point (x, y) on the generating curve is $2\pi y\tau ds$; and therefore the weight of it is $2\pi\rho g\tau y\, ds$: now imagine this weight to be condensed into a point at the centre of gravity of the ring, which is at M on the axis of x: the circumstances of pressure are not hereby changed: and let us imagine the weight of each ring to be similarly collected at its centre of gravity; then we have a series of weights arranged along the line Ox, of variable magnitude, the law of variation depending on the equation of the generating curve: but such that the weight at the distance x is equal to $2\pi y\rho g\tau\, ds$: hence we have to find the centre of gravity of this rod of variable density; and therefore, by virtue of equations (12),

$$\bar{x}\int 2\pi\rho g\tau y\, ds = \int 2\pi\rho g\tau xy\, ds,$$

and cancelling $2\pi g$,

$$\bar{x}\int \rho\tau y\, ds = \int \rho\tau xy\, ds. \tag{21}$$

Ex. 1. To find the centre of gravity of a thin shell of uniform thickness and density, the exterior surface of which is generated by the revolution of a quadrant of a circle about one of its bounding radii.

Let $\tau =$ thickness of shell; $\rho =$ density; then, fig. 50, the equation to the generating curve is

$$x^2 + y^2 = a^2;$$

$$\therefore \quad \frac{dx}{y} = \frac{-dy}{x} = \frac{ds}{a};$$

$$\therefore \quad \bar{x}\int_0^a a\, dx = \int_0^a ax\, dx;$$

$$\bar{x} = \frac{a}{2}.$$

This result is also manifest by the method of infinitesimals: in Vol. I (Differential Calculus), Art. 24, Ex. 7, it is shewn that each zone of the shell is equal to the corresponding zone of the cylinder of the same thickness circumscribing the spherical shell; and therefore as these zones are equal and equivalent as to the position of their centres of gravity, the latter may replace the former, and the centre of gravity of the hemispherical shell is the same as that of the cylindrical shell; and this latter is evidently on OA in the middle point of OA.

Ex. 2. To find the centre of gravity of a thin right conical shell of uniform thickness and density.

Let $\tau =$ the thickness of the shell; $\rho =$ the density; and let the equation to the generating straight line be

$$y = ax;$$

let the altitude of the shell $= a$: then $ds^2 = (1+a^2)\,dx^2$; and from (21) we have

$$\bar{x}\int_0^a x\,dx = \int_0^a x^2\,dx,$$

$$\bar{x} = \frac{2a}{3}.$$

This is also manifest by the following reasoning: the conical shell may be imagined to be resolved into a series of triangular plates all the vertices of which meet at the vertex of the cone, and the bases of which form the circular base of the conical shell: now the centre of gravity of a triangular plate is on the line which is drawn from the vertex to the middle point of the base, and is at a distance from the vertex equal to two-thirds of that line; and therefore the centre of gravity of the shell is on the axis at a distance from the vertex equal to two-thirds of the axis.

And suppose the thickness of the conical shell to vary as the distance from the vertex: then $\rho = k(1+a^2)^{\frac{1}{2}}x$;

$$\therefore \quad \bar{x}\int_0^a x^2\,dx = \int_0^a x^3\,dx,$$

$$\bar{x} = \frac{3a}{4}.$$

Ex. 3. To find the centre of gravity of a thin shell of uniform thickness and density formed by the revolution about its base of a wire bent into a semi-cycloid.

The equation to the generating curve is

$$x = a\,\text{versin}^{-1}\frac{y}{a} - (2ay-y^2)^{\frac{1}{2}};$$

$$\therefore \quad \frac{dx}{y} = \frac{dy}{(2ay-y^2)^{\frac{1}{2}}} = \frac{ds}{(2ay)^{\frac{1}{2}}};$$

$$\bar{x}\int_0^{2a}\frac{y\,dy}{(2a-y)^{\frac{1}{2}}} = \int_0^{2a}\frac{xy\,dy}{(2a-y)^{\frac{1}{2}}};$$

$$\therefore \quad \bar{x} = \frac{26a}{15}.$$

Ex. 4. The centre of gravity of a thin shell formed by the revolution of a semi-cycloidal wire about its axis is at a distance from the vertex

$$= \frac{2a}{15}\,\frac{15\pi-8}{3\pi-4}.$$

Ex. 5. If $\bar{x}$ determines the place of the centre of gravity of a thin shell formed by the revolution about the x-axis of a thin wire, of which the limiting abscissæ are 0 and x, and if $m\bar{x} = nx$, shew that the differential equation of the wire-curve is

$$y\,dy = \left\{ k^2 x^{\frac{4n-2m}{m-n}} - y^2 \right\}^{\frac{1}{2}} dx.$$

What curves are expressed (1) when $m = 2n$; (2) when $2m = 3n$?

135.] Centre of gravity of a thin curved shell.

Lastly, let us investigate the coordinates of the centre of gravity of a thin curved shell; of which let the thickness $= \tau$, the density $= \rho$; and let the equation to the bounding surface of the shell be $\mathrm{F}(x, y, z) = 0$. Then using the ordinary symbols, if $d\mathrm{A}$ is the surface-element at (x, y, z), $d\mathrm{V} = \tau\, d\mathrm{A}$; and

$$\begin{aligned} d\mathrm{A} &= \frac{\mathrm{Q}}{\mathrm{U}}\,dy\,dz = \frac{\mathrm{Q}}{\mathrm{V}}\,dz\,dx = \frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy, \qquad (22) \\ &= \{1+p^2+q^2\}^{\frac{1}{2}}\,dx\,dy; \end{aligned}$$

so that, taking for $d\mathrm{A}$ the last value of (22), (11) become,

$$\left.\begin{aligned} \bar{x}\iint \rho g\tau\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy &= \iint \rho g\tau x\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy, \\ \bar{y}\iint \rho g\tau\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy &= \iint \rho g\tau y\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy, \\ \bar{z}\iint \rho g\tau\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy &= \iint \rho g\tau z\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy. \end{aligned}\right\} \qquad (23)$$

If the surface of the shell is more conveniently referred to that system of polar coordinates in space which is explained in Art. 165, Vol. II (Integral Calculus), the general equations (11) instead of taking the form (23) will be modified according to it.

Ex. 1. To find the centre of gravity of the octant of a thin spherical shell of uniform thickness and density.

$$x^2+y^2+z^2 = a^2;$$

$$\mathrm{U} = 2x, \qquad \mathrm{V} = 2y, \qquad \mathrm{W} = 2z;$$

$$\begin{aligned} \therefore\quad \mathrm{Q}^2 &= 4(x^2+y^2+z^2) \\ &= 4a^2; \end{aligned}$$

so that if $(a^2-x^2)^{\frac{1}{2}} = \mathrm{Y}$, we have

$$\bar{x}\int_0^a\int_0^Y \frac{a\,dy\,dx}{(a^2-x^2-y^2)^{\frac{1}{2}}} = \int_0^a\int_0^Y \frac{ax\,dy\,dx}{(a^2-x^2-y^2)^{\frac{1}{2}}};$$

$$\bar{x}\int_0^a \frac{\pi}{2}\,dx = \int_0^a \frac{\pi}{2}\,x\,dx;$$

$$\therefore\quad \bar{x} = \frac{a}{2};$$

$$\bar{y}\int_0^a\int_0^Y \frac{a\,dy\,dx}{(a^2-x^2-y^2)^{\frac{1}{2}}} = \int_0^a\int_0^Y \frac{ay\,dy\,dx}{(a^2-x^2-y^2)^{\frac{1}{2}}};$$

$$\bar{y} = \frac{a}{2};$$

$$\bar{z}\int_0^a\int_0^Y \frac{a\,dy\,dx}{(a^2-x^2-y^2)^{\frac{1}{2}}} = \int_0^a\int_0^Y a\,dy\,dx;$$

$$\bar{z} = \frac{a}{2}.$$

Suppose the thickness of the shell to vary as the z-ordinate to any point of it; then $\tau = kz$, and

$$\bar{x}\int_0^a\int_0^Y ak\,dy\,dx = \int_0^a\int_0^Y akx\,dy\,dx;$$

$$\bar{x} = \frac{4a}{3\pi};$$

$$\bar{y}\int_0^a\int_0^Y ak\,dy\,dx = \int_0^a\int_0^Y aky\,dy\,dx;$$

$$\bar{y} = \frac{4a}{3\pi};$$

$$\bar{z}\int_0^a\int_0^Y ak\,dy\,dx = \int_0^a\int_0^Y ak\,(a^2-x^2-y^2)^{\frac{1}{2}}\,dy\,dx;$$

$$\bar{z} = \frac{2a}{3}.$$

136.] The following theorem, due to Pappus, expresses a relation between a plane area, the volume of the solid generated by it as it revolves about a line on its own plane, and the distance of the centre of gravity of the area from the axis, whereby, when any two of these quantities are given, we are able to discover the third.

Let the revolving area be of constant density and thickness, and be so thin as to be conceived to be a geometrical surface; then, if $\bar{y}$ is the distance of the centre of gravity of this area from the axis of x, we have,

$$\bar{y}\iint dy\,dx = \iint y\,dy\,dx;$$

$$\therefore \quad 2\pi\bar{y} \times \iint dy\,dx = \iint 2\pi y\,dy\,dx. \qquad (24)$$

Now these integrals being definite, the second factor of the left-hand member of the equation expresses the area in the plane (x, y), and the first factor is the length of the path described by the centre of gravity of that area, as it revolves through four right angles about the axis of x: and because $dy\,dx$ is the area-element, and $2\pi y$ is the path described by the area-element during a complete revolution of the area about the axis of x, the right-hand member is the product of all the area-elements of the given area and of their paths, and is therefore the volume described by the area during a complete revolution: if therefore the curve lies wholly on the same side of the axis of x, so that y does not change sign, the above equation expresses the following theorem:

If a plane area, lying wholly on the same side of a line in its own plane, revolves about that line, and thereby generates a solid of revolution, the volume of the solid thus generated is equal to the (geometrical) product of the revolving area and of the path described by its centre of gravity during the revolution.

As (24) is true for the whole revolution, a similar theorem is also true for any part of the revolution: and if the generating area is such as that described in fig. 46, where the axis of x is one of the bounding lines, then the limits of the y-integration in (24) are the ordinate to the curve and zero: therefore

$$2\pi\bar{y}\int y\,dx = \int \pi y^2\,dx,$$

and the right-hand member is the ordinary expression for the volume of a solid of revolution. In other cases the limits of y are given by the geometrical conditions of the problem.

Ex. 1. An ellipse revolves about a line in its own plane, the perpendicular distance of which from the centre is equal to c; it is required to find the volume of the ring generated during a complete revolution.

Let a and b be the semi-axes of the generating ellipse; then the generating area $= \pi ab$; and as $2\pi c$ is the path described by the centre of gravity,

$$\text{the volume} = 2\pi^2 abc.$$

It will be observed that the volume is the same, whatever direction the axis of revolution has with respect to the axes of the ellipse, provided that the perpendicular distance from the centre to the axis of revolution is the same.

Ex. 2. The volume of a sphere of radius a is $\frac{4\pi a^3}{3}$; and the area of a semicircle is $\frac{\pi a^2}{2}$: it is required to deduce from these data the position of the centre of gravity of the semicircle.

Let $\bar{y}$ be the distance of the centre of gravity of the semicircle from the diameter; then considering it as the generating area of the sphere, we have

$$2\pi\bar{y} \times \frac{\pi a^2}{2} = \frac{4\pi a^3}{3}; \qquad \therefore \quad \bar{y} = \frac{4a}{3\pi};$$

and by reason of the symmetry, the centre of gravity is on the line which is perpendicular to the diameter through the centre of the circle.

SECTION 4.—*Centre of gravity of heavy bodies bounded by plane and curved surfaces.*

137.] Before I proceed to the general case, I will consider that of a solid bounded by a surface of revolution, and refer the body to the axis of revolution as the axis of x: let the equation to the generating curve of the bounding surface be $y = f(x)$. Imagine the solid, (see fig. 51,) to be divided into thin circular slices by planes at an infinitesimal distance apart and perpendicular to the axis of revolution: of these let the circular slice PP′Q′Q be the type, and let OM $= x$, MN $= dx$, so that dx is the thickness of it. Of this slice take a particle at a distance r from the axis, and so that the plane passing through Ox and that particle may be inclined at an angle θ to the plane passing through Ox and Oy; then the volume of the element is equal to $r\,d\theta\,dr\,dx$. Let $\rho =$ the density of the body at the particle, then the mass-element $= \rho r\,dr\,d\theta\,dx$, and the weight-element $= \rho g r\,dr\,d\theta\,dx$.

Now if the constitution of the body as to density is symmetrical with respect to the axis of revolution, the centre of gravity is plainly on the axis of x, and therefore we have to find only $\bar{x}$; and we have from (11)

$$\bar{x}\int\!\!\int\!\!\int \rho g r\,d\theta\,dr\,dx = \int\!\!\int\!\!\int \rho g x r\,d\theta\,dr\,dx; \tag{25}$$

and performing the θ-integration through a whole revolution, so as to obtain the required result for a ring of radius r, and observing that the symmetry of the body renders ρ independent of θ, we have, dividing out $2\pi g$,

$$\bar{x}\iint \rho r\,dr\,dx = \iint \rho x r\,dr\,dx. \qquad (26)$$

And if the density is uniform throughout a complete slice, we may perform the r-integration between $r = 0$, and $r = y$, where y is the ordinate to the generating curve: and (26) becomes

$$\bar{x}\int \rho y^2\,dx = \int \rho y^2 x\,dx; \qquad (27)$$

the limits of integration depending on the circumstances of the problem.

Ex. 1. To find the centre of gravity of a paraboloid of revolution of uniform density, the length of whose axis is c.

Let the equation to the generating parabola be $y^2 = 4ax$; therefore from (27), as ρ is constant,

$$\bar{x}\int_0^c 4ax\,dx = \int_0^c 4ax^2\,dx; \qquad \therefore \quad \bar{x} = \frac{2}{3}c.$$

Ex. 2. To find the centre of gravity of a portion of a prolate spheroid of uniform density, the length of whose axis measured from the vertex is c.

Let the equation to the generating curve of the bounding surface be

$$y^2 = \frac{b^2}{a^2}(2ax - x^2);$$

then, as ρ is constant, (27) becomes

$$\bar{x}\int_0^c (2ax - x^2)\,dx = \int_0^c (2ax - x^2)\,x\,dx;$$

$$\therefore \quad \bar{x} = \frac{c}{4}\,\frac{8a - 3c}{3a - c}.$$

Thus for a hemi-spheroid, $c = a$, and we have

$$\bar{x} = \frac{5a}{8}.$$

As b does not enter into either of the last two values, they are the same for a spherical segment and for a hemisphere.

Ex. 3. To find the centre of gravity of a double convex lens of uniform density.

Let the equations to the generating circles of the two intersecting spheres be, fig. 52,

$$x^2 + y^2 = a^2, \qquad (x - c)^2 + y^2 = b^2,$$

where OA $= a$, BC $= b$, OC $= c$; then the equation to the plane of intersection of the spheres is

$$x = \frac{a^2+c^2-b^2}{2c} = k \text{ (say)};$$

then from (27),

$$\bar{x}\left\{\int_k^a (a^2-x^2)\,dx + \int_{c-b}^k \{b^2-(x-c)^2\}\,dx\right\}$$
$$= \int_k^a (a^2-x^2)\,x\,dx + \int_{c-b}^k \{b^2-(x-c)^2\}\,x\,dx;$$

whence may $\bar{x}$ be determined.

Ex. 4. To find the centre of gravity of a cone, the density of each circular slice of which varies as the nth power of its distance from a parallel plane through the vertex.

Let the vertex be the origin, and the equation to the generating line of the cone be $y = ax$; and let a be the altitude; then $\rho = kx^n$: and (27) becomes

$$\bar{x}\int_0^a x^{n+2}\,dx = \int_0^a x^{n+3}\,dx; \qquad \therefore \quad \bar{x} = \frac{n+3}{n+4}\,a.$$

Ex. 5. To find the centre of gravity of a cone, the density of every particle of which increases as its distance from the axis.

Let the vertex be the origin, a=the altitude, and let the equation of the generating line of the bounding surface be $y = ax$; then in equation (26) $\rho = kr$, so that

$$\bar{x}\int_0^a\int_0^{ax} r^2\,dr\,dx = \int_0^a\int_0^{ax} r^2 x\,dr\,dx; \qquad \bar{x} = \frac{4}{5}\,a.$$

Ex. 6. To find the centre of gravity of the volume of uniform density contained between a hemisphere and a cone whose vertex is the vertex of the hemisphere and base is the base of the hemisphere.

Let the common vertex, see fig. 53, be the origin; and let the equations to the bounding surfaces be

$$y^2 = 2ax-x^2 = \mathrm{Y}^2, \qquad y^2 = x^2;$$

so that Y and x are the limits of the r-integration in equation (26): then, as ρ is constant,

$$\bar{x}\int_0^a\int_x^{\mathrm{Y}} r\,dr\,dx = \int_0^a\int_x^{\mathrm{Y}} rx\,dr\,dx,$$

$$\bar{x}\int_0^a (2ax-x^2-x^2)\,dx = \int_0^a (2ax-x^2-x^2)\,x\,dx,$$

$$\bar{x} = \frac{a}{2}.$$

Ex. 7. If $\bar{x} = mx$, shew that the equation to the generating curve of the solid of revolution is $ky^2 = x^{\frac{2-m}{m-1}}$.

138.] Now let us take the most general case of a body in space; and first let it be referred to three rectangular axes originating at O: let (x, y, z) be the position of any particle of it, so that the volume-element abutting at it is $dx\,dy\,dz$; then

$$dV = dx\,dy\,dz:$$

let the density $= \rho$; so that equations (11) become

$$\left.\begin{aligned}\bar{x}\iiint \rho\,dx\,dy\,dz &= \iiint \rho x\,dx\,dy\,dz,\\ \bar{y}\iiint \rho\,dx\,dy\,dz &= \iiint \rho y\,dx\,dy\,dz,\\ \bar{z}\iiint \rho\,dx\,dy\,dz &= \iiint \rho z\,dx\,dy\,dz.\end{aligned}\right\} \qquad (28)$$

The integrals are of course definite and the extent of integration is assigned by the conditions of the problem.

Ex. 1. To find the centre of gravity of a homogeneous body in the form of the octant of an ellipsoid.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

and let $\quad Z = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}},$

$$Y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}};$$

then $\quad \bar{x}\int_0^a \int_0^Y \int_0^Z dz\,dy\,dx = \int_0^a \int_0^Y \int_0^Z x\,dz\,dy\,dx;$

$$\therefore \quad \bar{x} = \frac{3a}{8};$$

similarly, $\quad \bar{y} = \frac{3b}{8}, \qquad \bar{z} = \frac{3c}{8}.$

The integrals required in the preceding example have already been determined by Dirichlet's process of evaluation in Ex. 2, Art. 280, Vol. II (Integral Calculus).

Ex. 2. To find the centre of gravity of a body of uniform density bounded by the Cono-Cuneus of Wallis and by the planes $z = 0$, $y = c$.

The equation to the Cono-Cuneus is, equation (89), Art. 367, Vol. I, $\quad c^2 z^2 = y^2 (a^2 - x^2);$

and performing the z-, y-, x-integrations in order, the limits are $\frac{y}{c}(a^2-x^2)^{\frac{1}{2}}$ and 0, c and 0, a and 0; so that if

$$\mathrm{z} = \frac{y}{c}(a^2-x^2)^{\frac{1}{2}},$$

$$\bar{x}\int_0^a\int_0^c\int_0^{\mathrm{z}} dz\,dy\,dx = \int_0^a\int_0^c\int_0^{\mathrm{z}} x\,dx\,dy\,dz,$$

$$\bar{x}\int_0^a\int_0^c y(a^2-x^2)^{\frac{1}{2}}\,dy\,dx = \int_0^a\int_0^c xy(a^2-x^2)^{\frac{1}{2}}\,dy\,dx,$$

$$\bar{x}\int_0^a (a^2-x^2)^{\frac{1}{2}}\,dx = \int_0^a x(a^2-x^2)^{\frac{1}{2}}\,dx,$$

$$\bar{x} = \frac{4a}{3\pi};$$

$$\bar{y}\int_0^a\int_0^c\int_0^{\mathrm{z}} dz\,dy\,dx = \int_0^a\int_0^c\int_0^{\mathrm{z}} y\,dz\,dy\,dx,$$

$$\bar{y}\int_0^a\int_0^c y\,(a^2-x^2)^{\frac{1}{2}}\,dy\,dx = \int_0^a\int_0^c y^2(a^2-x^2)^{\frac{1}{2}}\,dx,$$

$$\bar{y} = \frac{2}{3}c;$$

$$\bar{z}\int_0^a\int_0^c\int_0^{\mathrm{z}} dz\,dy\,dx = \int_0^a\int_0^c\int_0^{\mathrm{z}} z\,dz\,dy\,dx,$$

$$\bar{z}\int_0^a\int_0^c \frac{y}{c}(a^2-x^2)^{\frac{1}{2}}\,dy\,dx = \int_0^a\int_0^c \frac{y^2(a^2-x^2)}{2c^2}\,dy\,dx,$$

$$\bar{z} = \frac{8a}{9\pi}.$$

139.] Again, let the curved bounding surface be referred to a system of polar coordinates of the construction of Art. 165, Vol. II; then

$$\left.\begin{aligned}
\bar{x}\iiint \rho r^2 \sin\theta\,dr\,d\theta\,d\phi &= \iiint \rho r^3(\sin\theta)^2\cos\phi\,dr\,d\theta\,d\phi,\\
\bar{y}\iiint \rho r^2 \sin\theta\,dr\,d\theta\,d\phi &= \iiint \rho r^3(\sin\theta)^2\sin\phi\,dr\,d\theta\,d\phi,\\
\bar{z}\iiint \rho r^2 \sin\theta\,dr\,d\theta\,d\phi &= \iiint \rho r^3 \sin\theta\cos\theta\,dr\,d\theta\,d\phi;
\end{aligned}\right\}\quad (29)$$

the integrals of course being definite, and the limits being assigned by the geometrical conditions of the problem.

Ex. 1. To find the centre of gravity of an octant of a sphere, the density of which varies as the nth power of the distance of any particle from the centre.

Let a = the radius of the sphere; and let $\rho = kr^n$; then equations (29) become

$$\bar{x}\int_0^a\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} r^{n+2}\sin\theta\, d\theta\, d\phi\, dr = \int_0^a\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} r^{n+3}(\sin\theta)^2\cos\phi\, d\theta\, d\phi\, dr;$$

$$\therefore\quad \bar{x} = \frac{n+3}{n+4}\,\frac{a}{2} = \bar{y} = \bar{z};$$

the last two values being inferred from the symmetry of the body.

Ex. 2. The vertex of a right circular cone is at the centre of a sphere; it is required to find the centre of gravity of a body of uniform density contained within the cone and the sphere.

Let the axis of z be the axis of the cone: and let α be the semi-vertical angle of the cone; a = the radius of the sphere; ρ = the constant density: then $\bar{x}$ and $\bar{y}$ are evidently equal to zero; and we have

$$\bar{z}\int_0^{2\pi}\int_0^{\alpha}\int_0^{a} r^2\sin\theta\, dr\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} r^3\sin\theta\cos\theta\, dr\, d\theta\, d\phi;$$

$$\bar{z}\,\frac{a^3}{3}(1-\cos\alpha)\,2\pi = \frac{a^4}{4}\,\frac{(\sin\alpha)^2}{2}\,2\pi,$$

$$\bar{z} = \frac{3a}{8}(1+\cos\alpha).$$

Ex. 3. The vertex of a right circular cone is on the surface of a sphere, and the axis of the cone passes through the centre of the sphere; if 2α is the vertical angle of the cone, and $\bar{z}$ is the distance of the centre of gravity from the vertex, shew that

$$\bar{z} = a\,\frac{1-(\cos\alpha)^6}{1-(\cos\alpha)^4}\,.$$

Ex. 4. If the equation to the cardioid is $r = a(1+\cos\theta)$, the distance from the origin of the centre of gravity of the solid formed by the revolution of the curve about the prime radius is equal to $\frac{4a}{5}$.

140.] I shall conclude this section with a few examples of determining the centres of gravity of bodies which do not come under any of the former methods, but to which the principles are equally applicable.

Ex. 1. To find the centre of gravity of a right pyramid of uniform density, whose base is any regular plane figure.

Let the vertex of the pyramid be the origin, and the axis of the pyramid the axis of x; divide the pyramid into slices of the thickness dx by planes perpendicular to the axis: then as the areas of the sections thus formed will vary as the squares of their homologous sides, and as these sides will vary as the distances from the vertex, so will the areas of the sections vary as the squares of the distances from the vertex; and therefore if the axis of the pyramid is divided into equal infinitesimal elements, the masses of the several slices will vary as the squares of the distance from the vertex. Now imagine each slice to be condensed into its centre of gravity, which point is on the axis of x; then if $a =$ the altitude of the pyramid, we shall have

$$\bar{x}\int_0^a x^2\,dx = \int_0^a x^3\,dx; \qquad \therefore \quad \bar{x} = \frac{3}{4}a.$$

Ex. 2. On the base of a hemisphere a right circular cone is constructed, the whole body being of uniform density; determine the altitude of the cone, so that the centre of gravity of the whole may be at the centre of the circular base of the hemisphere.

Let $a =$ the radius of the hemisphere, $c =$ the altitude of the cone: then if we imagine the hemisphere and the cone to be condensed into their centres of gravity, the moments of these weights must be equal about the centre of the circular base of the hemisphere: that is,

$$\int_0^a (a^2-x^2)\,x\,dx = \int_0^c \frac{a^2}{c^2}(c-x)^2 x\,dx;$$

$$\therefore \quad c^2 = 3a^2;$$

and therefore the vertical angle of the cone is 60°.

Ex. 3. When a heavy body with a convex surface rests on a horizontal plane, the vertical line through the centre of gravity also passes through the point of contact: because as the body is acted on by only two forces, viz. the weight acting downwards at the centre of gravity, and the reaction of the plane upwards at the point of contact, these forces cannot be in equilibrium unless they are equal, and act along the same line in opposite directions.

Hence it appears that the compound body of the last example will rest in any position on its convex spherical surface.

Hence also it follows that if a body is suspended from any point, the point of suspension and the centre of gravity are in the same vertical line.

A body in the form of a paraboloid of revolution of given altitude and uniform density is suspended from a point in the edge of its circular base; it is required to find the inclination of its axis to the vertical.

Let a = the altitude of the paraboloid; b = the radius of its circular base; θ = the angle between the axis of the paraboloid and the vertical: then, since the distance of the centre of gravity from the centre of the circular base $= \frac{a}{3}$, see Ex. 1, Art. 137,

$$\tan\theta = \frac{3b}{a}.$$

Ex. 4. If a heavy body is placed on a rough inclined plane, the friction of which is sufficient to prevent sliding, the body will be at rest so long as the vertical line through the centre of gravity passes within the part of the body which is in contact with the inclined plane; and if it falls beyond that part, the body will fall over; and if it passes through the edge of it, the body is just in its limiting position of rest.

A given cone rests with its base on an inclined plane: it is required to determine the inclination of the plane, when the cone is just on the point of falling over.

Let a = the altitude of the cone, and b = the radius of the base: then $\text{CG} = \frac{a}{4}$, see fig. 54: let $\text{CO}x = \alpha$:

$$\begin{aligned} \therefore\quad \tan\alpha &= \tan \text{CO}x, \\ &= \tan \text{CGB}, \\ &= \frac{4b}{a}; \end{aligned}$$

and when the angle of inclination of the plane exceeds this angle, the cone will fall over.

SECTION 5.—*Stability and instability of the equilibrium of heavy bodies.*

141.] The character of the equilibrium of heavy bodies, in respect of the stability or instability of the same, requires especial notice, although the discriminating conditions have already been investigated in the general case in Section 7 of the preceding Chapter. Let us refer at first to (280), Article 106, as in this case the action-lines of all the forces are parallel, and the axis of z may be taken parallel to these action-lines; and consequently, as a horizontal line may be taken for the axis of infinitesimal displacement of rotation, the equilibrium will be stable or unstable according as $\Sigma.\mathrm{P}\, z$ is positive or negative; that is, by Art. 107, according as $\Sigma.\mathrm{P}\, z$ is a maximum or a minimum. Hence in the case of a heavy body the equilibrium is stable or unstable for infinitesimal displacement about a horizontal axis according as $\Sigma.\rho g z\, d\mathrm{v}$ is a maximum or a minimum: but $\Sigma.\rho g z\, d\mathrm{v} = \bar{z}\, \Sigma.\rho g\, d\mathrm{v}$; consequently the equilibrium is stable or unstable according as $\bar{z}$ is a maximum or a minimum.

The theorem, however, may be demonstrated as follows by means of virtual velocities. Suppose a heavy body to be at rest on a horizontal plane, and no forces to act upon it, except gravity and the resistance of the plane; and suppose the body to have such an infinitesimal motion of displacement that it remains in contact with the plane; then as the virtual velocity of the reaction of the plane vanishes, the single condition of equilibrium is

$$\Sigma.\rho g\, d\mathrm{v}\, dz = 0. \tag{30}$$

But if $\bar{z}$ is the distance of the centre of gravity from the horizontal plane,

$$\bar{z}\, \Sigma.\rho g\, d\mathrm{v} = \Sigma.\rho g z\, d\mathrm{v}; \tag{31}$$

so that from (30) $\delta \bar{z} = 0$; consequently $\bar{z}$ is a maximum or a minimum; and as equilibrium is stable or unstable according as the radial moment is a maximum or a minimum, so observing that the action of all the weights is towards the plane of (x, y), the equilibrium is stable or unstable according as the position of the centre of gravity is the lowest or the highest.

This problem is that which is presented to us by rocking stones, and by many children's toys. We shall hereafter investigate the rocking motion of bodies thus placed.

142.] And to take a more general case. Let us consider that of a heavy body bounded by a convex surface resting on another body also with a convex surface. And let fig. 55 represent the bodies: the continuous lines indicating the position of the bodies when they are at rest at first, and the dotted lines the position of displacement. Let CAO be the vertical line passing through A the point of contact of the two surfaces when they are at rest, and through the centre of gravity of the upper body: let C be the centre of curvature of the lower body corresponding to the point A, and O that of the upper body; let G be the centre of gravity of the upper body: now suppose a small displacement of the upper body to take place by means of *rolling* on the lower one, so that there is no virtual velocity of the normal reactions of the surfaces: then if P is the new point of contact, and A′ is the point which was originally in contact with A, A′P=AP, the axis about which the rolling takes place being perpendicular to the plane of the paper. Let the curvature of the two surfaces be continuous about the points A and P; and by reason of the small displacement let O and G respectively be moved to O′ and G′; let $\text{CA} = \text{CP} = \rho_1$; $\text{OA} = \text{O}'\text{A}' = \text{O}'\text{P} = \rho_2$; $\text{ACP} = \theta$; $\text{OG} = \text{O}'\text{G}' = c$; therefore since the arc AP = the arc A′P;

$$\therefore \quad \rho_1\theta = \rho_2 \text{A}'\text{O}'\text{P}; \qquad \therefore \quad \text{A}'\text{O}'\text{P} = \frac{\rho_1}{\rho_2}\theta.$$

Let $h = \text{G}'\text{K}$ = vertical height of G′ above the horizontal line through C; therefore

$$h = (\rho_1+\rho_2)\cos\theta - c\cos\left(1+\frac{\rho_1}{\rho_2}\right)\theta;$$

and replacing the cosines by the first two terms of their equivalent series, because θ is small, we have

$$h = \rho_1+\rho_2-c-(\rho_1+\rho_2)\left(1-c\frac{\rho_1+\rho_2}{\rho_2^2}\right)\frac{\theta^2}{1.2};$$

$$\frac{dh}{d\theta} = -(\rho_1+\rho_2)\left(1-c\frac{\rho_1+\rho_2}{\rho_2^2}\right)\theta,$$

$$= 0, \text{ if } \theta = 0,$$

and changes sign from + to −, if c is less than $\dfrac{\rho_2^2}{\rho_1+\rho_2}$,

- - - - - - - - - - — to +, if c is greater than $\dfrac{\rho_2^2}{\rho_1+\rho_2}$;

and therefore h is a maximum or a minimum according as $\text{AG} = \rho_2 - c$ is greater or less than $\dfrac{\rho_1\rho_2}{\rho_1+\rho_2}$; that is, as

$$\frac{1}{\text{AG}} \text{ is less than or greater than } \frac{1}{\rho_1}+\frac{1}{\rho_2};$$

and therefore the equilibrium is stable or unstable according as

$$\frac{1}{\text{AG}} \text{ is greater than or less than } \frac{1}{\rho_1}+\frac{1}{\rho_2}. \tag{32}$$

If the equilibrium is neutral,

$$\frac{1}{\text{AG}} = \frac{1}{\rho_1}+\frac{1}{\rho_2}; \tag{33}$$

and in this case, for a small displacement, the centre of gravity of the upper body neither ascends nor descends.

If the lower surface is plane, $\rho_1 = \infty$, and the equilibrium is stable or unstable, according as AG is less or greater than ρ_2;

that is, according as the centre of gravity is below or above the centre of curvature corresponding to the point A.

If the lower surface is concave, ρ_1 is negative, and the equilibrium is stable or unstable according as

$$\frac{1}{\text{AG}} \text{ is greater or less than } \frac{1}{\rho_2} - \frac{1}{\rho_1}. \qquad (34)$$

143.] The values of ρ_1 and ρ_2 will of course depend on the position of the normal planes of the greatest and least curvature of the two surfaces, and therefore the stability will be different for the different rotation-axes which are perpendicular to the normal planes through A; the stability therefore will be greatest or least according as

$$\frac{1}{\rho_1} + \frac{1}{\rho_2}$$

is a minimum or a maximum.

If therefore in this latter case, which is the most unfavourable, the equilibrium is stable, it is also stable for every normal section passing through A, and therefore the position of the body is one of complete stability.

Suppose however that the upper and lower surfaces are so arranged, that α is the angle between the normal section of greatest curvature in the lowest, and that of the greatest curvature in the upper; and suppose that it is required to find the nature of the stability of any particular normal plane.

Let θ be the angle between the normal plane of displacement, and that of maximum curvature in the lowest surface: then if R_1 and r_1 are the principal radii of curvature of the lower surface, by Euler's theorem, Art. 403, Vol. I (Differential Calculus),

$$\frac{1}{\rho_1} = \frac{(\cos\theta)^2}{r_1} + \frac{(\sin\theta)^2}{\text{R}_1};$$

and if R_2 and r_2 are the principal radii of curvature of the upper surface,

$$\frac{1}{\rho_2} = \frac{\{\cos(\theta+\alpha)\}^2}{r_2} + \frac{\{\sin(\theta+\alpha)\}^2}{\text{R}_2};$$

therefore

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{(\cos\theta^2)}{r_1} + \frac{\{\cos(\theta+\alpha)\}^2}{r_2} + \frac{(\sin\theta)^2}{\text{R}_1} + \frac{\{\sin(\theta+\alpha)\}^2}{\text{R}_2}:$$

whereby the normal plane of least stability may be determined.

144.] The following are problems in which the stability of equilibrium is determined by the position of the centre of gravity;

the equilibrium being stable, neutral, or unstable according as the centre of gravity is in its lowest position, moves in a horizontal line, or is in its highest position.

Ex. 1. A heavy uniform beam rests against a smooth curve, and against a vertical wall, all of which are in the same vertical plane; it is required to find the nature of the curve so that the beam may be at rest in all positions.

Let the beam be QP, fig. 56, of which let G be the middle point and the centre of gravity; and let the horizontal line, in which the centre of gravity is in all positions of the beam, be the axis of x, and let it meet the vertical wall in the point O; let O be the origin, let the length of the beam be $2a$, so that the curve required meets the wall at a distance OA $(= a)$ below O; let OA be the axis of y; $\text{OM} = x$, $\text{MP} = y$, $\text{QGO} = \theta$;

$$\therefore \quad \frac{x}{2a} = \cos\theta, \qquad \frac{y}{a} = \sin\theta;$$

therefore squaring and adding,

$$\frac{x^2}{4a^2} + \frac{y^2}{a^2} = 1;$$

the equation to an ellipse, whose centre is O, horizontal semi-axis is $2a$, and vertical semi-axis is a.

The property of the curve required in the problem is evidently the same as that of the elliptic compasses.

Ex. 2. A heavy uniform beam rests against a smooth vertical wall, and on a smooth curve; determine the nature of the curve so that the beam may rest in all positions.

Let RQ be the beam of length $2a$, whose centre of gravity is G, fig. 57; P the point in the curve at which the beam touches it; let the horizontal line OMG, in which in all positions of the beam its centre of gravity is, be the axis of x; and let it meet the wall at O, and let O be the origin, $\text{OM} = x$, $\text{MP} = y$, $\text{QG} = \text{GR} = a$. Then, as the line RQ is a tangent to the required curve at P,

$$\tan \text{OGQ} = -\frac{dy}{dx}.$$

Therefore
$$a = \text{QP} + \text{PG},$$
$$= \frac{x\,ds}{dx} - \frac{y\,ds}{dy};$$
$$\therefore \quad y = x\frac{dy}{dx} - \frac{a\,dy}{(dx^2 + dy^2)^{\frac{1}{2}}};$$

which is a differential equation of Clairaut's form: and of which the singular solution is, $y^{\frac{2}{3}}+x^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ex. 3. To determine whether the position of the beam resting on two planes, as investigated in Ex. 2, Art. 60, is of stable or of unstable equilibrium.

In fig. 29 let GK $= h$; therefore

$$\begin{aligned} h &= \text{AC}\sin a - a\sin\theta, \\ &= 2a\frac{\sin a\sin(\beta+\theta)}{\sin(a+\beta)} - a\sin\theta, \\ &= \frac{a}{\sin(a+\beta)}\{\sin(a-\beta)\sin\theta + 2\sin a\sin\beta\cos\theta\}; \end{aligned}$$

$$\therefore \quad \frac{dh}{d\theta} = 0 = \frac{a}{\sin(a+\beta)}\{\sin(a-\beta)\cos\theta - 2\sin a\sin\beta\sin\theta\};$$

$$\therefore \quad \tan\theta = \frac{\sin(a-\beta)}{2\sin a\sin\beta}, \text{ (see Ex. 2, Art. 60)};$$

and $\frac{dh}{d\theta}$ changes sign from + to −; therefore h is a maximum, and the equilibrium is unstable.

SECTION 6.—*General properties of the centre of gravity.*

145.] THEOREM I. Of all points in space the centre of gravity is, with reference to a system of material particles, such that the sum of the products of the mass of each particle and the square of its distance from the point is a minimum.

Let (x, y, z) be the required point; $m_1, m_2, \dots m_n$ the masses of the particles; (x_1, y_1, z_1), $(x_2, y_2, z_2), \dots (x_n, y_n, z_n)$ their positions; then if

$$\left.\begin{aligned} u^2 = m_1&\{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2\} \\ +m_2&\{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2\} \\ +&\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ +m_n&\{(x-x_n)^2+(y-y_n)^2+(z-z_n)^2\}; \end{aligned}\right\}$$

and if u^2 is to be a minimum,

$$\begin{aligned} u\,\text{D}u = m_1&\{(x-x_1)\,dx+(y-y_1)\,dy+(z-z_1)\,dz\} \\ +m_2&\{(x-x_2)\,dx+(y-y_2)\,dy+(z-z_2)\,dz\} \\ +&\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ +m_n&\{(x-x_n)\,dx+(y-y_n)\,dy+(z-z_n)\,dz\}; \end{aligned}$$

and equating to zero the coefficients of dx, dy, dz, we have

$$x = \frac{\Sigma.mx}{\Sigma.m}, \qquad y = \frac{\Sigma.my}{\Sigma.m}, \qquad z = \frac{\Sigma.mz}{\Sigma.m}; \tag{35}$$

and as the function by the form of the expression admits of infinite increase, it evidently cannot be a maximum; (35) therefore render u a minimum; and these are the coordinates of the centre of gravity.

146.] THEOREM II. If a system of material particles is invariable in form, and its centre of gravity is at a constant distance from a fixed point, the sum of the products of the mass of each particle and the square of its distance from the fixed point is constant.

Let the fixed point be the origin, and let $(\bar{x}, \bar{y}, \bar{z})$ be the centre of gravity, and (x_1, y_1, z_1), (x_2, y_2, z_2), ... (x_n, y_n, z_n) the positions of the particles in a given position of the system, these coordinates being measured from the centre of gravity; also let

$$\bar{x}^2 + \bar{y}^2 + \bar{z}^2 = a^2;$$

and let $r_1, r_2, \ldots r_n$ be the distances of the particles from the fixed point: then

$$\begin{aligned}\Sigma.mr^2 &= m_1\{(\bar{x}+x_1)^2 + (\bar{y}+y_1)^2 + (\bar{z}+z_1)^2\} \\ &\quad + \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\ &\quad + m_n\{(\bar{x}+x_n)^2 + (\bar{y}+y_n)^2 + (\bar{z}+z_n)^2\}, \\ &= a^2(m_1 + m_2 + \ldots + m_n) \\ &\quad + 2\bar{x}\Sigma.mx + 2\bar{y}\Sigma.my + 2\bar{z}\Sigma.mz \\ &\quad + \Sigma.m\rho^2,\end{aligned}$$

if $\rho_1, \rho_2, \ldots \rho_n$ are the distances of m_1, m_2, ... m_n from the centre of gravity. But $\Sigma.mx = 0$, $\Sigma.my = 0$, $\Sigma.mz = 0$, because the centre of gravity is the origin; therefore

$$\Sigma.mr^2 = a^2\Sigma.m + \Sigma.m\rho^2;$$

and as the right-hand member is constant, so is the left-hand member, and the proposition is proved.

147.] THEOREM III. If there is a system of heavy material particles, the product of the sum of the masses and of the sum of the products of each mass and the square of its distance from the centre of gravity is equal to the sum of the product of every two masses and of the square of the distance between them.

Let the centre of gravity be the origin: then

$$\left.\begin{aligned} m_1x_1 + m_2x_2 + \ldots + m_nx_n &= 0, \\ m_1y_1 + m_2y_2 + \ldots + m_ny_n &= 0, \\ m_1z_1 + m_2z_2 + \ldots + m_nz_n &= 0. \end{aligned}\right\}$$

Let $\rho_1, \rho_2, \ldots \rho_n$ be the distances of $m_1, m_2, \ldots m_n$ from the origin; then squaring and adding the above, we have

$$\begin{aligned} m_1^2\rho_1^2 + m_2^2\rho_2^2 + \ldots + m_n^2\rho_n^2 & \\ & + 2m_1 m_2 (x_1 x_2 + y_1 y_2 + z_1 z_2) \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + 2m_{n-1} m_n (x_{n-1} x_n + y_{n-1} y_n + z_{n-1} z_n) = 0; \end{aligned}$$

$$\therefore \quad \Sigma.m^2\rho^2 + 2\,\Sigma.mm'\rho\rho'\cos(\rho, \rho') = 0, \tag{36}$$

if m, m' are the symbols for every two of the material particles, and (ρ, ρ') is the angle contained between ρ and ρ'. Now suppose u to be the distance between the positions of the two particles m and m', then

$$u^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\rho, \rho');$$

$$\therefore \quad 2\rho\rho'\cos(\rho, \rho') = \rho^2 + \rho'^2 - u^2.$$

Therefore (36) becomes

$$\Sigma.m^2\rho^2 + \Sigma.mm'(\rho^2 + \rho'^2 - u^2) = 0:$$

and when written at length

$$\begin{aligned} & m_1\{m_1\rho_1^2 + m_2\rho_2^2 + \ldots + m_n\rho_n^2\} \\ & + m_2\{m_1\rho_1^2 + m_2\rho_2^2 + \ldots + m_n\rho_n^2\} \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + m_n\{m_1\rho_1^2 + m_2\rho_2^2 + \ldots + m_n\rho_n^2\} - \Sigma.mm'u^2 = 0; \end{aligned}$$

and if $\text{M} = \Sigma.m = m_1 + m_2 + \ldots + m_n$; we have

$$\text{M}\Sigma.m\rho^2 = \Sigma.mm'u^2, \tag{37}$$

which is the proposition required*.

148.] THEOREM IV. If a material particle is in equilibrium under the action of many pressures which are represented as to intensity and line of action by straight lines meeting at the particle; and if at the extremities of each of these lines heavy particles equal in weight are placed, the centre of gravity of these is at the point which is at rest under the action of the impressed pressures.

By reason of equations (69), Art. 34, we have

$$\Sigma.\text{P}\cos\alpha = 0, \qquad \Sigma.\text{P}\cos\beta = 0, \qquad \Sigma.\text{P}\cos\gamma = 0: \tag{38}$$

let $s_1, s_2, \ldots s_n$ be the line-representatives of the impressed forces acting on the material particle, the place of which we will take to be the origin: so that the equations (38) become

$$\Sigma.s\cos\alpha = 0, \qquad \Sigma.s\cos\beta = 0, \qquad \Sigma.s\cos\gamma = 0. \tag{39}$$

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* In the "Mécanique Analytique" of Lagrange, Premiere partie, Section III, Art. 20, an extension of this Theorem is given.

Let (x_1, y_1, z_1), (x_2, y_2, z_2), ... (x_n, y_n, z_n) be the extremities of $s_1, s_2, \dots s_n$; so that

$$\begin{array}{lll} x_1 = s_1 \cos \alpha_1, & y_1 = s_1 \cos \beta_1, & z_1 = s_1 \cos \gamma_1, \\ x_2 = s_2 \cos \alpha_2, & y_2 = s_2 \cos \beta_2, & z_2 = s_2 \cos \gamma_2, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_n = s_n \cos \alpha_n; & y_n = s_n \cos \beta_n; & z_n = s_n \cos \gamma_n; \end{array}$$

whereby (39) become

$$\Sigma . x = 0, \qquad \Sigma . y = 0, \qquad \Sigma . z = 0;$$

and if the mass of the particle at the extremity of every line-representative is m, we have

$$\Sigma . mx = 0, \qquad \Sigma . my = 0, \qquad \Sigma . mz = 0;$$

and therefore the origin is the centre of gravity of all the particles.

CHAPTER V.

THE ACTION OF FORCES ON BODIES OF VARIABLE FORM.

SECTION 1.—*The action of forces on flexible and inextensible strings or cords.*

149.] Thus far the bodies or systems of material particles, on which the statical forces act, have been assumed to be rigid, and their forms, or the relative position of the particles, have been supposed not to change on account of the acting forces. We shall now extend the inquiry to the case of bodies whose form varies by the action of the pressures, but becomes permanent, and may be considered rigid, under the action of the impressed forces. I shall first shortly investigate the case of the Funicular Polygon.

Suppose a string or cord, fig. 58, AB to be fastened at the two points A, B; the cord being without weight, perfectly flexible, and perfectly inextensible; and suppose at Q_1, Q_2, Q_3, Q_4, definite points of it, pressures P_1, P_2, P_3, P_4 to act with definite intensities and along definite lines of action, so that the cord assumes the permanent position indicated in the figure; the object is the determination of the form of the polygonal figure which the cord of given length assumes under the action of these forces, and of the tensions of each of its component straight elements.

It is manifest that the tension is the same throughout each element; and that as each point $Q_1, Q_2, \dots Q_4$ is at rest, the forces acting at each are in equilibrium. Let the tensions along AQ_1, Q_1Q_2, ... Q_4B, be respectively $T_1, T_2, \dots T_5$, so that the pressures at the fixed points A and B are respectively T_1 and T_5; and let the angles between the successive straight parts of the cord be $a_1, a_2, \dots a_4$; then as the point Q_1 is kept at rest by the three forces T_1, P_1, and T_2, the lines of action of all are in the same plane, and we have

$$\frac{T_1}{\sin P_1Q_1Q_2} = \frac{P_1}{\sin a_1} = \frac{T_2}{\sin P_1Q_1A}. \qquad (1)$$

In the same way for the point Q_2 we have

$$\frac{T_2}{\sin P_2Q_2Q_3} = \frac{P_2}{\sin \alpha_2} = \frac{T_3}{\sin P_2Q_2Q_1}; \tag{2}$$

and so on for the other points; and therefore when the form of the polygon and the magnitudes and lines of action of the forces $P_1, P_2, \ldots\ldots$ are given, the tensions of the several connecting strings may be determined.

150.] Suppose that the lines of action of the forces $P_1, P_2, \ldots P_4$ bisect the angles $\alpha_1, \alpha_2, \ldots \alpha_4$; then $T_1 = T_2 = \ldots = T_5$; and

$$\frac{P_1}{\cos\frac{\alpha_1}{2}} = \frac{P_2}{\cos\frac{\alpha_2}{2}} = \ldots = \frac{P_4}{\cos\frac{\alpha_4}{2}}; \tag{3}$$

and this condition may be secured in two ways; (1) by fixing smooth pins at the points $Q_1 \ldots Q_4$, and passing the string round them, so that the tension of the string is the same on both sides of the pin, and the pressure on the pin is the resultant of these two equal forces, and therefore its line of action bisects the angle between their lines of action: and (2) by making the impressed forces act on the cord at the points $Q_1 \ldots$ by means of smooth rings which slide on the cord, and are at rest at these points; and the line of action of P_1 will manifestly under this arrangement bisect the angle AQ_1Q_2, because considering A and Q_2 to be fixed, and the cord to be also fastened at them, if the ring Q_1 slides, it can move only on the surface of a prolate spheroid, of the generating ellipse of which A and Q_2 are the foci, and the length AQ_1Q_2 of the cord is the major axis, and therefore the normal at Q_1 which is the line of action of P_1 bisects the angle between the focal distances.

If we suppose that the two sides of the polygon abutting at (say) Q_1 are equal; then if $AQ_1 = Q_1Q_2 = s_1$, and the radius of the circle passing through AQ_1Q_2 is ρ_1, we have

$$\cos\frac{\alpha_1}{2} = \frac{s_1}{2\rho_1}; \tag{4}$$

and therefore if all the sides are equal, from (3) it follows that each impressed force is inversely as the radius of the circle passing through its point of application and the two angular points of the polygon adjacent on each side.

Now of such a polygon with equal sides a curve is a particular case, when the length of the curve is the equicrescent variable; and the circle just mentioned is the circle lying in the osculating plane at the point, and its radius is the radius of absolute curvature of the curve at the point; and therefore when a funicular curve fastened at its two ends is acted on in all its equal elements by normal forces, the tension is the same throughout, and each normal force varies as the absolute curvature of the curve at the point where it is applied.

Thus suppose a cord to be stretched by given forces at its ends on a curved surface, then the pressure caused by the surface is at every point in the direction of the normal of the surface, and is therefore proportional to the absolute curvature of the curve which the cord assumes on the surface; and as the normal-line of the reaction is in the same plane with two consecutive elements of the funicular curve, the osculating plane of the curve is a normal plane to the surface at the common point; and therefore, see Art. 336, Vol. II (Integral Calculus), the curve is the geodesic line joining the two points: and this geodesic line may evidently be either the maximum or the minimum; thus, a cord stretched between two given points on a sphere will arrange itself along the geodesic line, which is a great circle; and as one great circle-arc abutting at the points will be a minimum, so will the remainder of the same great circle be the maximum.

151.] If the lines of action of all the forces acting on the funicular polygon are parallel, the cord is evidently in one plane. Let the lines of action of the forces be vertical; then $\sin \mathrm{P}_1 \mathrm{Q}_1 \mathrm{Q}_2 = \sin \mathrm{P}_2 \mathrm{Q}_2 \mathrm{Q}_1$, $\sin \mathrm{P}_2 \mathrm{Q}_2 \mathrm{Q}_3 = \sin \mathrm{P}_3 \mathrm{Q}_3 \mathrm{Q}_2, \ldots\ldots$; so that if $\beta_1, \beta_2, \ldots$ are the angles of inclination of the successive lengths to the horizontal line,

$$\mathrm{T}_1 \cos \beta_1 = \mathrm{T}_2 \cos \beta_2 = \mathrm{T}_3 \cos \beta_3 = \ldots; \tag{5}$$

and therefore the successive tensions are inversely as the cosines of the angles of inclination to the horizon of the sides along which they act; and therefore if T_0 is the tension of a side which is horizontal, and if T is the tension along any side whose inclination to the horizontal line is β,

$$\mathrm{T}_0 = \mathrm{T} \cos \beta. \tag{6}$$

Suppose however that the vertical forces are the weights of the several parts of the cord, so that $\mathrm{P}_1, \mathrm{P}_2, \ldots$ are proportional to the lengths $\mathrm{AQ}_1, \mathrm{Q}_1\mathrm{Q}_2, \ldots$; and moreover suppose that the lengths of the elements are infinitesimal, so that the polygon becomes a plane curve, then if the density and thickness, that is, the area of a transverse section, of the cord are constant throughout, and if $\rho =$ the density, and $\omega =$ the area of a transverse section, $\mathrm{P} = \rho\omega g\,ds$, $dx = ds\cos\beta$, $dy = ds\sin\beta$; and if T and T' are the tensions at the beginning and end of an element respectively,

$$\left.\begin{aligned}\mathrm{T}'\cos\beta' &= \mathrm{T}\cos\beta + d.\mathrm{T}\cos\beta,\\ \mathrm{T}'\sin\beta' &= \mathrm{T}\sin\beta + d.\mathrm{T}\cos\beta;\end{aligned}\right\} \qquad (7)$$

therefore taking vertical forces,

$$\mathrm{P} + \mathrm{T}\sin\beta = \mathrm{T}'\sin\beta',$$

and replacing P, T and $\mathrm{T}'\sin\beta'$ by their values,

$$\begin{aligned}\rho\omega g\,ds &= d.\mathrm{T}\sin\beta\\ &= d.\mathrm{T}\frac{dy}{ds};\end{aligned}$$

and if we consider T_0 to be known, and to be equal to the weight of a length $= c$ of the string of the string-curve, so that $\mathrm{T}_0 = \rho\omega cg$, then from (6) we have

$$\rho\omega g\,ds = d.\rho\omega cg\frac{dy}{dx};$$

and placing the origin at the lowest point of the curve,

$$s = c\frac{dy}{dx}; \qquad (8)$$

which expresses the property of the curve, that the length of it reckoned from the lowest point varies as the tangent of the angle at which the tangent of the string at the upper end is inclined to the horizon. This is a characteristic property of the curve, and from it all the other properties may be deduced. The equation in terms of x and y has been determined by means of (8) in Ex. 7, Art. 166, Vol. II (Integral Calculus). The curve which a heavy flexible and inextensible string thus takes is called *the catenary*. I propose however to investigate the form of string-curves under the action of given forces in a more general way, and in the course of the inquiry to return to the special form of the heavy catenary.

152.] Suppose a perfectly flexible and inextensible string to be in space, and to be at all its parts subject to the action of certain given forces; let it be referred to a system of coordinate axes, and at the point (x, y, z), let ρ be the density, ω the area of a transverse section of the cord, these quantities being in the general case functions of x, y, and z; and let ds be the length-element; and thus $\rho\omega ds$ is the mass-element of the cord. Let X, Y, Z be the axial components of the impressed forces acting on an unit of mass at that point; so that the pressures acting on the mass-element at the point are

$$\rho\omega\mathrm{X}\,ds, \qquad \rho\omega\mathrm{Y}\,ds, \qquad \rho\omega\mathrm{Z}\,ds. \tag{9}$$

Let T be the tension of the cord at the point (x, y, z); then as it acts along the curve, its resolved parts are

$$\mathrm{T}\frac{dx}{ds}, \qquad \mathrm{T}\frac{dy}{ds}, \qquad \mathrm{T}\frac{dz}{ds}; \tag{10}$$

and therefore at the point $(x+dx,\ y+dy,\ z+dz)$ the resolved parts of the tension are

$$\mathrm{T}\frac{dx}{ds}+d.\mathrm{T}\frac{dx}{ds}, \qquad \mathrm{T}\frac{dy}{ds}+d.\mathrm{T}\frac{dy}{ds}, \qquad \mathrm{T}\frac{dz}{ds}+d.\mathrm{T}\frac{dz}{ds}; \tag{11}$$

the tension varying continuously as we pass along the curve; let us suppose x, y, z, and s to increase simultaneously; then the element of the curve being in equilibrium under the action of the forces (9) (10) and (11), the conditions of equilibrium are

$$\left.\begin{aligned} d.\mathrm{T}\frac{dx}{ds} + \rho\omega\mathrm{X}\,ds &= 0, \\ d.\mathrm{T}\frac{dy}{ds} + \rho\omega\mathrm{Y}\,ds &= 0, \\ d.\mathrm{T}\frac{dz}{ds} + \rho\omega\mathrm{Z}\,ds &= 0; \end{aligned}\right\} \tag{12}$$

and from these equations all the properties of the curve are to be deduced.

First, integrating the equations, we have

$$\frac{\int\rho\omega\mathrm{X}\,ds}{dx} = \frac{\int\rho\omega\mathrm{Y}\,ds}{dy} = \frac{\int\rho\omega\mathrm{Z}\,ds}{dz} = -\frac{\mathrm{T}}{ds};$$

and therefore the numerators are proportional to the direction-cosines of the arc-element on which the forces act.

Also expressing at length the first terms of (12), and taking s to be equicrescent, we have

$$\left.\begin{aligned} \mathrm{T}d.\frac{dx}{ds} + \frac{dx}{ds}d\mathrm{T} + \rho\omega\mathrm{X}\,ds &= 0, \\ \mathrm{T}d.\frac{dy}{ds} + \frac{dy}{ds}d\mathrm{T} + \rho\omega\mathrm{Y}\,ds &= 0, \\ \mathrm{T}d.\frac{dz}{ds} + \frac{dz}{ds}d\mathrm{T} + \rho\omega\mathrm{Z}\,ds &= 0; \end{aligned}\right\} \tag{13}$$

Multiplying these equations severally by dx, dy, dz, and adding, we have

$$d\mathrm{T} + \rho\omega\{\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz\} = 0, \tag{14}$$

where $d\mathrm{T}$ is the total differential of T. This equation is evidently that of the tangential components of the forces. Let the integral of it be taken between the limits which carry the subscripts n and 0; and we have

$$\mathrm{T}_n - \mathrm{T}_0 + \int_0^n \rho\omega\{\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz\} = 0. \tag{15}$$

If therefore ρ, ω, X, Y, Z are functions of the coordinates of the point to which they apply, and are such that the quantity under the sign of integration is a complete differential, then the difference between the tensions at the limits is a function of the coordinates of those points only, and is independent of the form of the curve which joins them.

The analytical conditions which are satisfied when the second part of (15) is an exact differential have been investigated in Articles 373, 397, Vol. II (Integral Calculus), and the corresponding geometrical theorems have also been worked out. Many mechanical properties which satisfy the conditions will be exhibited hereafter; and it will be more convenient to consider the character of the preceding equations when they are under discussion. The tension of the string-curve is constant throughout its length, that is,

$$\mathrm{T}_n = \mathrm{T}_0, \tag{16}$$

whenever

$$\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz = 0; \tag{17}$$

and this occurs (1) when $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = 0$; that is, when the string is under the action of no force; (2) when the resultant force acts at every point along a line normal to the curve at the point.

153.] Also let us transfer the last term in each of (13) to the right-hand side of the equation, and take the squares of these equations, and add them: then if s is equicrescent, $\rho' =$ the absolute curvature of the curve at the point (x, y, z), and P is the impressed force on an unit-mass at (x, y, z); so that

$$\mathrm{P}^2 = \mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2,$$

$$\frac{\mathrm{T}^2 ds^2}{\rho'^2} + d\mathrm{T}^2 = \rho^2\omega^2\mathrm{P}^2 ds^2;$$

$$\therefore \quad \frac{\mathrm{T}^2}{\rho'^2} + \left(\frac{d\mathrm{T}}{ds}\right)^2 = \rho^2\omega^2\mathrm{P}^2; \tag{18}$$

and consequently, if the tension is constant throughout the curve,

$$\mathrm{P} = \frac{\mathrm{T}}{\rho\rho'\omega}; \tag{19}$$

and thus the impressed force varies inversely as the radius of absolute curvature at each point of the string, see Art. 150. Moreover, if the force is also constant, ρ' is constant, and the curvature is the same at all points; and if the string-curve is a plane-curve, it is also an arc of a circle.

Also from (13) eliminating T and dT, we have

$$\Big(dzd.\frac{dy}{ds}-dyd.\frac{dz}{ds}\Big)\mathrm{X}+\Big(dxd.\frac{dz}{ds}-dzd.\frac{dx}{ds}\Big)\mathrm{Y}+\Big(dyd.\frac{dx}{ds}-dxd.\frac{dy}{ds}\Big)\mathrm{Z}=0\,;$$

$$\therefore\ (dz\,d^2y-dy\,d^2z)\mathrm{X}+(dx\,d^2z-dz\,d^2x)\mathrm{Y}+(dy\,d^2x-dx\,d^2y)\mathrm{Z}=0\,; \tag{20}$$

and as the former factors of each term are proportional to the direction-cosines of the binormal, we conclude that the impressed force lies in the osculating plane of the string-curve.

Also if ϕ is the angle between the line of action of P and the arc-element, $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz = \mathrm{P}ds\cos\phi$;

therefore from (14),

$$d\mathrm{T}+\rho\omega\mathrm{P}ds\cos\phi = 0\,; \tag{21}$$

and substituting this value for dT in (18) we have

$$\mathrm{T} = \rho\rho'\omega\mathrm{P}\sin\phi\,; \tag{22}$$

these are the equations of the tangential and normal components.

154.] If the impressed forces all act in one plane, we may take that plane to be the plane of (x, y), and equations (12) become

$$\left.\begin{aligned} d.\mathrm{T}\frac{dx}{ds}+\rho\omega\mathrm{X}\,ds = 0,\\ d.\mathrm{T}\frac{dy}{ds}+\rho\,\omega\mathrm{Y}ds = 0\,;\end{aligned}\right\} \tag{23}$$

and taking the tangential and normal components, we have

$$d\mathrm{T}+\rho\omega\,(\mathrm{X}dx+\mathrm{Y}dy) = 0\,; \tag{24}$$

$$\frac{\mathrm{T}^2}{\rho'^2}+\Big(\frac{d\mathrm{T}}{ds}\Big)^2 = \rho^2\omega^2\mathrm{P}^2\,; \tag{25}$$

so that if T is constant,

$$\mathrm{T} = \rho\rho'\omega\mathrm{P}. \tag{26}$$

Of these general formulae the following are particular examples.

155.] Let us suppose gravity, or the earth's attraction, to be the only acting force, in which case the curve is the free catenary; and let the axis of x be horizontal, and that of y vertical; then $\mathrm{X} = 0$, $\mathrm{Y} = -g$; so that the equations (23) become

$$d.\mathrm{T}\frac{dx}{ds} = 0, \qquad d.\mathrm{T}\frac{dy}{ds}-g\omega\rho ds = 0\,; \tag{27}$$

$$\therefore\ \ \mathrm{T}\frac{dx}{ds} = \mathrm{T}_0, \tag{28}$$

when T_0 is the horizontal tension of the catenary; that is, it is the value of the tension, when $\frac{dx}{ds} = 1$. Thus the horizontal component of the tension is constant. It may be expressed more conveniently in the following form. Let σ = the density and a = the area of a transverse section of the string at the point where the string is horizontal; and let c = the length of a string of that density and thickness whose weight $= T_0$; so that

$$T\frac{dx}{ds} = T_0 = a\sigma cg. \tag{29}$$

Also from (27),
$$T\frac{dy}{ds} = \int g\rho\omega ds;$$

$$\therefore \quad a\sigma c\frac{dy}{dx} = \int \rho\omega ds; \tag{30}$$

and if the string is of the same thickness and density throughout, so that $\rho = \sigma$, $\omega = a$, then

$$c\frac{dy}{dx} = s, \tag{31}$$

if $s = 0$, when $\frac{dy}{dx} = 0$; that is, if s begins at the point at which the curve is horizontal. All the properties of the curve may be inferred from (31).

As the heavy catenary however has many remarkable geometrical properties, and has important applications to the theory of Suspension Bridges, I will also deduce its equation from first principles, so that it may be presented to the student in the clearest possible form.

156.] Suppose the curve, see fig. 59, to be suspended from two fixed points, A and B, in the plane of the paper, which is supposed to be vertical; let C be the lowest point of the catenary, and let a vertical line through it be taken for the axis of y, and let the horizontal line, which will also touch the curve at C, be the axis of x. Let $CM = x$, $MP = y$, the arc $CP = s$, ρ = density at P, ω = the area of the transverse section of the cord. Then the arc CP, after it has assumed its permanent form of equilibrium, may be considered as a rigid body kept at rest by three forces, (1) T the tension acting at P in the direction of the tangent, (2) the weight of the cord CP acting vertically downwards and which is equal to $\int_0^s g\rho\omega ds$, and (3) the horizontal tension at the lowest point C; as to the last force, let us suppose, as in the preceding Article, σ to be the density of the cord at C, a to be the area of a transverse section at the same point, and c to be the length of cord such that $ga\sigma c$ is equal to the tension at C; then by the triangle of forces, these forces are proportional to the three lines PT′, T′N, NP, which their lines of action are respectively parallel to; and therefore we have

$$\frac{\text{T}}{\text{PT}'} = \frac{\int_0^s g\rho\omega\, ds}{\text{T}'\text{N}} = \frac{ga\sigma c}{\text{NP}}; \tag{32}$$

but
$$\frac{\text{PT}'}{ds} = \frac{\text{T}'\text{N}}{dy} = \frac{\text{NP}}{dx};$$

$$\therefore \quad \frac{\text{T}}{ds} = \frac{\int_0^s g\rho\omega\, ds}{dy} = \frac{ga\sigma c}{dx}; \tag{33}$$

so that the equation to the curve is given by

$$a\sigma c\frac{dy}{dx} = \int_0^s \rho\omega\, ds; \tag{34}$$

and the tension at any point by the equation

$$\text{T} = ga\sigma c\,\frac{ds}{dx}; \tag{35}$$

which are the same equations as those found in the preceding Article.

157.] Now let us take a particular case, and suppose ω and ρ to be constant throughout the cord; so that $\rho = \sigma$, $\omega = a$, and the curve to become that of a cord of constant thickness and density, suspended from two given points A and B: therefore from (34),

$$\frac{dy}{dx} = \frac{s}{c}; \tag{36}$$

which is the same equation as (31); then differentiating, and making x equicrescent,

$$c\frac{d^2y}{dx^2} = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}};$$

$$\frac{d.\frac{dy}{dx}}{\left\{1 + \frac{dy^2}{dx^2}\right\}^{\frac{1}{2}}} = \frac{dx}{c};$$

and integrating, and taking the limits such that $\frac{dy}{dx} = 0$, when $x = 0$, we have

$$\log\left\{\frac{dy}{dx} + \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\} = \frac{x}{c};$$

$$\therefore \quad \frac{dy}{dx} + \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = e^{\frac{x}{c}};$$

$$\therefore \quad -\frac{dy}{dx} + \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = e^{-\frac{x}{c}};$$

$$\therefore \quad 2\frac{dy}{dx} = e^{\frac{x}{c}} - e^{-\frac{x}{c}}; \tag{37}$$

and integrating again, and observing that $y = 0$, when $x = 0$, we have

$$2y = c\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\} - 2c;$$

$$\therefore \quad y + c = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}. \tag{38}$$

Also equating the values of $\frac{dy}{dx}$ in (36) and (37) we have

$$s = \frac{c}{2}\left\{e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right\}; \tag{39}$$

and either (38) or (39) is the equation to the catenary of constant thickness and density, when the lowest point of the curve is the origin, and the horizontal tangent at it is the axis of x.

To simplify the equation, let the origin be moved to a point O, see fig. 60, at a distance c below C and on the axis of y, so that (38) becomes

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}; \tag{40}$$

and (39) is unaltered. The horizontal line through O is called *the directrix of the catenary*. Thus the ordinate of the catenary measured from the directrix is the sum of the ordinates of two logarithmic curves.

Now $c =$ OC is the length of a cord of the same thickness and density as the cord of the curve, the weight of which is equal to the tension of the curve at its lowest point: if therefore a smooth small pulley were placed at C, and if over it a cord of length c, and of the same thickness and density as the cord of the curve, and joined to the arc CP, were suspended, the curve would be in equilibrium.

Since from (39) $$\frac{ds}{dx} = \frac{1}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\} = \frac{y}{c}; \tag{41}$$

therefore from (35), $$\text{T} = g a \sigma y; \tag{42}$$

that is, the tension at every point of the curve is equal to the weight of a cord of the same thickness and density, the length of which is equal to the ordinate of the point. The tension therefore is the least at the lowest point of the catenary, and varies directly as the ordinate: it is consequently the same for the two points in the same horizontal line. And therefore if, see fig. 61, a cord of constant thickness and density is suspended over two small pulleys A and B, and is at rest by means of certain lengths hanging over the pulleys, the two ends H and K are in the same horizontal line, and the tension at the lowest point C is equal to the weight of a cord similar in all respects, and whose length is CO.

158.] Let us investigate some of the more prominent geometrical properties of the catenary. From (40) and (39) we have

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right\}, \qquad s = \frac{c}{2}\left\{e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right\};$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left\{e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right\} = \frac{s}{c}; \tag{43}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{2c}\left\{e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right\} = \frac{1}{c}\frac{ds}{dx} = \frac{y}{c^2}; \tag{44}$$

$$\therefore \quad y = c\frac{ds}{dx}.$$

Now as (40) is unaltered when x is replaced by $-x$, it follows that the catenary is symmetrical with respect to the axis of y.

Also squaring (39) and (40), and subtracting, we have

$$y^2-s^2 = c^2. \tag{45}$$

From the preceding equation it will be found that the radius of curvature of the catenary $= \frac{y^2}{c}$, and is equal to the normal; and that these lines are drawn from the curve in opposite directions; hence the radius of curvature at C is equal to c. Also from (42),

$$\begin{aligned}
\text{T}^2 &= g^2a^2\sigma^2y^2 \\
&= g^2a^2\sigma^2(c^2+s^2) \\
&= g^2a^2\sigma^2c^2+g^2a^2\sigma^2s^2 \\
&= (\text{tension of curve at lowest point})^2 \\
&\qquad\qquad + (\text{weight of cord of length} = s)^2.
\end{aligned}$$

Also let a tangent PΠ, fig. 60, be drawn to the catenary at the point P, and from M, the foot of the ordinate, let a perpendicular to PΠ be drawn; then since $\frac{dx}{ds}$ is the sine of ΠPM,

$$\Pi M = y\frac{dx}{ds}$$
$$= c; \qquad (46)$$

and therefore from (44) or (36) $P\Pi = s =$ the arc CP. Therefore the point Π is on the involute of the catenary which originates from the curve at C, and ΠM is a tangent to this involute; and as ΠM is the tangent to this last curve, and is equal to the constant quantity c, the involute is the equitangential curve or tractrix, the asymptote of which is the axis of x. Let therefore η and ξ be the current coordinates to this curve; $ON = \xi$, $N\Pi = \eta$; then

$$\begin{aligned}\frac{d\eta}{d\xi} &= \tan \Pi M x \\ &= -\tan \Pi M N \\ &= -\frac{\Pi N}{NM} = -\frac{\eta}{\{c^2-\eta^2\}^{\frac{1}{2}}},\end{aligned} \qquad (47)$$

which is the differential equation to the equitangential curve. And producing $P\Pi$, so that it cuts the axis of x in T, $P\Pi$ is the radius of curvature of the tractrix at the point Π, and ΠT is the normal; and therefore as PMT is a right angle, $P\Pi \times \Pi T = \Pi M^2$; therefore in the tractrix,

$$\text{the radius of curvature} \times \text{the normal} = c^2. \qquad (48)$$

The intrinsic equation of the catenary is

$$s = c \cot \psi. \qquad (49)$$

This may be derived analytically from the preceding equations by the process developed in Art. 168, Vol. II (Integral Calculus), see Ex. 5; or it may be proved geometrically: for $P\Pi = s$, $\Pi M = c$, $\Pi MT = \psi$; therefore $P\Pi = \Pi M \cot \Pi MT$. Also the catenary at its lowest point approximately coincides with a conical parabola. For taking the equation (38), the origin of which is at the lowest point,

$$\begin{aligned}y+c &= \frac{c}{2}\left\{e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right\} \\ &= \frac{c}{2}\left\{1+\frac{x}{c}+\frac{x^2}{1.2.c^2}+\frac{x^3}{1.2.3.c^3}+\dots\right. \\ &\qquad\left.+1-\frac{x}{c}+\frac{x^2}{1.2.c^2}-\frac{x^3}{1.2.3.c^3}+\dots\right\} \\ &= c\left\{1+\frac{x^2}{1.2.c^2}+\frac{x^4}{1.2.3.4.c^4}+\dots\right\};\end{aligned}$$

and omitting terms which involve powers of x higher than the second,

$$y = \frac{x^2}{2c};$$

$$\therefore \quad x^2 = 2cy;$$

the equation to a parabola, whose vertex is C, whose principal axis is Cy, and whose latus rectum is $2c$.

159.] The constant c which enters into the equations of the curve may be determined experimentally by means of the tension at the lowest point C. Suppose however that the data of the problem are different to those which we have taken. Suppose, for instance, that a homogeneous heavy cord of the length $2l$ is suspended from two points in the same horizontal line at a distance $2b$ apart, and that it is required to determine the equation of the catenary, the position of the lowest point, and the tension at every point.

Let the origin be taken at the point of bisection of the horizontal line which joints the two given points; see fig. 62; the horizontal line being the axis of x, and the vertical line reckoned positive downwards being the axis of y; OB $=$ OB$'$ $= b$; let OC $= h$; so that the equations become

$$h+c-y = \frac{c}{2}\left\{e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right\}; \qquad s = \frac{c}{2}\left\{e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right\}; \tag{50}$$

and in these we have to determine h and c in terms of l and b.

Let α be the angle at which the curve is inclined to OB at the point B; then we have $\sec\alpha+\tan\alpha = e^{\frac{b}{c}}$, and from (43) $\tan\alpha = \frac{l}{c}$;

$$\therefore \quad \frac{b}{l} = \cot\alpha \log(\sec\alpha+\tan\alpha)$$

$$= \cot\alpha \log\tan\left(45^\circ+\frac{\alpha}{2}\right);$$

whence α may be determined; and consequently c may be found. And from (50), if $y = 0$, we have

$$h+c = \frac{c}{2}\left\{e^{\frac{b}{c}}+e^{-\frac{b}{c}}\right\}; \qquad l = \frac{c}{2}\left\{e^{\frac{b}{c}}-e^{-\frac{b}{c}}\right\};$$

$$\therefore \quad h+l+c = ce^{\frac{b}{c}};$$

$$h = l\{\operatorname{cosec}\alpha-\cot\alpha\}$$

$$= l\tan\frac{\alpha}{2}.$$

$$\text{therefore the tension at the lowest point} = \rho g\omega l \cot a,$$

$$\text{and the tension at B and at B}' = \rho g\omega l \operatorname{cosec} a;$$

thus all the circumstances of the curve are determined.

Another problem of the same kind is, To determine the form and circumstances of the catenary when a heavy homogeneous string of given length is suspended over two smooth pulleys in the same horizontal line, and the ends of the string hang freely so that the string supports itself.

160.] To determine the position of the centre of gravity of the cord of the catenary of uniform thickness and density, beginning at the lowest point C; fig. 60.

$$\bar{x}\int g\rho\omega\,ds = \int g\rho\omega x\,ds; \qquad \therefore \quad \bar{x}\int ds = \int x\,ds;$$

$$\bar{x}s = xs - \int_0^x \frac{c}{2}\left\{e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right\}dx$$

$$= xs - \frac{c^2}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}} - 2\right\}$$

$$= xs - cy + c^2;$$

$$\therefore \quad \bar{x} = x - \frac{c(y-c)}{s}; \qquad (51)$$

$$\bar{y}\int ds = \int y\,ds;$$

$$\bar{y}s = \int_0^x \frac{c}{4}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)^2 dx$$

$$= \frac{c^2}{8}\left\{e^{\frac{2x}{c}} - e^{-\frac{2x}{c}}\right\} + \frac{cx}{2}$$

$$= \frac{sy}{2} + \frac{cx}{2};$$

$$\bar{y} = \frac{y}{2} + \frac{cx}{2s}. \qquad (52)$$

And by geometrical construction in fig. 60,

$$\bar{x} = \text{OT} + \text{T}\Pi, \qquad \bar{y} = \frac{\text{MP} + \text{OT}'}{2}.$$

In Art. 130 it has been proved that of all curves which a heavy wire or a flexible string of uniform thickness and density and of given length with its ends at fixed points can assume, the catenary is that of which the centre of gravity has the lowest position. The form therefore which a heavy flexible cord of uniform thickness and density assumes when suspended from two fixed points is that of stable equilibrium.

161.] Next let us consider the circumstances of a heavy string of varying thickness and density, under the action of gravity only.

From (33) we have

$$\frac{\mathrm{T}}{ds} = \frac{\int_0^s g\rho\omega\, ds}{dy} = \frac{ga\sigma c}{dx};$$

$$\therefore \quad ga\sigma c\frac{dy}{dx} = \int_0^s g\rho\omega\, ds; \tag{53}$$

and differentiating,

$$ga\sigma c\frac{d^2y}{dx^2} = g\rho\omega\frac{ds}{dx};$$

from which the variation of the density or of the thickness may be determined, when the catenarian curve is given; and the curve may be found, when the law of the thickness or of the density is given: also

$$\mathrm{T} = ga\sigma c\frac{ds}{dx}; \tag{54}$$

whereby the tension at any point of the curve may be found. Some examples are subjoined.

Ex. 1. It is required to determine the law of variation of the thickness of a heavy homogeneous string, that it may be in equilibrium in the form of a parabola with its vertex downwards and its axis vertical.

Let the equation be $x^2 = 4ay$;

$$\therefore \quad \frac{dx}{2a} = \frac{dy}{x} = \frac{ds}{(4a^2+x^2)^{\frac{1}{2}}}; \qquad \frac{d^2y}{dx^2} = \frac{1}{2a};$$

and therefore from (53), as ρ is constant and equal to σ,

$$\omega = \frac{ac}{(4a^2+x^2)^{\frac{1}{2}}}; \qquad \mathrm{T} = \frac{ga\sigma c}{2a}(4a^2+x^2)^{\frac{1}{2}};$$

so that ω varies inversely as, and T varies directly as, the square root of the distance of any element from the directrix: therefore when x is small, ω is constant, which fact has already been proved in Art. 158.

Ex. 2. It is required to find the law of variation of the density of a heavy string of uniform thickness that it may hang in the form of a semicircle with its diameter horizontal under the action of gravity.

$$x^2 = 2ay - y^2; \qquad \therefore \quad \frac{dx}{a-y} = \frac{dy}{x} = \frac{ds}{a}.$$

Therefore from (53), $\rho = \dfrac{\sigma ca}{(a-y)^2}$;

that is, the density varies inversely as the square of the depth below the horizontal diameter of the semicircle.

Also $$\text{T} = \frac{ga\sigma ca}{a-y}.$$

If therefore $y = a$, $\rho = \infty$, $\text{T} = \infty$: that is, the density and the tension are both infinite; and rightly so, because the string is vertical at the points of its support at the extremities of the horizontal diameter of the circle, and there is at them no counteracting horizontal force to balance the horizontal tension at the lowest point.

Ex. 3. To find the form of a heavy string, the thickness of which varies inversely as the square root of its length from the lowest point, when it is acted on by gravity.

In this case $$\omega = \mu s^{-\frac{1}{2}};$$
therefore from (53),
$$ga\sigma c \frac{dy}{dx} = \int_0^s g\rho\mu s^{-\frac{1}{2}} ds = 2g\rho\mu s^{\frac{1}{2}};$$
$$\frac{d\left(\frac{dy}{dx}\right)^2}{\left\{1+\frac{dy^2}{dx^2}\right\}^{\frac{1}{2}}} = \frac{4\rho^2\mu^2}{a^2\sigma^2c^2}dx; \qquad \therefore \quad \left\{1+\frac{dy^2}{dx^2}\right\}^{\frac{1}{2}} - 1 = \frac{2\rho^2\mu^2}{a^2\sigma^2c^2}x,$$
because the origin is at the lowest point, where the curve is horizontal; and making obvious substitutions,
$$a\frac{dy}{dx} = (x^2+2ax)^{\frac{1}{2}},$$
whence the equation to the curve will be found without difficulty. Also
$$\text{T} = ga\sigma c\frac{a+x}{a}.$$

Ex. 4. To find the equation to the catenarian curve, when the weight of each element of the curve varies as the horizontal projection of it.

This case is approximately that of suspension bridges, in which the weight of the chain and of the vertical suspending rods is neglected, and each element of the chain has to bear that part of the roadway which corresponds to the horizontal projection of it.

In this case $$\rho\omega g\, ds = \mu\rho\omega g\, dx;$$
therefore from (53),
$$ga\sigma c\frac{dy}{dx} = \int_0^x \mu\rho\omega g\, dx = \mu\rho\omega g x;$$
$$\therefore \quad dy = \frac{1}{2k}x\,dx; \qquad 4ky = x^2;$$

the equation of a parabola with its axis vertical, and vertex downwards.

Ex. 5. To determine the equation to the catenarian curve of uniform density, and the law of variation of the thickness, so that the thickness may be at all points proportional to the tension.

In this case
$$\omega = \mu \mathrm{T}; \tag{55}$$
therefore (33) becomes
$$\frac{\mathrm{T}}{ds} = \frac{\int_0^s g\rho\mu \mathrm{T}\, ds}{dy} = \frac{g\alpha\sigma c}{dx};$$
$$\therefore \quad g\alpha\sigma c d.\frac{dy}{dx} = g\rho\mu \mathrm{T}\, ds; \qquad d.\frac{dy}{dx} = g\rho\mu\frac{ds^2}{dx};$$
$$\frac{d.\frac{dy}{dx}}{1+\frac{dy^2}{dx^2}} = g\rho\mu\, dx; \tag{56}$$
$$\therefore \quad \log\sec \rho g\mu x = g\rho\mu y;$$
$$\sec g\rho\mu x = e^{g\rho\mu y}, \tag{57}$$
which is the equation to the required curve. This curve is called *the catenary of uniform strength*. If we substitute $\frac{1}{a}$ for $g\rho\mu$, we have $e^{\frac{y}{a}} = \sec\frac{x}{a}$; if $x=0$, $y=0$; and if $x=\pm\frac{\pi a}{2}$, $y=\infty$; so that the curve has two vertical asymptotes, equally distant from the origin, which are at a distance $= \pi a$ apart. Also
$$\mathrm{T} = g\alpha\sigma c \sec g\rho\mu x,$$
$$\omega = \mu g\alpha\sigma c \sec g\rho\mu x. \tag{58}$$

162.] In Art. 130 it is shewn that of all uniform and heavy curved lines of given length joining two given points in the same vertical plane, the catenary is that of which the centre of gravity has the lowest position; I propose to extend the problem to the case of heavy flexible strings of varying density and thickness, and to find the form of the curve so that the place of the centre of gravity of it may be the lowest possible.

Let the axis of z be vertical, and let a point on the curve be (x, y, z), and let the element ds begin at this point; let $\mu\, ds$ $=$ the mass-element of the string-curve, where μ is a function of x, y, z; then $\bar{z}$ is to be a minimum, where
$$\bar{z}\int_0^1 \mu\, ds = \int_0^1 \mu z\, ds. \tag{59}$$

Now $\int_0^1 \mu ds$ is the mass of the string, and this evidently is constant, so that the variation of the right-hand member of (59) is to vanish consistently with this condition;

$$\therefore \quad \delta.\int_0^1 \mu z ds = 0, \qquad \text{and} \qquad \delta.\int_0^1 \mu ds = 0; \qquad (60)$$

from the former we have

$$\begin{aligned} 0 &= \int_0^1 \delta.\mu z ds \\ &= \int_0^1 \left\{ \mu z \left(\frac{dx}{ds} d.\delta x + \frac{dy}{ds} d.\delta y + \frac{dz}{ds} d.\delta z \right) + ds \delta.\mu z \right\} \\ &= \left[\mu z \left(\frac{dx}{ds} \delta x + \frac{dy}{ds} \delta y + \frac{dz}{ds} \delta z \right) \right]_0^1 \\ &\qquad + \int_0^1 \left\{ \left(z ds \left(\frac{d\mu}{dx} \right) - d.\mu z \frac{dx}{ds} \right) \delta x + \left(z ds \left(\frac{d\mu}{dy} \right) - d.\mu z \frac{dy}{ds} \right) \delta y \right. \\ &\qquad\qquad \left. \left(\mu ds + z ds \left(\frac{d\mu}{dz} \right) - d.\mu z \frac{dz}{ds} \right) \delta z \right\}; \qquad (61) \end{aligned}$$

and from the latter of (60),

$$\begin{aligned} 0 &= \int_0^1 \delta.\mu ds \\ &= \left[\mu \left(\frac{dx}{ds} \delta x + \frac{dy}{ds} \delta y + \frac{dz}{ds} \delta z \right) \right]_0^1 \\ &\qquad + \int_0^1 \left\{ \left(ds \left(\frac{d\mu}{dx} \right) - \delta.\mu \frac{dx}{ds} \right) \delta x + \left(ds \left(\frac{d\mu}{dy} \right) - \delta.\mu \frac{dy}{ds} \right) \delta y \right. \\ &\qquad\qquad \left. \left(ds \left(\frac{d\mu}{dz} \right) - \delta.\mu \frac{dz}{ds} \right) \delta z \right\}. \qquad (62) \end{aligned}$$

Now for (61) and (62) to consist, it is necessary that

$$\begin{aligned} \frac{z ds \left(\frac{d\mu}{dx} \right) - d.\mu z \frac{dx}{ds}}{ds \left(\frac{d\mu}{dx} \right) - d.\mu \frac{dx}{ds}} &= \frac{z ds \left(\frac{d\mu}{dy} \right) - d.\mu z \frac{dy}{ds}}{ds \left(\frac{d\mu}{dy} \right) - d.\mu \frac{dy}{ds}} \\ &= \frac{\mu ds + z ds \left(\frac{d\mu}{dz} \right) - d.\mu z \frac{dx}{ds}}{ds \left(\frac{d\mu}{dz} \right) - d.\mu \frac{dx}{ds}} = \lambda, \qquad (63) \end{aligned}$$

where λ is an undetermined constant; and from these equations, when μ is given, the equation to the catenary is to be deduced. If $\mu = 1$, the equations (63) become (16), Art. 130.

163.] When the catenary is at rest under the action of forces, the action-lines of which pass all through a fixed point, and when that point is the source of the action of the force, so that the intensity of the force depends on the distance from that point of the particle on which the force acts, the equation and the properties of the catenary may be more conveniently investigated by the following process:

Let the point at which the forces originate, and which is called the centre of force, be taken for the origin, and let the central force acting on an unit of mass of the string be P; let the force be repulsive, so that its tendency is to remove the molecules of the string further from the origin, and therefore the string will be concave towards it; if the force is attractive P will be affected with a negative sign and the string-curve will be convex towards the origin. The components along the coordinate-axes of P acting on an unit-mass of the curve at the point (x, y, z) and at a distance r from the centre are

$$\text{X} = \frac{\text{P}x}{r}, \qquad \text{Y} = \frac{\text{P}y}{r}, \qquad \text{Z} = \frac{\text{P}z}{r};$$

so that the equations (12) become

$$\left.\begin{aligned} d.\text{T}\frac{dx}{ds} + \rho\omega ds\frac{\text{P}x}{r} &= 0, \\ d.\text{T}\frac{dy}{ds} + \rho\omega ds\frac{\text{P}y}{r} &= 0, \\ d.\text{T}\frac{dz}{ds} + \rho\omega ds\frac{\text{P}z}{r} &= 0; \end{aligned}\right\} \tag{64}$$

multiplying the second of these equations by z, and the third by y, and subtracting,

$$z\,d.\text{T}\frac{dy}{ds} - y\,d.\text{T}\frac{dz}{ds} = 0;$$

$$\left.\begin{aligned} \therefore \text{ integrating,} \qquad z\text{T}\frac{dy}{ds} - y\text{T}\frac{dz}{ds} &= h_1; \\ \text{and similarly} \qquad x\text{T}\frac{dz}{ds} - z\text{T}\frac{dx}{ds} &= h_2, \\ y\text{T}\frac{dx}{ds} - x\text{T}\frac{dy}{ds} &= h_3; \end{aligned}\right\} \tag{65}$$

and therefore multiplying these last equations severally by x, y, z, and adding,

$$h_1 x + h_2 y + h_3 z = 0; \tag{66}$$

which is the equation to a plane passing through the origin, which is the centre of force: whence we infer that the curve and the centre of force are in one and the same plane, and thus the catenary under the action of a central force is a plane curve.

164.] Let the plane in which the catenarian curve is be taken as the plane of reference; and let the curve be referred to a system of polar coordinates in it. Let (r, θ) be the place of the mass-element whose length is ds, and of which ρ and ω are respectively the density and the area of a transverse section. Also let P be the repulsive force and T the tension at this point. Then resolving along the tangent

$$\rho\omega\, ds\, \mathrm{P}\frac{dr}{ds} + d\mathrm{T} = 0;$$

$$\therefore \quad d\mathrm{T} + \rho\omega\mathrm{P}\, dr = 0; \tag{67}$$

which equation is also that of the virtual velocities, when the arbitrary displacement of the point of application of P and T takes place along the tangent. And resolving along the normal, if $d\psi$ is the angle contained between two consecutive normals, so that $ds = \rho' d\psi$, where ρ' is the radius of curvature and is equal to $r\dfrac{dr}{dp}$,

$$\rho\omega\, ds\, \mathrm{P}\frac{p}{r} + \mathrm{T}\, d\psi = 0;$$

$$\therefore \quad \rho\omega\mathrm{P}p + \mathrm{T}\frac{dp}{dr} = 0; \tag{68}$$

and if P is eliminated between (67) and (68),

$$\frac{d\mathrm{T}}{\mathrm{T}} + \frac{dp}{p} = 0;$$

$$\therefore \quad \mathrm{T}p = \mathrm{T}_0 p_0 = \text{a constant}, \tag{69}$$

if T_0 and p_0 are simultaneous given values of T and p.

Hence we conclude that the tension at any point of the curve varies inversely as the perpendicular from the centre of force on the tangent of the curve at that point.

The equation (69) is the equation of moments, with reference to the centre of force, of the forces acting on the element of the curve, and might have been deduced directly from (50), Art. 55.

If we eliminate T from (68) and (69) we have

$$\frac{dp}{p^2} - \frac{\omega\rho\mathrm{P}\, dr}{p_0 \mathrm{T}_0} = 0;$$

$$\therefore \quad \frac{1}{p} = -\int \frac{\omega\rho\mathrm{P}\, dr}{p_0\mathrm{T}_0}; \tag{70}$$

the limits of the integral being given by the conditions of the problem. From (70), when P is given, the equation to the curve may be found; and if the curve is given, P may be found; also from (69) the tension at any point of the curve may be found.

165.] In illustration of the preceding theorems let us take the following examples:

Ex. 1. If the central force is constant and is attractive, find the equation to the catenarian curve of constant thickness and density.

Let the force $= -f$; so that (70) becomes

$$\frac{1}{p} = \frac{\omega \rho f}{p_0 \mathrm{T}_0} \int dr$$
$$= \frac{\omega \rho f}{p_0 \mathrm{T}_0} r;$$

the curve being such that $r = \infty$, when $p = 0$; making an obvious substitution, we have

$$pr = k^2;$$

whence we have $k^2 = r^2 \cos 2\theta$, which is the equation of the equilateral hyperbola.

Also from (69), $\mathrm{T} = \omega \rho f r$.

Ex. 2. Find the equation to the curve of constant thickness and density when the central force is repulsive and varies as the distance.

Let $\mathrm{P} = \mu r$; so that from (70), if $p = 0$, when $r = \infty$,

$$\frac{1}{p} = -\frac{\mu \rho \omega}{2 \rho_0 \mathrm{T}_0} r^2$$
$$= -\frac{r^2}{c^3};$$

whence by integration we have

$$\frac{c^3}{r^3} = \cos 3(\theta - \gamma).$$

Ex. 3. Find the equation of the catenarian curve of constant thickness and density, when the central force is attractive and varies inversely as the square of the distance.

Let $\mathrm{P} = -\frac{\mu}{r^2}$; so that from (70),

$$\frac{1}{p} - \frac{1}{p_0} = \frac{\omega \rho \mu}{p_0 \mathrm{T}_0}\left(\frac{1}{r} - \frac{1}{r_0}\right);$$

and making obvious substitutions, and replacing $\frac{1}{r}$ by u, we have

$$c(u-k) = \frac{1}{p};$$

$$\text{therefore} \quad \frac{du^2}{d\theta^2} = (c^2-1)u^2 - 2c^2 ku + c^2 k^2; \tag{71}$$

and the integral of this equation will be of three different forms, according as c is greater than, equal to, or less than, unity.

(1) Let c^2 be greater than unity; then, if $c^2-1 = n^2$, the integral of (71) is of the form

$$u-a = \frac{b}{2}\{e^{n\theta}+e^{-n\theta}\}.$$

(2) Let $c^2 = 1$, then the integral is of the form

$$r = \frac{c}{1-\theta^2}.$$

(3) Let c^2 be less than unity; then, if $1-c^2 = n^2$,

$$u-a = b\cos n\theta.$$

Ex. 4. If the catenarian curve of uniform thickness and density is a parabola under the action of a central force in the focus, that force varies as $r^{-\frac{3}{2}}$.

Ex. 5. Prove that a parabola is the catenarian curve of constant density when the force varies inversely as the distance, and the thickness varies inversely as the square root of the distance from the centre of force.

Ex. 6. If the catenarian curve of uniform thickness and density is a circle, and has the centre of force in the circumference, shew that the force varies inversely as the cube of the distance.

166.] The catenary thus far has been considered a free curve. If however the string is stretched on a curved surface, and is also under the action of given forces by which it is kept on the surface, the equations of equilibrium may be investigated in the following manner:

Let us in the first place consider the surface to be smooth.

Let the equation to it be $F(x, y, z) = 0$; and let its partial derived functions be U, V, W; and let $Q^2 = U^2+V^2+W^2$: let $R\,ds$ be the pressure of the surface against the mass-element whose length is ds, so that the equations of equilibrium are

$$\left.\begin{aligned} d.T\frac{dx}{ds} + \rho\omega X\,ds + R\frac{U}{Q}\,ds &= 0, \\ d.T\frac{dy}{ds} + \rho\omega Y\,ds + R\frac{V}{Q}\,ds &= 0, \\ d.T\frac{dz}{ds} + \rho\omega Z\,ds + R\frac{W}{Q}\,ds &= 0. \end{aligned}\right\} \tag{72}$$

Multiply these equations severally by dx, dy, dz, and add, and let s be equicrescent; then because

$$U\,dx + V\,dy + W\,dz = 0,$$

we have
$$dT + \rho\omega\,\{X\,dx + Y\,dy + Z\,dz\} = 0; \tag{73}$$

which assigns the tension in terms of the impressed forces, and shews that it is independent of the reaction of the surface; and if X, Y, Z are functions of the coordinates of ds, and such that $\rho\omega(X\,dx + Y\,dy + Z\,dz)$ is an exact differential, then T depends on the coordinates of the extreme points of the string, and is independent of the form of the surface.

If $X\,dx + Y\,dy + Z\,dz = 0$, T is constant throughout the length of the string, whatever is the form of the surface.

Again, differentiating the first terms of (72), and multiplying the equations severally by $\frac{U}{Q}, \frac{V}{Q}, \frac{W}{Q}$, and adding, we have

$$T\left\{\frac{U}{Q}\,d.\frac{dx}{ds} + \frac{V}{Q}\,d.\frac{dy}{ds} + \frac{W}{Q}\,d.\frac{dz}{ds}\right\} + \rho\omega\,ds\left\{\frac{XU + YV + ZW}{Q}\right\} + R\,ds = 0; \tag{74}$$

and therefore if $\theta =$ the angle between the normal to the surface and the principal normal to the curve at a common point, and if $\phi =$ the angle between the normal to the surface and the line of action of the resultant of the impressed forces, viz. P, and if $\rho' =$ the radius of absolute curvature of the curve, we have

$$\frac{T\cos\theta}{\rho'} + \rho\omega P\cos\phi + R = 0; \tag{75}$$

so that from (73) and (75) R may be determined. And since $R\,ds$ is the pressure of an element of the curve against the surface,

$$\text{the whole pressure} = \int R\,ds. \tag{76}$$

Again, suppose that $X = Y = Z = 0$, and that we differentiate the first terms of each of the equations (72), and eliminate T and dT by cross-multiplication, then

$$(dz\,d^2y - dy\,d^2z)U + (dx\,d^2z - dz\,d^2x)V + (dy\,d^2x - dx\,d^2y)W = 0; \tag{77}$$

and therefore the binormal of the curve is perpendicular to the normal of the surface; the curve therefore along which the string is laid is a geodesic line on the surface.

167.] If the string rests on a smooth plane curve, we may take the plane of the curve to be that of (x, y), and $F(x, y) = 0$ to be the equation to the curve; in which case the equations are

$$\left.\begin{aligned} d.\text{T}\frac{dx}{ds}+\rho\omega\text{X}\,ds-\text{R}\,dy &= 0,\\ d.\text{T}\frac{dy}{ds}+\rho\omega\text{Y}\,ds+\text{R}\,dx &= 0;\end{aligned}\right\} \tag{78}$$

whence we have

$$d\text{T}+\rho\omega\,(\text{X}\,dx+\text{Y}\,dy) = 0; \tag{79}$$

$$\frac{\text{T}}{\rho'}+\rho\omega\left(\text{X}\frac{dy}{ds}-\text{Y}\frac{dx}{ds}\right) = \text{R}; \tag{80}$$

whereby T and R may be found.

If gravity is the only acting force, we may take the plane of (x, y) to be vertical, and take the horizontal line to be the x-axis, and the y-axis to be positive upwards: then, if the string is of uniform thickness and density,

$$d\text{T}-\rho\omega g\,dy = 0; \qquad \therefore \quad \text{T}-\text{T}_0 = \rho\omega g(y-y_0); \tag{81}$$

$$\text{R} = \frac{\text{T}}{\rho'}+\rho\omega g\frac{dx}{ds}. \tag{82}$$

The following are examples in which the pressure of strings on smooth surfaces and curves is calculated:

Ex. 1. On the smooth surface of a circular cylinder whose radius $= a$, and whose axis is horizontal, a heavy homogeneous string of given length rests in a vertical plane: determine the tension at any point and the whole pressure on the cylinder.

Let the section of the cylinder be represented in fig. 64. Let θ_1 and θ_0 be the angles corresponding to the ends of the string, θ being measured from the horizontal line through the centre of the circle. Let the place of ds be (a, θ); then, if θ_0 is the angle corresponding to the lower end of the string, $\text{T}_0 = 0$; and the tension at any point is equal to the sum of the weights of the successive elements of the string resolved along the curve; so that

$$\begin{aligned}\text{T} &= \int_{\theta_0}^{\theta} a\rho\omega g\cos\theta\,d\theta\\ &= a\rho\omega g\,(\sin\theta-\sin\theta_0);\end{aligned} \tag{83}$$

$$\therefore \quad \text{T}_1 = a\rho\omega g\,(\sin\theta_1-\sin\theta_0). \tag{84}$$

Hence if the string reaches from the highest point to the horizontal line, $\theta_0 = 0$, $\theta_1 = \frac{\pi}{2}$, and the tension at the highest point $= a\rho\omega g$; but the weight of the string $= \frac{\pi\rho a\omega g}{2} = \text{W}$, say;

$$\therefore \quad \text{T} = \frac{2\text{W}}{\pi};$$

so that if a weight $= \text{W}'$ is suspended to the string at the lowest point where it touches the cylinder,

$$\text{T} = \frac{2\text{W}}{\pi} + \text{W}'.$$

The pressure on the surface may thus be found. It is due (1) to the weight of the element of the string which corresponds to it, and this $= a\rho\omega g \sin\theta\, d\theta$; (2) to the tension; let the tension at $ds = \text{T}$, and let ds subtend an angle $= d\theta$ at the centre of the circle; the action-lines of T at both ends of ds coincide with the tangents at these points, and R acts along the line which joins the centre of the circle to the point of intersection of these two tangents; consequently

$$\text{R}\, a d\theta = 2\text{T} \sin\frac{d\theta}{2} = \text{T} d\theta; \qquad \therefore\ \text{R} = \frac{\text{T}}{a};$$

and we have
$$\text{R} = \frac{\text{T}}{a} + \rho\omega g \sin\theta; \tag{85}$$

which result is the same as (82). Hence

$$\begin{aligned}\text{the whole pressure} &= \int_{\theta_0}^{\theta_1} a\text{R}\, d\theta \\ &= \int_{\theta_0}^{\theta_1} \{a\rho\omega g(\sin\theta - \sin\theta_0) + a\rho\omega g \sin\theta\}\, d\theta \\ &= 2a\rho\omega g(\cos\theta_0 - \cos\theta_1) - a\rho\omega g \sin\theta_0\, (\theta_1 - \theta_0). \quad (86)\end{aligned}$$

Hence if the string reaches from the highest point to the horizontal line the whole pressure $= 2a\rho\omega g$; that is, the whole pressure is equal to twice the tension at the highest point.

The preceding investigation shews that the part of the pressure due to the tension varies inversely as the radius of the cylinder; and as the investigation involves only the infinitesimal angles at which two consecutive normals are inclined to each other, the result is true for any cylinder of continuous curvature; so that, if ρ' is the radius of curvature,

$$\text{R} = \frac{\text{T}}{\rho'}; \tag{87}$$

this being that part of the normal pressure which is due to the tension of the string.

Hence also for a given pressure the tension varies inversely as the curvature of the cylinder.

Ex. 2. If a string, whose mass is so small that it may be neglected in comparison of the tension which acts on it, rests on a smooth surface, what are the circumstances of pressure and tension?

In this case, all the terms involving $\omega\rho$ are to be omitted; so that from (73) $d\text{T} = 0$; and T is constant throughout the length of the string.

Also from (75),
$$\text{R} = \frac{\text{T}\cos\theta}{\rho'}. \tag{88}$$
If the string lies in a plane curve, $\cos\theta = 1$; and we have, as also from (82),
$$\text{R} = \frac{\text{T}}{\rho'}. \tag{89}$$
Let $d\psi$ be the angle of contingence at the point (x, y); so that $ds = \rho' d\psi$;
$$\therefore \quad \text{R}\,ds = \text{T}\,d\psi;$$
$$\begin{aligned} \therefore \quad \text{the whole pressure} &= \int_0^1 \text{R}\,ds \\ &= \text{T}(\psi_1 - \psi_0). \end{aligned} \tag{90}$$
Thus the whole pressure along the curve between the given limits varies as the angle between the normals at the ends of the curve.

Thus, if over a smooth horizontal cylinder a fine string is suspended, which has at its ends weights, each of which $= \text{W}$, and these hang vertically downwards,
$$\text{the whole pressure} = \pi\text{W}.$$

168.] Suppose however the surface on which the string rests to be rough, and the string to be on the point of motion along its length, so that friction arises from the roughness; then this friction is a force which acts along the string in the direction contrary to that of the motion: and if $\text{R}\,ds$ is the pressure on the surface of a length-element of the string, and $\text{F}ds$ is the friction corresponding to ds, and μ is the coefficient of friction, see Art. 118,
$$\text{F}\,ds = \mu\text{R}ds;$$
and as F acts in the direction of the string along which motion is about to take place, the components of $\text{F}\,ds$ are
$$\text{F}\,dx, \qquad \text{F}\,dy, \qquad \text{F}\,dz;$$
$$\text{or} \quad \mu\text{R}\,dx, \qquad \mu\text{R}\,dy, \qquad \mu\text{R}\,dz;$$
so that the equations of pressure are
$$\left.\begin{aligned} d.\text{T}\frac{dx}{ds} + \rho\omega\text{X}\,ds + \mu\text{R}dx + \text{R}\frac{\text{U}}{\text{Q}}\,ds &= 0, \\ d.\text{T}\frac{dy}{ds} + \rho\omega\text{Y}\,ds + \mu\text{R}dy + \text{R}\frac{\text{V}}{\text{Q}}\,ds &= 0, \\ d.\text{T}\frac{dz}{ds} + \rho\omega\text{Z}\,ds + \mu\text{R}dz + \text{R}\frac{\text{Z}}{\text{Q}}\,ds &= 0; \end{aligned}\right\} \tag{91}$$
and from these equations general properties may be deduced.

As the investigation, however, presents no difficulties, and is similar to those of the preceding Articles, we need not occupy our space with it; and I will take a particular form which gives some practical results of considerable interest.

Over the surface of a rough circular cylinder, whose axis is horizontal, a fine inextensible string, whose mass may be neglected, is placed in a vertical plane, and given forces act at the ends of the string. What are the circumstances of pressure and tension?

Let fig. 64 represent the string resting on the cylinder, of which the plane of the paper is a section perpendicular to the axis of the cylinder: let the string be in contact with the cylinder over an arc which subtends at the centre the angle $\text{ACB} = a$; and let the forces at the ends of the string be T_0 and T'; and these are also the tensions at A and B. Let $\text{AC} = a$, $\text{ACP} = \theta$, $\text{PCQ} = d\theta$; then resolving normally and tangentially, we have

$$\text{T} = a\text{R}; \qquad d\text{T} = \text{F}\,ds = \mu \text{R}\, a\, d\theta: \tag{92}$$

$$\therefore \quad \frac{d\text{T}}{\text{T}} = \mu d\theta; \qquad \text{T} = \text{T}_0\, e^{\mu\theta}, \tag{93}$$

as T_0 is the tension when $\theta = 0$; hence as θ increases in arithmetical progression, T increases in geometrical progression. The value of T is the greatest just as the rope begins to slip; let T_1 be the value of T at B just as the slipping begins; then

$$\text{T}_1 = \text{T}_0\, e^{\mu a}; \tag{94}$$

so that if the force at B is less than the value of T_1 thus determined, the rope will not move. Thus, if a rope were wound twice round the cyclinder,

$$\text{T}_1 = \text{T}_0\, e^{4\pi\mu};$$

and if $\mu = 4$, which is an usual value of μ, we have approximately $\text{T}_1 = 165\,\text{T}_0$, which shews how great is the force which one man may exert by merely coiling a rope round a post.

From the first of (92) we have $\text{R} = \frac{\text{T}}{a} = \frac{\text{T}_0}{a} e^{\mu\theta}$; consequently

$$\text{the normal pressure on the cylinder} = \int_0^a \text{T}_0\, e^{\mu\theta} d\theta$$

$$= \frac{\text{T}_0}{\mu}(e^{\mu a} - 1). \tag{95}$$

169.] Ex. 1. A string passes over three rough cylindrical horizontal bars which are at equal distances apart, and the lower two of which are in the same horizontal plane; and at the ends of the string weights are suspended: find the difference between them just as motion begins to take place.

As the cord is in contact with the surfaces through an angle $\frac{\pi}{6}$ at each of the lower bars, and through an angle $\frac{2\pi}{3}$ at the upper bar,

$$T_1 = T_0 e^{\mu\pi}. \tag{96}$$

Ex. 2. A string passes over a rough horizontal cylinder; and two weights P and Q are suspended at its ends so that P is just beginning to descend: what weight must be added to Q, so that Q may be beginning to descend?

Let Q′ be the additional weight required; then we have

$$P = Qe^{\mu\pi}, \qquad Q + Q' = Pe^{\mu\pi};$$

$$\therefore \quad Q' = \frac{P^2 - Q^2}{Q}.$$

Ex. 3. A heavy uniform chain is hung over a rough horizontal cylinder; how much lower will one end of the chain be than the other, just when the chain begins to move?

Let c be the length of chain which hangs down on one side, and $c+x$ the length of that which hangs on the other, just when the chain begins to move, so that the pressures at the ends of the horizontal diameters are $c\omega\rho g$ and $(c+x)\omega\rho g$ respectively: then, taking account of the weight of the chain, and resolving tangentially and normally, we have

$$dT = \rho\omega g\, dy + \mu R\, ds; \tag{97}$$

$$R = \frac{T}{a} + \rho\omega g \sin\theta; \tag{98}$$

$$\therefore \quad dT - \mu T\, d\theta = \rho\omega g a\, d\theta(\cos\theta + \mu\sin\theta); \tag{99}$$

and integrating, and introducing the values at the given limits, we have

$$x = \frac{2\mu a}{\mu^2+1}(e^{\mu\pi}+1) + c(e^{\mu\pi}-1). \tag{100}$$

If $c = 0$, no string hangs on one side of the cylinder; and x then determines the force which must be applied at the other end to make the string move round the cylinder.

SECTION 2.—*The equilibrium of elastic strings.*

170.] Our knowledge of the internal constitution of bodies is doubtless very imperfect; but so far as it goes, there is no material substance in nature, the relative positions of the particles of which are not changed when the matter is acted on by external pressures: if a force acts on a body at a certain point, and in the way of pressure against it, the particles of the body at, or about the point of application, approach to each other; and if the force is a pulling force, the distances between the constituent molecules of the body, at and about the point of application, are increased. It seems indeed that a body is made up of a system of molecules, infinitesimal in volume, and at an infinitesimal distance apart, and that these are held in a state of relative rest by forces acting reciprocally from one to another; and that these forces are functions of the distances between the molecules; and that when an external force acts on the system, the molecules are either separated farther from, or are brought nearer to, each other, by reason of the action of the force; so that either a compression or a dilatation of the system takes place; all bodies, that is, are compressible and extensible to a certain degree: the relative position of the molecules is not the same when the body is free from, and when it is subject to, external pressures. Into the particular mode of action of such forces on the constitution of a body, or the change of molecular action of the internal forces under the influence of such external force, I shall enter only briefly, and generally, and reserve the special study of the subject to a subsequent portion of this course, where I hope fully to enter into it; and also now we have not data sufficient for the full solution of the problem. But I would observe, that our previous results of forces acting on rigid bodies, that is, on bodies the constituent molecules of which are in a state of relative rest, are not hereby falsified, because the molecules of the body though disturbed at first are ultimately in relative rest. It is the amount of this disturbance which we shall generally calculate: and upon the hypothesis of the truth of certain laws, which are for the most part empirical, and will not be deduced from more remote principles of the structural constitution of bodies.

The disturbances or displacements which the molecules undergo are of three kinds: there may be (1) a longitudinal compression or dilatation; I shall calculate the effects of this on a bar or a string: (2) a flexure or a bending, as of a thin flexible membrane, or plate or spring; this I shall also consider: (3) a twisting or a torsion, as of a twisted bar. Now in all these, as in all similar displacements, one result is the same; no disturbance or disarrangement, at least within certain limits, takes place, unless there is also called into action a force of restitution, whereby the body tends to recover its former state; the molecular forces are such that, so long as temperature, &c., remain the same, they tend to bring the body back again into that state which it had before the disturbance due to the external force: this energy of restitution is called *Elasticity;* "La force elastique," says D'Alembert, "est une propriété ou puissance des corps, au moyen de laquelle ils se rétablissent dans la figure et l'étendue, qu'une cause extérieure leur avait fait perdre." Thus elasticity in the first of the three cases mentioned above, is the tendency which a stretched string has to return to its former and unstretched length: in the second case it is the force of a spring, as that of a coil which is the motive power of a watch: in the third case it is the force of return which a twisted wire exhibits, as in Coulomb's Torsion Balance, or in Cavendish's experiment with leaden balls. Let this term then be plainly distinguished from expansibility, extensibility, compressibility, and so on: it is consequent upon these last, but expresses a property quite distinct from them; and the greater or less perfectness of elasticity of a given substance depends on the degree with which it recovers the state, as to the arrangement of its molecules, whence it has been displaced: if the state is altogether recovered, elasticity is perfect: if the body remains in the state into which it has been put by the disturbing force, it is said to be wholly inelastic: neither of these conditions is ever fully satisfied in nature. Thus much as to elasticity is sufficient for our present purpose.

171.] I will in the first place take the most simple case of an extensible string, which is stretched by the action of certain forces in the direction of its length.

The law to which the extension is subject, and which is commonly called Hooke's law, is, The extension is as the tension: that is, the length added to an extensible string by means of a stretching force varies as the force. Also the same law may be supposed to be applicable to compression, that is, the compression varies as the compressing force. Suppose the length of an extensible string of an unit-length, and the area of whose transverse section is an unit-area, to be by the action of an unit-force increased by a length e, so that 1 becomes $1+e$; then, by reason of the preceding law, under the action of a force T, the length is increased by $e\text{T}$, so that 1 becomes $1+e\text{T}$; and therefore, the circumstances as to thickness, density, &c., of the string being the same throughout, the length of a string of length a becomes $a(1+e\text{T})$; e is called *the coefficient of elasticity*. If the stretching force is not the same throughout the length of the string, this formula is inapplicable as it stands; but we may resolve the string into infinitesimal parts, and apply the law to each of these.

It is sometimes convenient to express e in another form. Let a' be the length of a when stretched by the constant force T throughout; so that

$$a' = a(1+e\text{T}); \tag{101}$$

and let E be the value of T, when a is stretched so that its length is doubled:

$$\text{then} \qquad 2a = a(1+e\text{E}); \qquad \therefore \quad e = \frac{1}{\text{E}}; \tag{102}$$

$$\text{and (101) becomes} \qquad a' = a\left(1+\frac{\text{T}}{\text{E}}\right): \tag{103}$$

E is called *the modulus of elasticity*.

172.] Ex. 1. A heavy extensible string of constant thickness and density is suspended by one end, and hangs vertically; it is required to find the length of it thus stretched.

Let O, fig. 68, be the end by which it is suspended: $a =$ the length of it when unstretched: $\text{OA} = a' =$ the length when stretched: $\rho =$ the density: $\omega =$ the area of a transverse section: $g =$ earth's attraction on an unit-mass: $\text{OP} = x'$, $\text{PQ} = dx'$: and suppose x to be the distance of P from O, when the string is not stretched: then the weight of $\text{PA} = \rho g \omega (a-x)$: and this is the stretching force on PQ: therefore

$$dx' = dx\,\{1+e\rho g\omega(a-x)\};$$

$$\therefore \quad \Big[x'\Big]_0^{a'} = \int_0^a \{1+e\rho g\omega(a-x)\}\,dx;$$

$$a' = a + \frac{e\rho g\omega a^2}{2}$$

$$= a\left\{1+\frac{e\rho g\omega a}{2}\right\}.$$

If W is the weight of the chain, $\text{W} = \rho\omega g a$, and if E is the modulus of elasticity,

$$a' = a\left\{1+\frac{\text{W}}{2\text{E}}\right\}.$$

If the whole weight of the string had been collected at the lowest point, then

$$a' = a\{1+e\text{W}\};$$

and therefore by its own weight the string is stretched only half as much as it would be, if that weight were collected at its lowest point.

If ρ or ω varies, the corresponding alteration must be made in the preceding integral.

Ex. 2. A heavy extensible string of constant thickness and density is suspended by one end, and hangs vertically; at a given point in it a weight is fixed: it is required to find the length of the string thus stretched.

Let O be the end by which it is suspended: let A be the point at which the weight, say W, is placed, OA $= a$, AB $= b$, a and b referring to the string unstretched: then, using the same symbols as in the preceding example, we have

$$a'+b' = \int_0^a \{1+\rho g\omega e(a+b-x)+e\text{W}\}\,dx + \int_0^b \{1+\rho g\omega e(b-x)\}\,dx$$

$$= a+b+e\text{W}a+\frac{\rho g\omega e}{2}(a+b)^2.$$

Ex. 3. Two weights P and Q resting on two inclined planes, fig. 69, are connected by an elastic string PQ; it is required to find the position of equilibrium.

Let CP $= x$, CQ $= y$; let the inclinations to the horizon of CA, CB, PQ be α, β, θ; let the tension of PQ = T, and the unstretched length $= a$: $\therefore$ PQ $= a\{1+e\text{T}\}$.

Then resolving along the planes, and eliminating T, we have

$$\tan\theta = \frac{\text{Q}\sin\beta\cos\alpha - \text{P}\sin\alpha\cos\beta}{(\text{P}+\text{Q})\sin\alpha\sin\beta},$$

$$\text{PQ} = a\left\{1+\frac{e\text{P}\sin\alpha}{\cos(\theta+\alpha)}\right\}.$$

Ex. 4. A heavy string whose density varies as the distance from one end is suspended by that end and stretched by its own weight: find the extension.

Employing the same notation as before, and replacing ρ by kx, see fig. 68,

$$\text{the weight of AP} = \int_x^a k\omega g x\,dx = \frac{k\omega g}{2}(a^2-x^2);$$

and this is the stretching weight of dx;

$$\therefore \quad dx' = dx\left\{1 + \frac{ek\omega g}{2}(a^2 - x^2)\right\}$$

$$\int_0^{a'} dx' = \int_0^{a}\left\{1 + \frac{ek\omega g}{2}(a^2 - x^2)\right\}dx$$

$$a' = a + \frac{ek\omega g a^3}{3}.$$

If W is the weight of the string,

$$\begin{aligned} W &= \int_0^a k\omega g x\,dx \\ &= \frac{k\omega g a^2}{2}; \end{aligned}$$

$$\therefore \quad a' = a\left\{1 + \frac{2eW}{3}\right\}.$$

Ex. 5. A heavy elastic ring is placed round a smooth vertical cone, and descends by its own weight; it is required to find the position of equilibrium.

Consider the cone to be the limiting form of a regular pyramid of n sides, of which two adjacent ones are the triangles APQ, AQR in fig. 70: and let PQ and QR be two adjoining elements of the string which rest on these sides: let the triangles APQ and AQR be bisected by the lines Ap and Aq drawn to the middle points of their bases; and so that the string contained between p and q is the nth part of the whole ring. Let W = the weight of the ring, a = the radius of it unstretched; r = the radius of it stretched; 2α = the vertical angle of the cone; then the weight of pQq $= \frac{W}{n}$, and this resolved along AQ $= \frac{W}{n}\cos\alpha$: now the other forces acting on pQq are the two tensions along pQ and Qq, and these are equal to each other and to T (say); let AQP = AQR = β; and resolving along AQ, we have

$$\frac{W}{n}\cos\alpha = 2T\cos\beta;$$

$$\text{now} \quad 2\pi r = 2\pi a\{1 + eT\};$$

$$\therefore \quad r = a\left\{1 + \frac{eW}{2n}\frac{\cos\alpha}{\cos\beta}\right\};$$

$$\text{but} \qquad \cos\beta = \frac{pQ}{AQ} = \frac{\pi OQ}{nOQ}\sin\alpha;$$

$$\therefore \quad r = a\left\{1 + \frac{eW\cot\alpha}{2\pi}\right\};$$

and this determines the position of the ring.

173.] The last example of the preceding Article differs from the others, in that the string, by reason of its increased length, also undergoes a change of curvature; and this change of form is doubtless to a certain extent resisted, or favoured, as the case may be, by the elastic forces of the matter of the string: that is, by those forces of elasticity which affect the curvature of the string. And of these forces no account has been taken; the ring is supposed to be perfectly flexible, and yet extensible.

We may however consider, in a more general form, the curve which is taken by a string, perfectly flexible, and extensible according to Hooke's Law, under the action of given forces.

Let ds' be a length-element of the curve before it is stretched, and ds the corresponding length-element in its stretched state: then, if T is the tension,

$$ds = ds'(1+e\mathrm{T}); \qquad (104)$$

also let the thickness and density of the curve when stretched be the same throughout the length; this supposition is of course in applications generally only approximately true; and let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ be the impressed forces acting on an unit-mass of the string before it is stretched; then the equations (12), Art. 152, become

$$\left.\begin{aligned} \rho\omega\mathrm{X}\,ds + (1+e\mathrm{T})\,d.\mathrm{T}\frac{dx}{ds} &= 0, \\ \rho\omega\mathrm{Y}\,ds + (1+e\mathrm{T})\,d.\mathrm{T}\frac{dy}{ds} &= 0, \\ \rho\omega\mathrm{Z}\,ds + (1+e\mathrm{T})\,d.\mathrm{T}\frac{dz}{ds} &= 0; \end{aligned}\right\} \qquad (105)$$

from which the general properties of the curve are to be deduced, and the properties of any particular curve when the impressed forces are given.

Let s be equicrescent; then, expanding the last terms of each of the equations (105), and multiplying the equations severally by dx, dy, dz, and adding, we have

$$\rho\omega\,\{\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz\} + (1+e\mathrm{T})d\mathrm{T} = 0. \qquad (106)$$

And from (104), $\qquad e\,d\mathrm{T} = d.\frac{ds}{ds'}$;

$$\therefore \quad \rho\omega\,\{\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz\} + \frac{1}{e}\frac{ds}{ds'}\,d.\frac{ds}{ds'} = 0;$$

$$\therefore \quad \rho\omega\,\{\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz\} + \frac{1}{2e}\,d.\left(\frac{ds}{ds'}\right)^2 = 0; \qquad (107)$$

whence by integration $\frac{ds}{ds'}$, and thence the extension of the string, may be determined.

174.] Suppose however the string to be heavy, and gravity to be the sole acting force: the string-curve will manifestly be wholly in one vertical plane. Let the plane be that of (x, y), and let the axes of x and y be respectively horizontal and vertical: and let the curve be above the axis of x: then

$$\mathrm{X} = \mathrm{Z} = 0, \qquad \mathrm{Y} = -g;$$

therefore from the first of (105),

$$d.\mathrm{T}\frac{dx}{ds} = 0; \tag{108}$$

and therefore the horizontal tension is constant throughout the curve: let it be equal to the weight of a string of length c, the thickness and density of which are the same as those of the string-curve: then integrating (108) we have

$$\mathrm{T}\frac{dx}{ds} = \rho\omega c g. \tag{109}$$

Again, from the second of (105),

$$\rho\omega g\, ds = (1+e\mathrm{T})\, d.\mathrm{T}\frac{dy}{ds};$$

$$\therefore \quad ds = c\Big(1+e\rho\omega c g\frac{ds}{dx}\Big)\, d.\frac{dy}{dx}. \tag{110}$$

To integrate this, let $\frac{dy}{dx} = \tan\tau$; therefore

$$ds = c\,(1+e\rho\omega c g \sec\tau)\, d.\tan\tau; \tag{111}$$

$$\left.\begin{aligned} \therefore \quad dx &= ds\cos\tau = c\,(\cos\tau + e\rho\omega c g)\, d.\tan\tau, \\ dy &= ds\sin\tau = c\,(\sin\tau + e\rho\omega c g\tan\tau)\, d.\tan\tau; \end{aligned}\right\} \tag{112}$$

and integrating,

$$x = c\left\{\log\tan\Big(\frac{\pi}{4}+\frac{\tau}{2}\Big) + e\rho\omega c g\tan\tau\right\}, \tag{113}$$

$$y = c\left\{\sec\tau + \frac{e\rho\omega c g}{2}(\tan\tau)^2\right\}, \tag{114}$$

the limits of integration being such that $x = 0$, $y = c$, when $\tau = 0$; so that the axis of x is the directrix of the curve, and at a distance c below the lowest point of the curve; and the axis of y passes through the lowest point. Also from (111) and (109) we have,

$$s = c\left\{\tan\tau + \frac{e\rho\omega cg}{2}\left(\tan\tau\sec\tau + \log(\tan\tau+\sec\tau)\right)\right\}, \tag{115}$$

$$T = \rho\omega cg \sec\tau. \tag{116}$$

If τ is eliminated by means of (113) and (114), the resulting equation is that of the string-curve: the expression however is so complicated that it is not worth while to write it at length. But in the case wherein e is small, and the second and higher powers of it may be neglected without appreciable error, from (113) we have,

$$\sin\tau = \frac{e^{\frac{x}{c}} - e^{-\frac{x}{c}}}{e^{\frac{x}{c}} + e^{-\frac{x}{c}}};$$

and therefore from (114),

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}} + \frac{e\rho\omega cg}{4}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)^2\right\}, \tag{117}$$

which is the equation to the catenary of slight extensibility. Also to determine the increase of the length of the arc, in this case we have from (104),

$$ds' = \frac{ds}{1+eT} = ds(1-eT);$$

therefore from (116), and neglecting terms in (114) involving e,

$$\begin{aligned} ds' &= ds - e\rho\omega cg\sec\tau\, ds \\ &= ds - e\rho\omega gy\, ds, \\ \therefore\quad ds - ds' &= e\rho\omega gy\, ds, \\ \therefore\quad s - s' &= e\rho\omega g\int y\, ds. \end{aligned} \tag{118}$$

Now if $\bar{y}$ is the distance from the directrix of the centre of gravity of the arc s,

$$\bar{y}s = \int y\, ds;$$

$$\therefore\quad s - s' = e\rho\omega g s\bar{y};$$

that is, the increase of the arc s due to the tension varies as the distance from the directrix of the centre of gravity of the arc.

SECTION 3.—*The equilibrium of the elastic plates or springs.*

175.] In this section I propose to take only a few cases of a simple character, and to select those which not only exemplify the general mode of investigating problems of elastic plates, but are also useful as establishing the principles on which the strength of materials is estimated by civil engineers. And as the first example I will consider the effects of forces applied to the bending of a flexible and elastic thin plate whose bounding outline in its plane and original form is a rectangle.

Imagine a rectangular plate of an uniform elastic action and of constant density, of a finite length and breadth, a and b; and of infinitesimal thickness 2τ, which however is such as to develop forces of elasticity when the lamina is bent in the direction of its length by the action of certain external forces. Also imagine the plate to be resolved into a series of rods, all of which are parallel to the length a of the plate, and are of infinitesimal depth dz; so that of each of these the thickness is 2τ and the length is a. When the flexure takes place each of these rods may undergo three different kinds of change: (1) the length may be contracted or increased; (2) the absolute curvature may be altered; (3) one element of a rod may be twisted upon the consecutive element of the same rod: the first two effects I shall consider: the latter will not enter into the investigation, as the material is supposed to be of a non-crystalline texture, and, as such, to be incapable of developing forces which would cause the twisting.

Suppose the rectangular plate, fig. 73, to be that whose length is a and breadth is b; and suppose it to be perpendicular to the plane of the paper, and in its original unbent form to pierce the plane of the paper along the axis of x: also suppose it to be fixed throughout its breadth at the extremity passing through O, so that when the plate is bent, that end of it which is intersected by the plane of the paper at O may be unchanged as to position; and suppose the end of the plate at A to be stiff throughout its breadth, so that the plate may be bent by a single force applied at that extremity; and thus that its surfaces, which were originally plane and parallel and at a distance 2τ apart, may be the two surfaces of a cylinder: and thus all the rods, into which we have imagined the plate to be divided, will be rods, equal and similarly bent, of the form delineated in the figure; and where X and Y are the pressures parallel to the axes, applied at the extremity A and causing the flexure of the plate.

Let us consider the bent rod of infinitesimal depth dz, and whose under-surface in the figure is OAB; and let us assume that the molecules, which in the bent state are along the normal common to both surfaces, were originally in a line normal to the two plane faces; so that NP″P′ is the common normal to the two curves OP′ and BP″; let another consecutive normal be drawn to these curves, and let it meet the former normal in N, so that N is the centre of curvature. Again, let the rod be resolved into other smaller rods or fibres, the depth of each of which is the same as that of the rod, and the sum of the several breadths of which is 2τ: then each of those contained within the space P′Q′Q″P″ is of course parallel to P′Q′ and to P″Q″; and of these let us consider PQ. N is the common centre of curvature of all: let ρ be the radius of curvature of that one which is equally distant from P′ and P″, and which I shall call *the mean fibre;* and let σ be the length of this mean fibre contained between P′P″ and Q′Q″, and let the angle at N $= d\psi$; so that

$$\sigma = \rho\, d\psi. \qquad (119)$$

Let σ' be the length of PQ, and let PQ be at a distance u from the mean fibre,

$$\therefore \quad \sigma' = (\rho + u)\, d\psi; \qquad (120)$$

u being positive or negative according as PQ is nearer to or farther from the convex side of the plate than the mean fibre, and the limits of u being τ and $-\tau$.

Now in the process of bending, the fibres on the side towards the convexity of the bent plate will undergo dilatation, and those towards the concavity will undergo contraction. For assuming the coefficients of dilatation (or contraction, as the case may be,) to be different for different fibres, if s is the original length of the fibre contained between P′P″ and Q′Q″, we have by Hooke's law,

$$\sigma = s(1+e), \qquad \sigma' = s(1+e'), \qquad (121)$$

and which correspond to dilatation or contraction according as e is positive or negative. Therefore from the last three equations

$$\frac{1+e'}{1+e} = \frac{\rho+u}{\rho};$$

whence, as e and e' are infinitesimal,

$$e' = e + \frac{u}{\rho}. \qquad (122)$$

Whence it appears that if the length of the mean fibre is not changed, that is, if $e = 0$, then e' and u have the same sign; and therefore the fibres undergo dilatation or contraction according as they are on the side towards P′Q′ or P″Q″; and in either case the change of length is proportional to the distance from the mean fibre.

176.] And imagining the bent lamina to be in a rigid state under the action of the several forces, let us investigate the elastic forces which act on the part $\mathrm{A}\mathrm{Q}'\mathrm{Q}''$ by means of the section $\mathrm{Q}''\mathrm{Q}\mathrm{Q}'$. Now as any fibre PQ has undergone expansion or contraction, so does it tend to contract or expand; let us suppose that this elastic force, corresponding to an unit of surface, varies as the extent of displacement; that is, as the coefficient of elasticity; so that the force acting on an unit of surface $= ke'$; and let us suppose the thickness of the plate to be, with the exception of a variation infinitesimal in comparison with the thickness, the same as before the flexure, so that its thickness is 2τ; and its depth is dz; then if $\mathrm{T} =$ the whole force, and this acts in a line normal to $\mathrm{Q}''\mathrm{Q}\mathrm{Q}'$,

$$\mathrm{T} = \int_{-\tau}^{\tau} ke'\,dz\,du; \tag{123}$$

and if we replace e' by its value from (122), and integrate,

$$\mathrm{T} = 2\,ke\tau\,dz. \tag{124}$$

Also let L be the moment of these elastic forces about an axis perpendicular to the plane of the paper and passing through the mean fibre; then

$$\begin{aligned} \mathrm{L} &= \int_{-\tau}^{\tau} ke'\,dz\,u\,du \\ &= k\,dz\int_{-\tau}^{\tau}\left(e+\frac{u}{\rho}\right)u\,du \\ &= \frac{2k\tau^3\,dz}{3\rho}. \end{aligned} \tag{125}$$

Hence it appears (1) that T varies as the contraction or expansion of the mean fibre, and is independent of its curvature; (2) that L is independent of the extension, and varies directly as the curvature of the mean fibre; (3) that T varies directly as the thickness, and L varies as the cube of the thickness.

Also when the length of the mean fibre is not changed by the bending, $e = 0$, $\mathrm{T} = 0$, and L remains the same.

And because similar results are true for each rod into which the plate is divided, so for a section parallel to the side whose length is b and through the whole breadth of the plate,

$$\mathrm{T} = 2bke\tau, \qquad \mathrm{L} = \frac{2kb\tau^3}{3\rho}; \tag{126}$$

and if ω is the area of the section, $\omega = 2b\tau$; therefore

$$\text{T} = ke\omega, \qquad \text{L} = \frac{k\omega\tau^2}{3\rho}.$$

177.] The preceding investigations also enable us to find the equation to the curves which the fibres take. Since the force T acts on the element P′Q′Q″P″ at the side Q′Q″, an equal and opposite force acts on the side P′P″, because the mass-element is at rest, and no other force acts. And as the same result is true for all the elements of the lamina, T is constant throughout, and is therefore equal to the parts of X and Y which are normal to the end of the plate at A; e is also constant, and by virtue of equation (124) is proportional to this force, and is positive or negative according as the impressed forces act to dilate or contract the mean fibre.

Let (x_0, y_0) be the point A at which is applied the force which causes the bending of the plate; let X and Y be the axial components of this force; then these forces, together with those applied on that section of the plate whose intersection with the plane of the paper is Q′Q″, keep in equilibrium the part of the plate between A and Q′Q″. Now if (x, y) is the point P, x and y will also, neglecting infinitesimals, be the coordinates to the point of intersection of the mean fibre and Q′Q″; and therefore, taking moments about that point, we have

$$\text{L} + \text{X}(y_0 - y) - \text{Y}(x_0 - x) = 0; \tag{127}$$

and substituting for L from (126), and replacing ρ by its equivalent expression, we have for the differential equation of the curve,

$$\frac{2kb\tau^3}{3}\frac{d^2y}{dx^2} + \{\text{X}(y_0 - y) - \text{Y}(x_0 - x)\}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = 0; \tag{128}$$

the integral of which will contain two arbitrary constants: and these will be determined by the condition that $\frac{dy}{dx} = 0$, when $x = y = 0$, and by the length of the curve between the origin and (x_0, y_0) which is given.

If the lamina is not fixed at O, a force equal and opposite to the resultant of X and Y must be supplied at it.

178.] Let us consider the two particular cases of (128), in which the forces act, (1) wholly perpendicular to, (2) along, the plate in its original unbent state. In the first case $\text{X} = 0$; so that (128) becomes

$$\frac{2kb\tau^3}{3}\frac{d^2y}{dx^2} - \text{Y}(x_0 - x)\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} = 0. \tag{129}$$

Let $\frac{2kb\tau^3}{3\text{Y}} = c$; $\quad \therefore \; c\frac{d^2y}{dx^2} = (x_0 - x)\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}$;
therefore integrating, and observing that at the inferior limit, $x = 0$, when $\frac{dy}{dx} = 0$,

$$2c\frac{dy}{dx} = (2x_0x - x^2)\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}; \tag{130}$$

$$\therefore \quad dy = \frac{(2x_0x - x^2)\,dx}{\{4c^2 - (2x_0x - x^2)^2\}^{\frac{1}{2}}};$$

$$ds = \frac{2c\,dx}{\{4c^2 - (2x_0x - x^2)^2\}^{\frac{1}{2}}};$$

neither of which expressions can be integrated further.

If however the elastic force of the lamina is very great compared with the deflecting force the bending is slight, and thus $\frac{dy}{dx}$ is very small throughout, and neglecting the second and higher powers of it we have from (130),

$$2c\,dy = (2x_0x - x^2)\,dx,$$

$$\therefore \quad 6cy = 3x_0x^2 - x^3; \tag{131}$$

which is the equation to the curve taken by the lamina: that is, the lamina is bent into a cylindrical surface, the trace of which on the plane of (x, y) is given by (131). This equation however expresses the form of the lamina only as long as $\frac{dy}{dx}$ is small.

Let y be replaced by y_0 in (131); then

$$y_0 = \frac{x_0^3}{3c}:$$

and therefore replacing c by its equivalent,

$$y_0 = \frac{x_0^3\text{Y}}{2kb\tau^3}; \tag{132}$$

that is, the distance through which the end of the lamina has been moved varies as the deflecting force, as the cube of the length (approximately), inversely as the breadth of the plate, and inversely as the cube of the thickness.

179.] Another form of the problem of the preceding Article is that of a heavy thin flexible rectangular plate fixed to a horizontal edge along one of its edges, which is placed in a horizontal position, and is then bent by its own weight.

Let a be the length of the plate, b its breadth, and 2τ its thickness; and let it be placed in a horizontal position, with the side 2τ vertical: then as the deflexion is very small, we may, in equation (127), consider $x_0 = a$, $\mathrm{x} = 0$, and $\frac{dy}{dx}$ to be very small, so that its second and higher powers may be neglected. Taking the moments of the section Q′Q″, in fig. 73,

$$\mathrm{L} = \frac{2kb\tau^3}{3}\,\frac{d^2y}{dx^2};$$

and as the weight of that part of the plate which lies beyond Q′Q″ is in equilibrium with this force of elasticity, we have,

$$\frac{2kb\tau^3}{3}\,\frac{d^2y}{dx^2} = b\tau\rho g(a-x)^2;$$

$$\therefore\ \frac{dy}{dx} = \frac{\rho g}{2k\tau^2}\{a^3-(a-x)^3\},$$

because $\frac{dy}{dx} = 0$, when $x = 0$; therefore

$$y = \frac{\rho g}{2k\tau^2}\left\{\frac{3a^2x^2}{2} - ax^3 + \frac{x^4}{4}\right\}; \tag{133}$$

and therefore the whole deflexion at the extremity is

$$\frac{3\rho g a^4}{8k\tau^2};$$

that is, the deflexion of the extremity of the plate varies as the fourth power of the length of the beam, and inversely as the square of the depth of the beam.

180.] Let us now take the second case of Art. 180; viz. that in which $\mathrm{Y} = 0$; and let us suppose X to act, like a crushing pressure, towards O: then if the lamina under the action of such a force is bent at all, its deflexion from a straight line is very slight, and thus $\frac{dy}{dx}$ is very small: I shall neglect therefore the second and higher powers of it: also $y_0 = 0$, since the force X acts still along the axis of x: therefore (128) becomes

$$\frac{2kb\tau^3}{3}\,\frac{d^2y}{dx^2} + \mathrm{X}y = 0:$$

and if we put

$$\frac{3\mathrm{X}}{2kb\tau^3} = c^2, \tag{134}$$

we have

$$\frac{d^2y}{dx^2} + c^2y = 0;$$

and supposing the end of the plate to press against a rough fixed plane at O, but not to be fixed as heretofore, we have

$$\frac{dy^2}{dx^2} + c^2(y^2 - a^2) = 0; \tag{135}$$

so that $\frac{dy}{dx} = ca$, when $y = 0$, and where a is undetermined: and integrating again,

$$y = a \sin cx, \tag{136}$$

because $y = 0$, when $x = 0$. The plate therefore takes a corrugated form, the section of which, by a plane perpendicular to it, and parallel to its length, is the curve of sines. And from (136) a is the greatest amplitude of this curve, and therefore is very small in comparison of the length of the curve, because the deflection of the plate from a plane is supposed to be very small.

If $a = 0$, the plate will continue plane, and its length will, by reason of Art. 173, be slightly diminished, and become $a\left(1 - e\frac{\mathrm{X}}{2b\tau}\right)$. If a is not equal to zero, the plate, which is like a rectangular piece of watch-spring, takes the corrugated form, and the number of undulations on the cylindrical surface will depend on c; let h be the distance OA, then since $y = 0$, when $x = h$, c and h must be related by the equation, $ch = \pi i$, where i is any whole number;

$$\therefore \quad y = a \sin \frac{\pi i}{h} x; \tag{137}$$

and therefore also $y = 0$, when $x = \frac{nh}{i}$, where n is any number from 0 up to i, so that the curve cuts the axis of x in $i+1$ points, and therefore the surface has i elevations or depressions. Also, if l is the length of the curve,

$$l = \int_0^h \left\{1 + \frac{\pi^2 i^2 a^2}{h^2}\left(\cos \frac{\pi i}{h} x\right)^2\right\}^{\frac{1}{2}} dx; \tag{138}$$

and omitting the fourth and higher powers of a, we have

$$l = h + \frac{\pi^2 i^2 a^2}{4h}; \tag{139}$$

$$\therefore \quad a = \frac{2h}{\pi i}\left\{\frac{l}{h} - 1\right\}^{\frac{1}{2}}; \tag{140}$$

whereby a is given in terms of h and l.

181.] The greatest value of the compressing force X which can be applied at the end of a spring, and not bend it, is called *the vertical strength of the spring;* in this case $l = h$, $i = 1$: therefore from (134),

$$\mathrm{X} = \frac{2kb\tau^3 c^2}{3} = \frac{2kb\tau^3 \pi^2}{3l^2};$$

and using for l its approximate value a,

$$\mathrm{X} = \frac{2k\pi^2 b\tau^3}{3a^2}; \tag{141}$$

so that, other incidents being the same, the vertical strength of the spring varies inversely as the square of the length.

Hereby also are we enabled to calculate the greatest weight that a vertical pillar of a given form and height can bear without being bent by the weight.

Suppose the pillar to be of a height h and its transverse section to be rectangular, the sides of the rectangle being a and b; then the greatest weight which it will bear, without being bent perpendicularly to the side b, is

$$\frac{k\pi^2 ab^3}{12h^2};$$

and, without being bent perpendicularly to the side a, the greatest weight is $\frac{k\pi^2 a^3 b}{12h^2}$;

and if the transverse section is a square, $a = b$, and the strength of the beam perpendicularly to either of the sides is

$$\frac{k\pi^2 a^4}{12h^2}, \tag{142}$$

and varies therefore as the fourth power of the side.

Suppose a transverse section to be square and to be hollow, so that the side of the external square is a and of the internal square b; then

$$\text{the vertical strength of the beam} = \frac{k\pi^2(a^4 - b^4)}{12h^2}.$$

182.] We may also approximately investigate the vertical strength of beams, the transverse sections of which are of forms other than rectangles; and let us assume, as the most probable hypothesis, that the mean fibre is that which passes through the centres of gravity of all similar transverse sections; then L must be calculated in each case, as in Art. 176, so that we may substitute in equation (127); let ξ and η be the coordinates to any element of the area of the transverse section; and let us consider the following examples:

Ex. 1. The section of the beam is a circle, of which the radius is a.

Suppose the mean fibre of the cylindrical beam originally to be coincident with the axis of x; and ultimately, if bent, to be in the plane of (x, y), so that the bending takes place about an axis perpendicular to the plane of (x, y): let ξ be taken in, and η perpendicular to, the plane of (x, y); then

$$\begin{aligned} \text{L} &= \int_{-a}^{+a} 2k(a^2-\xi^2)^{\frac{1}{2}} e' \xi \, d\xi \\ &= 2k\int_{-a}^{+a} (a^2-\xi^2)^{\frac{1}{2}} \left(e + \frac{\xi}{\rho}\right) \xi \, d\xi \\ &= \frac{k\pi a^4}{4\rho}: \end{aligned}$$

so that (127) becomes

$$\frac{\pi k a^4}{4} \frac{d^2y}{dx^2} + \text{X}y = 0; \qquad \therefore \quad \text{X} = \frac{\pi^3 k a^4}{4h^2};$$

a comparison of this result with (142) shews, that if the areas of the transverse sections are equal in the two cases, the vertical strengths of the square and the circular beams are as $\pi : 3$; there is therefore a small advantage in favour of the square beam.

Ex. 2. Let the beam be circular and hollow: let a be the radius of the external, b the radius of the internal surface: then by the last result, if X is the vertical strength,

$$\text{X} = \frac{\pi^3 k}{4h^2}(a^4 - b^4).$$

Ex. 3. Let the transverse section of the beam be an isosceles triangle, of which the base is a and the altitude c: then if the altitude lies in the plane of (x, y) and the base of the triangle becomes convex,

$$\text{L} = \frac{kac^3}{36\rho}; \qquad \therefore \quad \text{X} = \frac{k\pi^2 ac^3}{36h^2}.$$

Such are the principles on which is founded the mathematical theory of the strength of materials: for a more complete investigation I must refer the reader to treatises wherein the subject is specially discussed; because the constants, which are left undetermined in the preceding expressions, are to be found by experiment; and particular and very delicate apparatus, the construction of which requires minute explanation, is needed for their determination.

CHAPTER VI.

ON ATTRACTIONS.

SECTION 1.—*The direct investigation of the attraction of bodies.*

183.] In the following chapter, amongst many properties of matter which will be formally stated as axiomatic principles of the science of motion, will occur one which is called *the law of inertia,* and which declares that matter has no power to change the state in which itself is; and experiment amply verifies it in the phaenomena of nature: it is not however hence to be inferred that matter has no power of acting on, or of influencing, other matter: on the contrary, matter does act on other matter in the way of either attraction or repulsion, and according to certain laws: and this action is not impeded by the presence or the intervention of other matter; every particle of matter attracts or repels every other particle in the same way as if the two existed alone. Nature presents to us many phaenomena in evidence of this active power of matter. There is, in the first place, that universal law of gravitation, by reason of which every material particle of the celestial system exercises on every other particle a force which varies as the product of the masses of the particles, and inversely as the square of the distance between them; and which acts along the line joining the two particles, and tends to draw them nearer together. So again in the explanation of magnetic and electrical phaenomena, there are doubtless two states in which particles active with the influence may be: and the attraction or repulsion which mutually acts between them varies as the product of the intensities of the two particles, and inversely as the square of the distance between them; and the force is attractive or repulsive according as the particles are in opposite or in the same magnetic states; and the line of action is that which joins the two particles. There are also other phaenomena where the attraction varies inversely as the square of the distance, but where the line of action is not that which joins the two particles. These and similar cases require investigation, and for this reason: when two single material particles attract or repel each other, it is easy to conceive the force which mutually acts from one to the other; we can easily imagine the tendency of the one to move towards or from the other in the straight line which joins the two. But when one material particle is attracted simultaneously by many others, aggregated into a finite body of a given form and density, the determination of the intensity and of the line of action of the resultant force requires investigation; and perhaps also the density of the attracting body may vary, in which case the difficulty is increased. The following inquiry will be for the most part confined to the cases where the law of attraction is that of the product of the two attracting particles, and of the inverse square of the distance between them, because this is the law of gravitation, and generally rules in cosmical phaenomena: but it will also embrace other laws; so that by operating with general laws we may determine the results which they necessitate, and by a comparison of these with the works of Nature, may obtain a knowledge of the special laws which rule therein.

184.] As to the attraction varying as the product of the masses of the attracting and the attracted particles: let there be two particles m and m' at a distance r apart; and let the law of attraction, which is a function of the distance between them, be $f(r)$; so that the attraction of an unit-particle in the position of m' on an unit-particle in the position of m is $f(r)$: now m' contains m' unit-particles; and each one of these attracts the unit-particle in the position of m with a force $f(r)$; therefore the whole force of m' on the unit-particle in the position of m is $m'f(r)$: but m also contains m unit-particles, and each of these is attracted with equal force by m'; therefore the whole attractive (or repulsive) force of m' on m is

$$m m' f(r). \tag{1}$$

If the attraction varies inversely as the square of the distance, $f(r) = \frac{1}{r^2}$, and

$$\text{the attractive force} = \frac{m m'}{r^2}; \tag{2}$$

and in all cases which we shall investigate, the line of action of the force lies along r.

Suppose now m to be the mass of an attracted particle, and $d\text{v}$ to be a volume-element of the attracting body, and ρ to be the density of $d\text{v}$, and r the distance between m and $d\text{v}$, then the attraction of $d\text{v}$ on m is

$$m\rho\, d\text{v} f(r); \tag{3}$$

and the attraction of the whole body on m will be found by means of the Integral Calculus.

In this section I propose to investigate the attraction of bodies on particles, and in some cases on other bodies, directly by integration. An indirect method for the investigation of these attractions will be given in a following section.

Whenever the law of attraction is not expressly stated, it is assumed to be that of gravitation.

185.] The attraction of a straight rod or wire of uniform thickness and density on an external particle; fig. 74.

Let O be the attracted particle whose mass is m, and let AB be the attracting bar: of which let the density be ρ, and let the area of a transverse section be ω; from O draw OC perpendicular to AB; let CA$=a$, CB$=b$: OC$=c$. Let PQ be a volume-element of the bar, CP $= y$, PQ $= dy$: therefore the mass-element at P $= \rho\omega\, dy$; and let the attractions be calculated along, and perpendicular to, OC; let the attraction of the bar on O along OC and towards C = X, and let the attraction at right angles to OC and towards A = Y. Then

$$\text{the attraction of P on O along OP} = \frac{m\omega\rho\, dy}{c^2+y^2};$$

therefore the attraction of P on O in the direction OC

$$= \frac{m\omega\rho\, dy}{c^2+y^2}\cos \text{POC},$$

$$= \frac{m\omega\rho c\, dy}{(c^2+y^2)^{\frac{3}{2}}}.$$

The attraction of P on O at right angles to OC

$$= \frac{m\omega\rho\, dy}{c^2+y^2}\sin \text{POC},$$

$$= \frac{m\omega\rho y\, dy}{(c^2+y^2)^{\frac{3}{2}}};$$

$$\therefore \quad \text{X} = \int_{-b}^{a} \frac{m\omega\rho c\,dy}{(c^2+y^2)^{\frac{3}{2}}}$$

$$= \frac{m\omega\rho}{c}\left[\frac{y}{(c^2+y^2)^{\frac{1}{2}}}\right]_{-b}^{a}$$

$$= \frac{m\omega\rho}{c}\left\{\frac{a}{(c^2+a^2)^{\frac{1}{2}}} + \frac{b}{(c^2+b^2)^{\frac{1}{2}}}\right\}$$

$$= \frac{m\omega\rho}{c}\left\{\frac{\text{CA}}{\text{OA}} + \frac{\text{CB}}{\text{OB}}\right\}$$

$$= \frac{m\omega\rho}{c}\{\sin\text{AOC} + \sin\text{BOC}\}; \qquad (4)$$

$$\text{Y} = \int_{-b}^{a} \frac{m\omega\rho y\,dy}{(c^2+y^2)^{\frac{3}{2}}}$$

$$= m\omega\rho\left\{\frac{1}{(b^2+c^2)^{\frac{1}{2}}} - \frac{1}{(a^2+c^2)^{\frac{1}{2}}}\right\}$$

$$= m\omega\rho\left\{\frac{1}{\text{OB}} - \frac{1}{\text{OA}}\right\}. \qquad (5)$$

If $a = b$, so that C is the middle point of the bar, $\text{Y} = 0$, and $\text{X} = \frac{2m\omega\rho}{c}\sin\text{AOC}$; that is, the attraction of the bar acts only in a direction at right angles to its length.

186.] By the following geometrical construction we obtain a remarkable equivalent for the attraction of a rod on a particle outside of it, as in the last Article.

From centre O, fig. 75, and radius OC, describe an arc of a circle meeting OA, OB, OP, OQ in the points a, b, p, q; and suppose a bar of the same material, density, and thickness as AB to be bent into an arc of a circle, and to coincide with the arc ab; then the attraction of this bent bar on O is the same as that of the straight bar AB.

From O as a centre, and with the radius OP, describe a small arc PR; then

$$\frac{\text{PQ}}{pq} = \frac{\text{PR}\sec\text{QPR}}{pq} = \frac{\text{OP}}{\text{O}p}\sec\text{POC},$$

$$= \frac{\text{OP}^2}{\text{O}p^2};$$

$$\therefore \quad \frac{\text{PQ}}{\text{OP}^2} = \frac{pq}{\text{O}p^2}.$$

Now the attraction of PQ on O in the direction $\text{OP} = \frac{\mu\omega\rho\text{PQ}}{\text{OP}^2}$, and therefore is equal to $\frac{\mu\omega\rho\, pq}{\mathrm{o}p^2}$; that is, is the same as that of the element pq of the circular arc: and as a similar result is true for all the elements of the circular arc, so the total attraction of the bar AB on O is the same as that of the circular bar ab. If the angle AOB is bisected by the line OD, the line of action of the whole attraction of the bar ab manifestly is OD: OD is therefore also the line of action of the whole attraction of the bar AB on O.

Hence it follows that if O is capable of moving towards AB, each element of its path will bisect the angle AOB, and the path will be a hyperbola of which A and B are the two foci; and the particle will ultimately meet the bar at a point, the difference between whose distances from A and B is equal to OA—OB. Thus if O is a particle of iron filings and AB is a magnetized bar, the path which O will take in moving towards AB is a hyperbola.

Hence also if from A and B as foci, an ellipse is described passing through O, OD will bisect the focal distances, and is evidently a normal to the ellipse at O; thus the action-line of the force on the particle at O will be perpendicular to the ellipse, and the particle will rest in equilibrium on the ellipse. We shall speak on this subject more at length in the following section.

Hence also if three bars of the same thickness and density, and attracting with a force varying inversely as the square of the distance, are arranged as a triangle, a particle placed in the centre of the circle inscribed in the triangle is equally attracted in all directions.

The preceding process of integration is also applicable when the density of the attracting bar is variable.

187.] Also let the following results be proved:

(1) The attraction of a bar of uniform thickness and density, when the attraction varies directly as the distance, on a particle in contact with it at distances a and b respectively from the ends of the bar is

$$\frac{m\rho\omega}{2}(a^2-b^2);$$

and therefore if the attracted particle is placed at the end of a bar whose length is a, so that $b = 0$,

$$\text{the attraction} = \frac{m\rho\omega a^2}{2},$$

and is the same as if the whole bar were condensed into a particle at its centre of gravity in the middle point of a.

(2) The attraction of a bar of uniform thickness and density on a particle in the same straight line with it, and at distances a and b severally from the ends of the bar, is

$$m\rho\omega\frac{a-b}{ab}.$$

(3) The attraction of two straight bars, each of which is of uniform thickness and density, on each other, in the same straight line, of the lengths a and b, and at a distance c apart, is

$$\rho\rho'\omega\omega'\log\frac{(a+c)(b+c)}{c(a+b+c)};$$

and this is of course the force which is required to keep the bars asunder.

Since the result involves the anharmonic ratio of the four points which are the ends of the bars, it follows that if AB and CD are the bars, and if through any point V lines VA, VB, VC, VD are drawn of any length, and any line A′B′C′D′ is drawn cutting them, the mutual attraction of A′B′ and C′D′ is the same as that of AB and CD.

(4) Two straight bars of lengths $2a$ and $2b$ and of constant thickness and density, and each particle of which attracts with a force varying inversely as the square of the distance, are placed parallel to each other at a distance c apart, and so that the line joining their middle points is perpendicular to each of them: it is required to shew that the force necessary to keep them apart is

$$\frac{2\rho\rho'\omega\omega'}{c}\left\{\{c^2+(b+a)^2\}^{\frac{1}{2}}-\{c^2+(b-a)^2\}^{\frac{1}{2}}\right\}.$$

188.] The attraction of a bent rod of uniform thickess and density on a given particle.

Let us first investigate the attraction of a bar bent into the form of a circular arc on a particle at the centre.

Let $\rho =$ the density, $\omega =$ the area of a transverse section of the bar: $a =$ the radius of the circle, $2\alpha =$ the angle subtended at the centre by the bar; fig. 76. Now it is manifest that the resultant attraction acts along the line OC bisecting the subtended angle, for the resultant attraction which is perpendicular to that line vanishes. Let POC $= \theta$, AOC $=$ BOC $= \alpha$: then the attraction of the bar in the direction OC

$$= \frac{m\omega\rho}{a}\int_{-\alpha}^{\alpha}\cos\theta\, d\theta$$

$$= \frac{2m\omega\rho\sin\alpha}{a}, \tag{6}$$

and therefore varies directly as the sine of half the subtended angle and inversely as the radius of the arc.

Hence the whole attraction of the bar AB, in Art. 186, on O, and along the line OD, see fig. 75, is

$$\frac{2m\omega\rho}{\text{OC}}\sin\frac{\text{AOB}}{2}.$$

Hereby we are enabled to solve the following problems:

Ex. 1. Three bars, each of which is of uniform density and thickness, form a triangle; find the position of a particle placed within the triangle which is equally attracted in all directions.

Let the densities of the bars be respectively ρ, σ, τ, and let the transverse sections of all three be the same; let the perpendiculars from the attracted particle on the sides be p, q, r; and let the sides subtend at the attracted particle angles 2α, 2β, 2γ; then the particle is kept at rest by the three forces

$$\frac{2m\omega\rho\sin\alpha}{p},\ \frac{2m\omega\sigma\sin\beta}{q},\ \frac{2m\omega\tau\sin\gamma}{r},$$

the angles between the lines of action of which are $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$; or $180°-\alpha$, $180°-\beta$, $180°-\gamma$, because $\alpha+\beta+\gamma = 180°$; and therefore by the triangle of forces, Art. 21, the forces are proportional to the sines of these angles; therefore

$$\frac{\rho}{p} = \frac{\sigma}{q} = \frac{\tau}{r}.$$

And if $\rho = \sigma = \tau$, then $p = q = r$, and the attracted particle is at the centre of the circle inscribed in the triangle.

Ex. 2. Two bars CA and CB of the same constant thickness and density meet at right angles and attract a particle placed at the foot of the perpendicular from C on AB; it is required to find the magnitude and the line of action of the resultant attraction.

Let CA $= a$, CB $= b$, $a^2+b^2 = c^2$; and let P be the position of the attracted particle. Then the attraction of CA on P in the line bisecting the angle APC is $\frac{2m\omega\rho c^2}{a^2 b}\sin 45°$; and, similarly, the attraction of CB on P in the line bisecting the angle BPC is

$\frac{2m\omega\rho c^2}{ab^2}\sin 45°$; and as these two lines of action are perpendicular to each other,

$$\text{the resultant attraction} = \frac{2^{\frac{1}{2}} m\omega\rho c^2}{a^2 b^2};$$

and the line of action of it is inclined at 45° to each of the lines CA and CB.

From (6) it appears, that the attraction of a circular rod on a particle at its centre is the greatest when $a = 90°$, that is, when the arc is a semicircle; and if $a = 180°$, that is, if the circle is complete, the attraction vanishes.

Suppose however the ring to be complete, and the attracted particle to be in the plane of the ring, and at a small distance x from the centre; then we have the following problem.

189.] To find the attraction of a circular ring on a particle in its plane, and near to its centre.

Let ρ be the density, and ω the area of a transverse section of the ring: $a =$ the radius, fig. 77, $\text{CO} = x$, which is very small, and such that we shall neglect the third and higher powers of it; $\text{PCA} = \theta$, $\text{QCP} = d\theta$; $m =$ the mass of O, $\text{OM} = a\cos\theta - x$, $\text{OP}^2 = a^2 - 2ax\cos\theta + x^2$. It is manifest that the ring attracts O along the line COA alone; and the attraction

$$= 2m\int_0^{\pi} \frac{\rho\omega a\, d\theta}{\text{OP}^2}\cos\text{POM}$$

$$= 2m\rho\omega a\int_0^{\pi} \frac{(a\cos\theta - x)\, d\theta}{(a^2 - 2ax\cos\theta + x^2)^{\frac{3}{2}}}$$

$$= \frac{2m\rho\omega}{a^2}\int_0^{\pi} (a\cos\theta - x)\Big(1 - \frac{2x\cos\theta}{a} + \frac{x^2}{a^2}\Big)^{-\frac{3}{2}} d\theta$$

$$= \frac{2m\rho\omega}{a^2}\int_0^{\pi} \Big\{a\cos\theta + \frac{3\cos 2\theta + 1}{2}x + \frac{9\cos\theta + 15\cos 3\theta}{8a}x^2\Big\}\, d\theta$$

$$= \frac{\pi m\rho\omega}{a^2}x;$$

and therefore the attraction varies directly as the distance of the particle from the centre of the ring.

190.] To find the attraction of a circular ring of uniform thickness and density on a particle m at a given distance from its plane, and in the line perpendicular to the plane and passing through the centre of the ring.

Let a be the radius, ρ the density, ω the area of a transverse section of the ring, c the distance of the attracted particle from the plane of the ring; see fig. 78; suppose the plane of the ring to be perpendicular to that of the paper; let P be an element of it, and let the line PC drawn from P to C, the centre of the ring, make an angle θ with the plane of the paper; then the mass of the element at P $= \rho\omega a\, d\theta$; and the attraction of P on O along the line OC

$$= \frac{m\rho\omega a\, d\theta}{c^2+a^2} \cos \text{POC};$$

therefore the attraction of the ring $= \dfrac{m\rho\omega a c}{(c^2+a^2)^{\frac{3}{2}}} 2\pi.$

Hereby we can solve the following problems:

Ex. 1. To determine the attraction of a hollow cylindrical tube on a particle at a given point in its axis.

Let $r =$ the radius of the interior surface of the tube, $\tau =$ the thickness, $\rho =$ the density; and let the distances of the attracted particle m from the ends of the axis of the tube be a and b; and let the tube be resolved into a series of rings of infinitesimal depth by means of planes perpendicular to the axis of the tube: then if x is the distance from m of any ring whose thickness is dx, the attraction of the whole tube

$$= 2\pi m\rho\tau r \int_{-b}^{a} \frac{x\,dx}{(r^2+x^2)^{\frac{3}{2}}}$$

$$= 2\pi m\rho\tau r \left\{ \frac{1}{(r^2+b^2)^{\frac{1}{2}}} - \frac{1}{(r^2+a^2)^{\frac{1}{2}}} \right\}.$$

Ex. 2. To prove that the attraction of a thin paraboloidal shell or cup, limited by a plane through the focus perpendicular to the axis of the shell, on a particle at the focus is equal to $\dfrac{8\pi m\rho\tau}{3}(2^{\frac{1}{2}}-1)$.

191.] The following are other problems on the attraction of thin wires.

Ex. 1. The attraction of a thin wire in the form of a parabola on a particle in its focus $= \dfrac{4m\rho\omega}{3a}$, where $4a$ is the latus rectum of the curve.

Ex. 2. The attraction of a semicircular ring, on a particle at the extremity of the diameter which bisects the ring $= \dfrac{m\rho\omega}{a} \log\tan\dfrac{3\pi}{8}$, where a is the radius of the ring.

192.] The attraction of a thin circular plate on a particle m in the line passing through the centre of the plate and perpendicular to it.

Let the attracted particle and the centre of the plate be in the plane of the paper, fig. 79; and let the plane of the plate be perpendicular to it. Let $\rho =$ the density, $\tau =$ the thickness of the plate: $a =$ the radius, and OC $= c$, the distance of the attracted particle from the plate. Resolve the plate into concentric circular rings, of which let the radius of that containing the element P be r, and the depth be dr; then if ACP$=\theta$, PCQ$=d\theta$,

$$\text{the mass of the element at P} = \rho\tau r\,dr\,d\theta;$$

therefore the attraction of the plate on O in the direction OC

$$= m\rho\tau c\int_0^a\int_0^{2\pi}\frac{r\,d\theta\,dr}{(c^2+r^2)^{\frac{3}{2}}}$$

$$= 2\pi m\rho\tau c\left[-\frac{1}{(c^2+r^2)^{\frac{1}{2}}}\right]_{r=0}^{r=a}$$

$$= 2\pi m\rho\tau\left\{1-\frac{c}{(a^2+c^2)^{\frac{1}{2}}}\right\}. \qquad (7)$$

The attraction of the plate in a direction at right angles to OC is evidently zero.

Since in (7) $\dfrac{c}{(a^2+c^2)^{\frac{1}{2}}} = \cos \text{AOC}$, it follows that the attraction of all circular plates of the same thickness and density on a particle in the line passing through their centres and perpendicular to their planes is the same, if their diameters subtend the same angle at the attracted particle. Hence if a right cone is divided into a series of circular plates, all of which are of the same thickness, by means of planes perpendicular to the axis of the cone, the attraction of each of these on a particle at the vertex is the same.

In (7) if the radius of the plate is infinite, that is, if $a = \infty$,

$$\text{the attraction} = 2\pi m\rho\tau,$$

which is independent of the distance of the attracted particle from the attracting plate: therefore the attraction of a plate of infinite extent on a particle outside of it is the same, whatever is the distance from the plate at which the particle is placed.

Hence for particles near to the surface of the earth, the earth's attraction is constant; because the earth may be conceived to be divided into a series of thin plates by planes perpendicular to the vertical line passing through the attracted particle, the radius of each of which is very large in comparison of the distance of the particle.

If the law of attraction varies as the nth power of the distance, the attraction of the circular plate on a particle outside of it in the line passing through its centre, and perpendicular to its plane,

$$= \frac{2\pi m\rho\tau c}{n+1}\left\{(c^2+a^2)^{\frac{n+1}{2}} - c^{n+1}\right\}; \tag{8}$$

which is the same as (7), if $n = -2$.

193.] To determine the attraction of a solid of revolution on a particle in its axis.

Let the solid, fig. 80, be resolved into circular slices of infinitesimal thickness by means of planes perpendicular to the axis of revolution. Let O be the attracted particle, of which the mass is m; and let $y = f(x)$ be the equation to the generating curve of the bounding surface of the solid.

Let $\text{OM} = x$, $\text{MP} = y$, $\text{OA} = a$, $\text{OB} = b$; and let the thickness of the circular slice PMP' be dx; then by (7), the attraction on O of the differential circular slice

$$= 2\pi m\rho\left\{1 - \frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right\}dx; \tag{9}$$

therefore the attraction of the solid on m

$$= 2\pi m\rho\int_a^b\left\{1 - \frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right\}dx; \tag{10}$$

y having been replaced by its equivalent value in terms of x by means of the equation to the generating curve.

Similarly may the whole attraction be found from (8), when the attraction varies as the nth power of the distance.

Ex. 1. To find the attraction of a homogeneous circular cylinder of length a and radius b on a particle in its axis at a distance c from one end.

$$\text{The attraction} = 2\pi m\rho\int_c^{c+a}\left\{1 - \frac{x}{(b^2+x^2)^{\frac{1}{2}}}\right\}dx$$

$$= 2\pi m\rho\,\{a - (b^2+(c+a)^2)^{\frac{1}{2}} + (b^2+c^2)^{\frac{1}{2}}\};$$

so that if the particle m is in contact with [illegible] cylinder, $c = 0$, and the attraction $= 2\pi m\rho\,\{a+b-(a^2+b^2)^{\frac{1}{2}}\}$.

If the cylinder is of infinite length in the direction from the attracted particle, $a = \infty$, and the attraction

$$= 2\pi m\rho\,\{(b^2+c^2)^{\frac{1}{2}} - c\};$$

and if the particle is in contact with the end of the cylinder of infinite length $c = 0$, and the attraction $= 2\pi m\rho b$, and varies as the radius of the cylinder.

Ex. 2. To find the attraction of a homogeneous right cone on a particle at its vertex.

Let the vertical angle $= a$; so that the equation to the generating line is

$$y = x\tan a;$$

and let a be the altitude of the cone: then from (10),

$$\begin{aligned} \text{the attraction} &= 2\pi m\rho\int_0^a (1-\cos a)\,dx \\ &= 2\pi m\rho(1-\cos a)\,a. \end{aligned}$$

If the attraction varies as the nth power of the distance, the attraction of the cone on a particle at its vertex is

$$\frac{2\pi m\rho}{(n+1)(n+3)}\{(\sec a)^{n+1}-1\}\,a^{n+3}.$$

Ex. 3. The attraction of a circular cylinder of length a and radius c, whose density is constant, on a particle in the centre of its circular end, is, if the attraction varies as the nth power of the distance,

$$\frac{2\pi m\rho}{(n+1)(n+3)}\left\{(a^2+c^2)^{\frac{n+3}{2}}-a^{n+3}-c^{n+3}\right\}.$$

Ex. 4. To find the attraction of a homogeneous sphere on a particle external to it.

Let $a =$ the radius, $\rho =$ the density of the sphere; $m =$ the mass of the attracted particle; $c =$ the distance of the particle from the centre of the sphere, so that the equation to the generating circle of the sphere is

$$y^2+(x-c)^2 = a^2.$$

The attraction of the sphere on m

$$= 2\pi m\rho\int_{c-a}^{c+a}\left\{1-\frac{x}{(a^2-c^2+2cx)^{\frac{1}{2}}}\right\}dx$$

$$= \frac{4\pi m\rho a^3}{3c^2} \tag{11}$$

$$= \frac{m\text{M}}{c^2}, \tag{12}$$

if M $=$ the mass of the sphere: but as c is the distance of the attracted particle from the centre of the sphere, this result expresses the attraction on each other of two particles m and M at the distance c apart: consequently the attraction of a sphere on a particle external to it is the same as if the mass of the sphere were condensed into its centre.

This result is physically of great importance; because in the investigation of the circumstances of a particle moving under the attraction of a sphere, every particle of which attracts it with a force varying inversely as the square of the distance, the attracting sphere may be supposed to be condensed into its centre; and the problem becomes reduced to that of the mutual attraction of two particles.

Also if two spheres attract each other, the action is the same as that of two particles whose masses are equal to those of the spheres, and placed at the centres of the spheres: and therefore the force which acts mutually on them is equal to the product of their masses divided by the square of the distance between their centres.

Ex. 5. To find the attraction of a homogeneous sphere on a particle on its surface.

In this case, $y^2 = 2ax - x^2$; therefore

$$\text{the attraction} = 2\pi m\rho \int_0^{2a} \left\{1 - \left(\frac{x}{2a}\right)^{\frac{1}{2}}\right\} dx$$

$$= 2\pi m\rho \frac{2a}{3}$$

$$= \frac{4\pi m\rho a}{3};$$

and therefore the attraction varies directly as the radius of the sphere.

Ex. 6. To find the attraction of a homogeneous sphere on a particle within it.

If c is the distance of the particle from the centre of the sphere, and a is the radius of the sphere, the attraction of the larger segment of the sphere whose base is the plane through the attracted particle and perpendicular to the line joining it and the centre is

$$\frac{2\pi m\rho}{3c^2}\{a^3 + c^3 - (a^2 - c^2)^{\frac{3}{2}}\};$$

and the attraction of the lesser segment is

$$\frac{2\pi m\rho}{3c^2}\{a^3 - c^3 - (a^2 - c^2)^{\frac{3}{2}}\},$$

and the attraction of the whole sphere, being the excess of the former of these over the latter, is

$$\frac{4\pi m\rho}{3}c;$$

and varies therefore as the distance of the attracted particle from the centre of the sphere. But by the last example this would be the case if the particle were on the surface of a sphere whose radius is c; therefore the spherical shell, of the thickness $a-c$, exerts no attraction on the particle.

Ex. 7. By similar processes let it be proved that the attraction of a homogeneous oblate spheroid on a particle m at its pole is

$$\frac{4\pi\rho m b}{e^2}\left\{1-\frac{(1-e^2)^{\frac{1}{2}}}{e}\sin^{-1}e\right\},$$

where b and e are respectively the semi-minor axis and the eccentricity of the generating ellipse. And that the attraction of a prolate spheroid on a particle m at its pole is

$$\frac{4\pi\rho m a(1-e^2)}{e^2}\left\{\frac{1}{2e}\log\frac{1+e}{1-e}-1\right\};$$

where a and e are respectively the semi-major axis and the eccentricity of the generating ellipse.

194.] The Calculus of Variations enables us to solve the following problem :

To determine the form of the bounding surface of revolution of a homogeneous mass of given volume, so that the attraction of it on a particle in its axis may be a maximum.

Let u be the attraction, and let πc^3 be the volume of the given mass, which is to be contained between x_1 and x_0; and thus, if $r^2 = x^2+y^2$,

$$u = 2\pi m\rho\int_0^1\left\{1-\frac{x}{r}\right\}dx; \tag{13}$$

$$\pi c^3 = \pi\int_0^1 y^2\,dx; \tag{14}$$

$$\therefore\quad \delta u = 0 = \Big[\Big(1-\frac{x}{r}\Big)\delta x\Big]_0^1-\int_0^1\left\{\frac{xy}{r^3}\,dy\,\delta x-\frac{xy}{r^3}\,dx\,\delta y\right\}, \tag{15}$$

$$\delta.c^3 = 0 = \big[y^2\delta x\big]_0^1-\int_0^1\{2y\,dy\,\delta x-2y\,dx\,\delta y\}; \tag{16}$$

therefore, if k^2 is an arbitrary constant,

$$k^2\frac{xy}{r^3} = y; \qquad \therefore\quad k^2x = (x^2+y^2)^{\frac{3}{2}},$$

$$y^2 = x^{\frac{2}{3}}(k^{\frac{4}{3}}-x^{\frac{4}{3}}). \tag{17}$$

Hence it appears that the curve, which by its revolution about the axis of x generates the solid, cuts the axis of x at the origin, and also when $x = k$; thus $x_0 = 0$, $x_1 = k$, and the integrated part of (15) shews that it cuts it in both points at right angles. Also substituting from (17) in (14) we have

$$c^3 = \int_0^k (x^{\frac{2}{3}} k^{\frac{4}{3}} - x^2)\, dx$$

$$= \frac{4}{15} k^3;$$

$$\therefore \quad k^3 = \frac{15}{4} c^3:$$

substituting from which in (17), the equation to the curve is completely determined. And the attraction of it on m

$$= 2\pi m \rho \int_0^k \left\{ 1 - \frac{x^{\frac{2}{3}}}{k^{\frac{2}{3}}} \right\} dx$$

$$= \frac{4}{5} \pi m \rho k$$

$$= \left(\frac{48}{25}\right)^{\frac{1}{3}} \pi m \rho c = \text{A}_1 \text{ (say)}.$$

Now the attraction of a sphere whose mass is $\pi \rho c^3$ on a particle m at its surface is

$$\left(\frac{4}{3}\right)^{\frac{2}{3}} \pi m \rho c = \text{A}_2 \text{ (say)}; \qquad \therefore \quad \frac{\text{A}_1}{\text{A}_2} = \left(\frac{27}{25}\right)^{\frac{1}{3}}.$$

Every particle on the surface of the solid of given mass and of greatest attraction attracts m with equal force in the direction of the axis; for if r is the distance of any particle on the surface from m, and if θ is the angle between r and the axis, from the preceding equations we have,

$$\frac{\cos\theta}{r^2} = \frac{1}{k^2},$$

which is constant, and therefore is the same for all particles on the surface.

195.] To find the attraction of a spherical shell of infinitesimal thickness, and of constant density on an external particle, when the law of attraction is represented by f (distance).

Let the centre of the shell be the origin; and let the shell be referred to that system of polar coordinates in space, which is explained in Art. 165, Vol. II, and let the attracted particle be on the axis of z at a distance c from the centre of the shell; let

$r =$ the radius, $dr =$ the thickness, $\rho =$ the density of the spherical shell: so that

$$\text{the mass-element of the shell} = \rho r^2 \sin\theta\, dr\, d\theta\, d\phi :$$

let $u =$ the distance of this mass-element from the attracted particle m; then the attraction of the mass-element on the attracted particle along the line joining m and the centre of the shell is

$$\frac{\rho m r^2 \sin\theta (c - r\cos\theta) f(u)}{u}\, dr\, d\theta\, d\phi; \tag{18}$$

and therefore the attraction of the shell on m

$$= \rho m r^2\, dr \int_0^{\pi}\int_0^{2\pi} \frac{\sin\theta (c - r\cos\theta)}{u} f(u)\, d\phi\, d\theta$$

$$= 2\pi\rho m r^2\, dr \int_0^{\pi} \frac{\sin\theta (c - r\cos\theta)}{u} f(u)\, d\theta. \tag{19}$$

But since $\qquad u^2 = r^2 - 2rc\cos\theta + c^2;$

$$\therefore \quad u\, du = rc \sin\theta\, d\theta,$$

and $\qquad 2c(c - r\cos\theta) = u^2 + c^2 - r^2,$

and when $\theta = \pi$, $u = c + r$, when $\theta = 0$, $u = c - r$; therefore substituting in (19), the attraction of the shell on m

$$= \frac{\pi\rho m r\, dr}{c^2} \int_{c-r}^{c+r} (u^2 + c^2 - r^2) f(u)\, du. \tag{20}$$

Ex. 1. Let the law of attraction be that of the inverse square of the distance; $\quad \therefore \; f(u) = \dfrac{1}{u^2};$

$$\therefore \quad \text{the attraction} = \frac{\pi\rho m r\, dr}{c^2}\left[u - \frac{c^2 - r^2}{u}\right]_{c-r}^{c+r}$$

$$= \frac{4\pi\rho m r^2\, dr}{c^2}; \tag{21}$$

and the mass of the shell $= 4\pi\rho r^2\, dr = M$ (say);

$$\therefore \quad \text{the attraction of the shell} = \frac{mM}{c^2};$$

and is therefore the same as if the mass of the shell were condensed into its centre.

Ex. 2. Let the attraction vary directly as the distance: then $f(u) = u$, and

$$\text{the attraction of the shell} = \frac{\pi\rho m r\, dr}{c^2}\left[\frac{u^4}{4} + \frac{c^2 - r^2}{2} u^2\right]_{c-r}^{c+r}$$

$$= 4\pi\rho m r^2 c\, dr \tag{22}$$

$$= mMc;$$

and therefore the attraction is the same as if the mass of the shell were condensed into its centre.

Similarly may the attraction be determined for any other law of attraction.

196.] Hereby the attraction of a sphere on an external particle can be determined by considering it as resolved into a series of concentric spherical shells of infinitesimal thickness.

Ex. 1. Let the sphere be homogeneous, and let a be its radius: then if the law of attraction is that of the inverse square of the distance, from (21),

$$\text{the attraction on } m = \frac{4\pi\rho m}{c^2}\int_0^a r^2 dr$$

$$= \frac{4\pi\rho m a^3}{3c^2}; \qquad (23)$$

and is the same as if the sphere were condensed into its centre: and if the particle m is on the surface, $c = a$, and

$$\text{the attraction} = \frac{4\pi\rho m}{3} a, \qquad (24)$$

and varies directly as the radius of the sphere.

Now, under certain conditions, this gives a value to g, which in Art. 123 has been defined to be the weight of a mass-unit; that is, g is the earth's attraction on a mass-unit placed at its surface. Consequently if the mass of the earth is homogeneous, and its figure is a sphere of radius a,

$$g = \frac{4}{3}\pi\rho a,$$

which gives g in terms of the radius and mean density of the earth.

Let the law of attraction be that of the direct distance: then from (22),

$$\text{the attraction on } m = 4\pi\rho m c\int_0^a r^2\, dr$$

$$= \frac{4\pi\rho m c a^3}{3}; \qquad (25)$$

and therefore is the same as if the sphere were condensed into its centre.

Ex. 2. Let us assume the density of a particle of the sphere to vary as some power of the distance of the particle from the centre: so that the sphere is composed of a series of homogeneous concentric shells, the density of which is different for different shells.

Thus suppose the density to vary inversely as the distance from the centre, and the law of attraction to be that of the inverse square of the distance; then by reason of (21), since $\rho = \frac{k}{r}$,

$$\text{the attraction on } m = \frac{4\pi m k}{c^2}\int_0^a r\,dr$$

$$= \frac{2\pi m k a^2}{c^2}. \qquad (26)$$

And if the attraction varies directly as the distance,

$$\text{the attraction on } m = 2\pi\rho m c k a^2.$$

Ex. 3. Let us suppose the density of the concentric spherical shells to decrease in arithmetic progression from the centre of the sphere: so that if ρ_0 is the density at the centre, $\rho_0 - kr$ is the density of the shell whose radius is r, where k is a constant. Then, if the attracted particle is within the sphere and at a distance c from the centre, the matter of the sphere which lies outside of the spherical surface passing through the attracted particle exercises no attraction on it: and of that within this spherical surface

$$\text{the attraction} = \frac{4\pi m}{c^2}\int_0^c (\rho_0 - kr) r^2\,dr$$

$$= 4\pi m\left\{\frac{\rho_0 c}{3} - \frac{kc^2}{4}\right\};$$

and consequently varies partly as the distance and partly as the square of the distance of the particle from the centre of the sphere.

In each of the cases, (21) and (22), as the attraction of the shell on m is the same as if the shell were condensed into its centre, so will the attraction of the whole full sphere be the same as if it were condensed into its centre.

Now in celestial mechanics this fact is of great importance: for the planetary bodies are nearly spherical, and the density of each of them is variable; and they are probably composed of concentric shells, each of which is of uniform density, and the density of which decreases as we pass from the centre to the surface. Thus by this property we can avoid the difficulty of investigating the attracting properties of them as solid bodies, and we can treat them as single attracting material particles.

From the preceding results also it follows that supposing the earth to be a sphere, the attraction of it on particles external to it varies inversely as the square of their distance from the centre of the earth. Thus if g and g' are the attractions of the earth on the same particle respectively at the mean surface which corresponds to the radius r, and on the top of a mountain whose height is h, then

$$\frac{g}{g'} = \frac{(r+h)^2}{r^2}.$$

197.] And the preceding results suggest another important question: Are there any other laws of attraction, besides those of the inverse square of the distance, and of the direct distance, for which the attraction of a spherical shell on a particle without it is the same as if the shell were condensed into its centre?

If $\rho =$ the density of the shell, and c is the distance of its centre from m, the attraction of the shell, condensed into its centre, is $4\pi\rho m r^2 dr f(c)$; and as this is equal to its attraction in its actual form, we have from (20),

$$4\pi\rho m r^2 dr f(c) = \frac{\pi\rho m r dr}{c^2}\int_{c-r}^{c+r}(u^2+c^2-r^2)f(u)\,du, \tag{27}$$

whence the form of f is to be determined. Omitting common factors, and integrating by parts the right-hand member of (27),

$$4rc^2 f(c) = \Big[(u^2+c^2-r^2)\int f(u)\,du\Big]_{c-r}^{c+r} - 2\int_{c-r}^{c+r}\Big\{u\int f(u)\,du\Big\}\,du. \tag{28}$$

$$\text{Let}\quad \int f(u)\,du = \phi(u), \qquad \text{and let}\quad \int u\,\phi(u)\,du = \psi(u); \tag{29}$$

$$\therefore\quad \phi'(u) = f(u), \qquad \psi'(u) = u\,\phi(u); \tag{30}$$

and therefore from (28),

$$\begin{aligned}4rc^2 f(c) &= \Big[(u^2+c^2-r^2)\phi(u) - 2\psi(u)\Big]_{c-r}^{c+r}\\ &= 2c(c+r)\phi(c+r) - 2c(c-r)\phi(c-r) - 2\psi(c+r) + 2\psi(c-r)\\ &= 2c^2\frac{d}{dc}\Big\{\frac{\psi(c+r)-\psi(c-r)}{c}\Big\};\end{aligned}$$

$$\therefore\quad 2rf(c) = \frac{d}{dc}\Big\{\frac{\psi(c+r)-\psi(c-r)}{c}\Big\}; \tag{31}$$

which is a functional equation to be satisfied by the form of f. Now expanding by Taylor's series, we have

$$2rf(c) = \frac{d}{dc}\cdot\frac{2}{c}\Big\{\psi'(c)\frac{r}{1} + \psi'''(c)\frac{r^3}{1.2.3} + \dots\Big\}; \tag{32}$$

and as no relation exists between r and c, the coefficients of the several powers of r must vanish separately; therefore

$$f(c) = \frac{d}{dc}\frac{\psi'(c)}{c}; \tag{33}$$

$$0 = \frac{d}{dc}\frac{\psi'''(c)}{c}; \text{ and so on}; \tag{34}$$

but from (30), $\psi'(c) = c\phi(c)$; therefore from (33),

$$f(c) = \frac{d}{dc}\phi(c)$$
$$= f(c),$$

which is only an identity. Also since from (30)

$$\psi'(c) = c\phi(c);$$
$$\therefore \quad \psi''(c) = \phi(c) + c\phi'(c)$$
$$= \phi(c) + cf(c), \text{ from (30)};$$
$$\therefore \quad \psi'''(c) = \phi'(c) + f(c) + cf'(c)$$
$$= 2f(c) + cf'(c);$$

therefore from (34), if 3A is an arbitrary constant,

$$2f(c) + cf'(c) = 3\text{A}c,$$
$$\therefore \quad 2cf(c) + c^2 f'(c) = 3\text{A}c^2;$$
$$c^2 f(c) = \text{A}c^3 + \text{B},$$

where B is another arbitrary constant: therefore

$$f(c) = \text{A}c + \frac{\text{B}}{c^2}; \tag{35}$$

and this value of $f(c)$ also makes to vanish the coefficients of all the other powers of r in (32): it is therefore the complete solution of the equation (31).

Thus the only laws which satisfy the requirements of the problem are (1) that of the inverse square of the distance, when $\text{A} = 0$ and B is finite; (2) that of the direct distance, when $\text{B} = 0$ and A is finite; and (3) that of these laws in combination, when A and B are both finite.

These are of course the only laws of attraction for which a sphere can attract an external particle with the same force as if it is condensed into its centre; because the sphere may be resolved into a series of concentric homogeneous shells, each of which will attract with the same force as if it is condensed into its centre.

Hence also two homogeneous shells external to each other will attract each other with the same force as if each is condensed into its centre.

198.] To investigate the attraction of a homogeneous spherical shell of infinitesimal thickness on a particle m placed within it, when the law of attraction is represented by f (distance).

Let all the quantities and symbols be the same as in Art. 195: in this case however c is less than r, and the limits of integration in the expression corresponding to (20) are $r+c$ and $r-c$; so that the attraction of the shell on m

$$= \frac{\pi\rho m r\,dr}{c^2}\int_{r-c}^{r+c}(u^2+c^2-r^2)f(u)\,du. \tag{36}$$

Ex. 1. Let the law of attraction be that of the inverse square of the distance; so that $f(u) = \frac{1}{u^2}$;

$$\begin{aligned}\text{the attraction} &= \frac{\pi\rho m r\,dr}{c^2}\left[u + \frac{r^2-c^2}{u}\right]_{r-c}^{r+c} \\ &= \frac{\pi\rho m r\,dr}{c^2}\{2c-2c\} \\ &= 0; \qquad (37)\end{aligned}$$

therefore the attraction of the shell on an interior particle is zero, and the particle is equally attracted in all directions.

The geometrical proof of this proposition is so simple that it is desirable to insert it. In fig. 81, let the centre of the shell and the attracted particle be in the plane of the paper, and let the circular ring APBP′ be the section of the shell by the same plane. At O, the place of m, let solid angles be formed, which occupy all space about it: and let each be considered with reference to an equal opposite and vertical one; let ω be the area of a spherical surface, described from O at the radius = unity, which is intercepted by one of these solid angles: then the area of the spherical surface intercepted at the distance r is $r^2\omega$: thus a mass-element of P$pq$Q at the distance r from O $= \rho r^2\omega\,dr$; and as the attraction of this on m

$$= \frac{m}{r^2}\rho r^2\omega\,dr = m\rho\omega\,dr,$$

so will the attraction of all the mass at P$pq$Q be $m\rho\omega \times$ Pp: similarly, the attraction at P′$p'q'$Q′ is $m\rho\omega\times$ P′p': but by the geometry of the circle Pp = P′p'; therefore the attractions of these masses are equal: and acting on O in opposite directions, they neutralize each other. And because the same result is true of every pair of such opposite small masses into which the whole shell may be divided, the attraction exercised by it on the particle m at O is zero.

The shell has been considered to be of finite thickness, but it is obvious that the same result is true for a shell of infinitesimal thickness.

Hence it follows that the attraction of a full homogeneous sphere on a particle within it varies as the distance of the particle from the centre: for if a concentric spherical surface is described passing through the attracted particle, the shell lying outside of that sphere has no attraction on the particle; and it is attracted only by the mass lying within that sphere; and that varies directly as its radius: see equation (24).

If the sphere is composed of a series of concentric homogeneous shells, the density of which however varies, then the attraction of all those lying outside of the concentric sphere passing through the attracted particle is zero: and as the attraction of each of the others is the same as if it were condensed into its centre, so if $c =$ the distance of m from the centre and $\text{M} =$ the mass of all those shells lying nearer than m to the centre,

$$\text{the attraction of the sphere on } m = \frac{m\text{M}}{c^2};$$

and if the matter of M is homogeneous, so that M varies as c^3, the attraction varies directly as the distance from the centre.

Ex. 2. Let the attraction vary directly as the distance; then

$$f(u) = u;$$

and from (36) we have

$$\begin{aligned}\text{the attraction} &= \frac{\pi\rho m r\,dr}{c^2}\left[\frac{u^4}{4} - \frac{(r^2-c^2)u^2}{2}\right]_{r-c}^{r+c} \\ &= 4\pi\rho m r^2 c\,dr,\end{aligned}$$

and this is the same as if the shell were condensed into its centre.

199.] The result in (37) leads us to inquire whether there are any other laws besides that of the inverse square of the distance, for which the attraction of a homogeneous spherical shell on a particle within it is zero.

In this case from (36),

$$0 = \int_{r-c}^{r+c} (u^2+c^2-r^2) f(u)\,du$$

$$0 = \Big[(u^2+c^2-r^2)\int f(u)\,du\Big]_{r-c}^{r+c} - 2\int_{r-c}^{r+c}\Big\{u\int f(u)\,du\Big\}\,du;$$

now making the substitutions of (29), we have

$$0 = \Big[(u^2+c^2-r^2)\phi(u) - 2\,\psi(u)\Big]_{r-c}^{r+c};$$

$$\therefore\quad 0 = c(r+c)\phi(r+c) + c(r-c)\phi(r-c) - \psi(r+c) + \psi(r-c)$$

$$= \frac{d}{dc}\left\{\frac{\psi(r+c)-\psi(r-c)}{c}\right\};$$

$$\therefore\quad 0 = \frac{d}{dc}\left\{\psi'(r) + \psi'''(r)\frac{c^2}{1.2.3} + \cdots\right\},$$

and integrating,

$$\psi'(r) + \psi'''(r)\frac{c^2}{1.2.3} + \cdots = \text{a constant} = \text{A}; \tag{38}$$

and as no relation exists between r and c,

$$\psi'(r) = \text{A}, \tag{39}$$

$$\psi'''(r) = 0, \text{ and so on.}$$

And since from (30), $\psi'(r) = r\phi(r)$,

$$\therefore\quad r\phi(r) = \text{A}, \qquad \phi(r) = \int f(r)\,dr = \frac{\text{A}}{r};$$

$$\therefore\quad f(r) = -\frac{\text{A}}{r^2};$$

therefore the law of the inverse square of the distance is the only one for which the attraction of a homogeneous spherical shell on an internal particle is zero.

200.] To determine the attraction of a rectangular plate on a particle at a given distance from the plate in the line passing through the centre and perpendicular to the plane of the plate.

Let $2a$ and $2b$ be the sides of the plate, ρ its density, and τ its thickness; let $c =$ the distance of the particle from the plate; then the attraction of the plate on the particle

$$= 4m\rho\tau c\int_0^a\int_0^b \frac{dy\,dx}{(x^2+y^2+c^2)^{\frac{3}{2}}}$$

$$= 4m\rho\tau bc\int_0^a \frac{dx}{(c^2+x^2)(b^2+c^2+x^2)^{\frac{1}{2}}}$$

$$= 4m\rho\tau\tan^{-1}\frac{ab}{c(a^2+b^2+c^2)^{\frac{1}{2}}}.$$

Hence, if the plate is a square plate, $b = a$; and the attraction

$$= 4m\rho\tau\tan^{-1}\frac{a^2}{c(2a^2+c^2)^{\frac{1}{2}}} = 4m\rho\tau\sin^{-1}\frac{a^2}{a^2+c^2}.$$

If $a = \infty$, the extent of the plate is infinite, and as the right-hand member $= 2\pi m\rho\tau$, the attraction is constant; and thus the attraction on an external particle of a square plate of infinite extent is the same, whatever is the distance of the particle from the plate. This result is the same as that already found for a circular plate in Art. 192.

Also, if $c = 0$, that is, if the particle is on the plate, the attraction is equal to $2\pi m\rho\tau$, and is the same as that of an infinite square plate.

Hereby also we can solve the following problems:

Ex. 1. To find the attraction of a homogeneous prism, whose transverse section is a square, on a particle in its axis at a given distance from one end.

Let $2a =$ the side of the square transverse section of the prism; $l =$ the length of the prism; $c =$ the distance of m from one end: then the attraction of the prism on m

$$= 4m\rho\left[x\sin^{-1}\frac{a^2}{a^2+x^2} - 2a\log\frac{a+(x^2+2a^2)^{\frac{1}{2}}}{(x^2+a^2)^{\frac{1}{2}}}\right]_{x=c}^{x=l+c}.$$

Ex. 2. The attraction of a pyramid on a square base, of which the altitude is a and $2b$ is the side of the base, on a particle at its vertex

$$= 4m\rho a\sin^{-1}\frac{b^2}{a^2+b^2}.$$

201.] The attraction of thin plates on particles in the plane of the plate.

Ex. 1. To find the attraction of a thin rectangular plate on a particle external to it and in its own plane.

Let $a-a'$, and $b-b'$ be the sides of the plate; then, if the axes are parallel to the sides of the plate, and (a', b') is the angle nearest to the origin, and X, Y are the axial-components of the attraction,

$$\text{X} = m\rho\tau\int_{b'}^{b}\int_{a'}^{a}\frac{x\,dx\,dy}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= m\rho\tau\int_{b'}^{b}\left\{\frac{1}{(y^2+a'^2)^{\frac{1}{2}}} - \frac{1}{(y^2+a^2)^{\frac{1}{2}}}\right\}dy$$

$$= m\rho\tau\left\{\log\frac{b+(b^2+a'^2)^{\frac{1}{2}}}{b'+(b'^2+a'^2)^{\frac{1}{2}}} - \log\frac{b+(b^2+a^2)^{\frac{1}{2}}}{b'+(b'^2+a^2)^{\frac{1}{2}}}\right\};$$

$$\text{Y} = m\rho\tau\left\{\log\frac{a+(a^2+b'^2)^{\frac{1}{2}}}{a'+(a'^2+b'^2)^{\frac{1}{2}}} - \log\frac{a+(a^2+b^2)^{\frac{1}{2}}}{a'+(a'^2+b^2)^{\frac{1}{2}}}\right\}.$$

If $a' = b' = 0$; that is, if the attracted particle is at an angle of the rectangle $X = Y = \infty$, and the attraction is infinite. If one, however, of these quantities is zero, the attraction is still finite.

Ex. 2. The attraction of an elliptical plate on a particle at the focus, when the attraction varies directly as the distance is equal to $m M a e$, if $M =$ the mass of the plate.

202.] The attraction of various bodies on particles.

Ex. 1. The vertex of a right circular cone is at the centre of a sphere: find the attraction of the part of the sphere intercepted by the cone on a particle at the vertex.

If we refer to the system of polar coordinates in space, and take the axis of the cone to be the z-axis, then

$$\text{the attraction} = m\rho \int_0^{2\pi} \int_0^{a} \int_0^{a} \sin\theta \cos\theta \, dr \, d\theta \, d\phi$$

$$= \pi m \rho a (\sin a)^2.$$

If $a = 90°$, the portion becomes a hemisphere, and consequently the attraction of a hemisphere on a particle at its centre $= \pi m \rho a$.

Ex. 2. The vertex of a cone is on the surface of a sphere, and the axis of the cone passes through the centre of the sphere: find the attraction of the intercepted mass on a particle at the vertex of the cone.

Let the vertex of the cone be the origin, and the axis of the cone the axis of z; so that according to the notation of Art. 165, Vol. II (Integral Calculus), the equation to the sphere is $r = 2a\cos\theta$; then, if Z is the attraction along the axis of z, and $2a\cos\theta = R$,

$$Z = m\rho \int_0^{2\pi} \int_0^{a} \int_0^{R} \sin\theta \cos\theta \, dr \, d\theta \, d\phi$$

$$= \frac{4\pi\rho m a}{3} \{1 - (\cos\theta)^3\}.$$

Ex. 3. To find the attraction of a hemisphere on a particle at its edge.

Let the place of the particle be the origin, and let the line passing through it and the centre be the z-axis, the plane circular base of the hemisphere being in the plane of (y, z); then, taking the system of coordinates given in Art. 165, Vol. II (Integral Calculus), if $2a\cos\theta = R$, and X, Y, Z are the axial-components of the attraction,

$$\text{X} = \rho m \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\text{r}} (\sin\theta)^2 \cos\phi \, dr \, d\theta \, d\phi$$

$$= \frac{4\rho m a}{3}.$$

$$\text{Y} = \rho m \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\text{r}} (\sin\theta)^2 \sin\phi \, dr \, d\theta \, d\phi$$

$$= 0.$$

$$\text{Z} = \rho m \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\text{r}} \sin\theta \cos\theta \, dr \, d\theta \, d\phi$$

$$= \frac{2\pi\rho m a}{3}.$$

203.] The attraction of a homogeneous ellipsoid.

Let the place of the attracted particle m be the origin; and let the coordinate axes be parallel to the principal axes of the ellipsoid, the centre of the ellipsoid being at (α, β, γ), so that the equation to its surface is

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} + \frac{(z-\gamma)^2}{c^2} = 1; \tag{40}$$

and let us refer it to the system of polar coordinates explained in Art. 165, Vol. II; so that, if

$$\left.\begin{aligned} \left(\frac{\sin\theta\cos\phi}{a}\right)^2 + \left(\frac{\sin\theta\sin\phi}{b}\right)^2 + \left(\frac{\cos\theta}{c}\right)^2 &= \text{A}, \\ \frac{\alpha\sin\theta\cos\phi}{a^2} + \frac{\beta\sin\theta\sin\phi}{b^2} + \frac{\gamma\cos\theta}{c^2} &= \text{B}, \\ \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 &= \text{C}, \end{aligned}\right\} \tag{41}$$

(40) becomes $$\text{A}r^2 - 2\text{B}r + \text{C} = 0; \tag{42}$$
then if r_1 and r_2 are the roots of this equation, r_1 being the greater, and r_2 the less,

$$r_1 = \frac{\text{B} + (\text{B}^2 - \text{AC})^{\frac{1}{2}}}{\text{A}}, \qquad r_2 = \frac{\text{B} - (\text{B}^2 - \text{AC})^{\frac{1}{2}}}{\text{A}}. \tag{43}$$

If ρ is the density of the ellipsoid,

its mass-element $= \rho r^2 \sin\theta \, dr \, d\theta \, d\phi$;

and as the direction-cosines of r are $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, and $\cos\theta$, so the resolved attractions on m of the mass-element are

$$m\rho(\sin\theta)^2 \cos\phi \, dr \, d\theta \, d\phi, \qquad m\rho(\sin\theta)^2 \sin\phi \, dr \, d\theta \, d\phi,$$
$$m\rho \sin\theta\cos\theta \, dr \, d\theta \, d\phi;$$

the integrals of which for limits assigned by the geometrical conditions of the problem are the axial-components of the total attraction.

Let X, Y, Z be the axial-components of the attraction; and let us first consider the attracted particle to be within the ellipsoid; so that the limits of the r-integration are r_1 and $-r_2$; then

$$\text{X} = \int_0^{\pi}\int_0^{\pi}\int_{-r_2}^{r_1} m\rho (\sin\theta)^2 \cos\phi \, dr\, d\theta\, d\phi, \tag{44}$$

$$\text{Y} = \int_0^{\pi}\int_0^{\pi}\int_{-r_2}^{r_1} m\rho (\sin\theta)^2 \sin\phi \, dr\, d\theta\, d\phi, \tag{45}$$

$$\text{Z} = \int_0^{\pi}\int_0^{\pi}\int_{-r_2}^{r_1} m\rho \sin\theta \cos\theta \, dr\, d\theta\, d\phi. \tag{46}$$

Of these three I shall consider the last, because it is the most simple; and results which are derived from it may be extended to the other two by an exchange of letters only.

Performing the r-integration, and replacing r_1 and r_2 by their values in (43),

$$\text{Z} = \int_0^{\pi}\int_0^{\pi} m\rho \sin\theta \cos\theta \frac{2\text{B}}{\text{A}} \, d\theta\, d\phi. \tag{47}$$

Now

$$\frac{\sin\theta\cos\theta\,\text{B}}{\text{A}} = \frac{\sin\theta\cos\theta\,(b^2c^2\alpha\sin\theta\cos\phi + c^2a^2\beta\sin\theta\sin\phi + a^2b^2\gamma\cos\theta)}{(bc\sin\theta\cos\phi)^2 + (ca\sin\theta\sin\phi)^2 + (ab\cos\theta)^2};$$

and observing that the denominator is a rational function of $\sin\theta$ and $(\cos\theta)^2$, and that the first two of the three terms contained in the numerator are rational functions of the same quantities; and observing that the limits of the θ-integration are π and 0; by virtue of (40), Art. 88, Vol. II (Integral Calculus), the integrals of the quantities corresponding to these two terms vanish; and we have

$$\text{Z} = 2m\rho a^2b^2\gamma \int_0^{\pi}\int_0^{\pi} \frac{\sin\theta(\cos\theta)^2 \, d\theta\, d\phi}{(bc\sin\theta\cos\phi)^2 + (ca\sin\theta\sin\phi)^2 + (ab\cos\theta)^2}; \tag{48}$$

and performing first the ϕ-integration, we have

$$\text{Z} = 2\pi m\rho ab\gamma \int_0^{\pi} \frac{\sin\theta(\cos\theta)^2 \, d\theta}{\{c^2(\sin\theta)^2 + a^2(\cos\theta)^2\}^{\frac{1}{2}} \{c^2(\sin\theta)^2 + b^2(\cos\theta)^2\}^{\frac{1}{2}}};$$

and therefore by virtue of Art. 88, Vol. II (Integral Calculus),

$$\text{Z} = 4\pi m\rho ab\gamma \int_0^{\frac{\pi}{2}} \frac{\sin\theta(\cos\theta)^2 \, d\theta}{\{c^2(\sin\theta)^2 + a^2(\cos\theta)^2\}^{\frac{1}{2}} \{c^2(\sin\theta)^2 + b^2(\cos\theta)^2\}^{\frac{1}{2}}}. \tag{49}$$

Let $\cos\theta = t$; $\therefore$ $\sin\theta\, d\theta = -dt$; and since $t = 0$, when $\theta = \frac{\pi}{2}$, and $t = 1$, when $\theta = 0$, we have, substituting in (49),

$$\text{Z} = 4\pi m\rho ab\gamma \int_0^{1} \frac{t^2\, dt}{\{c^2 + (a^2 - c^2)t^2\}^{\frac{1}{2}} \{c^2 + (b^2 - c^2)t^2\}^{\frac{1}{2}}}. \tag{50}$$

Similarly,

$$\text{X} = 4\pi m\rho bca \int_0^1 \frac{t^2\,dt}{\{a^2+(b^2-a^2)t^2\}^{\frac{1}{2}}\{a^2+(c^2-a^2)t^2\}^{\frac{1}{2}}}; \tag{51}$$

$$\text{Y} = 4\pi m\rho ca\beta \int_0^1 \frac{t^2\,dt}{\{b^2+(c^2-b^2)t^2\}^{\frac{1}{2}}\{b^2+(a^2-b^2)t^2\}^{\frac{1}{2}}}; \tag{52}$$

which definite integrals represent the axial-components of the attraction of a homogeneous ellipsoid on an internal particle.

Now these three expressions involve elliptic integrals which cannot be expressed in circular or logarithmic functions. The problem however is reduced to simple quadrature; and the required integration involves the summation of the attractions of a series of conical shells, whose common vertex is the attracted point, and the thickness of which increases directly as the distance from the vertex, because the r- and the ϕ-integrations have been taken between limits which give a double conical shell; and therefore the element-function in (51) &c. is the attraction of such a double shell.

t, it will be observed, is the cosine of the semi-vertical angle of the cone: the axis of the cone being parallel to that principal axis of the ellipsoid parallel to which the attraction is resolved.

204.] Jacobi has put the three preceding expressions for X, Y, Z under an elegant form by means of the following substitution:

$$\text{Let} \quad t^2 = \frac{a^2}{u+a^2}; \qquad \therefore \quad dt = -\frac{a\,du}{2(a^2+u)^{\frac{3}{2}}};$$

$$\therefore \quad a^2+(b^2-a^2)t^2 = a^2\frac{u+b^2}{u+a^2},$$

$$a^2+(c^2-a^2)t^2 = a^2\frac{u+c^2}{u+a^2};$$

and when $t=0$, $u=\infty$; when $t=1$, $u=0$: therefore

$$\begin{aligned} \text{X} &= 2\pi m\rho abca \int_0^\infty \frac{du}{(u+a^2)\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{1}{2}}} \\ &= \frac{3\text{M}ma}{2}\int_0^\infty \frac{du}{(u+a^2)\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{1}{2}}}, \end{aligned} \tag{53}$$

if M = the mass of the ellipsoid. Similarly,

$$\text{Y} = \frac{3\text{M}m\beta}{2}\int_0^\infty \frac{du}{(u+b^2)\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{1}{2}}}; \tag{54}$$

$$\text{Z} = \frac{3\text{M}m\gamma}{2}\int_0^\infty \frac{du}{(u+c^2)\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{1}{2}}}. \tag{55}$$

These values may also be expressed in the following form: Let

$$\text{U} = \int_0^\infty \frac{du}{\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{1}{2}}}$$

then

$$\text{X} = -3\text{M}m\alpha\left(\frac{d\text{U}}{d.a^2}\right), \quad \text{Y} = -3\text{M}m\rho\left(\frac{d\text{U}}{d.b^2}\right), \quad \text{Z} = -3\text{M}m\gamma\left(\frac{d\text{U}}{d.c^2}\right).$$

And they also give the following remarkable relation:

$$\begin{aligned} \frac{d\text{X}}{d\alpha} + \frac{d\text{Y}}{d\beta} + \frac{d\text{Z}}{d\gamma} &= \frac{3\text{M}m}{2}\int_0^\infty \frac{(u+b^2)(u+c^2)+(u+c^2)(u+a^2)+(u+a^2)(u+b^2)}{\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{3}{2}}}\,du \\ &= -\frac{3\text{M}m}{abc} \\ &= -4\pi\rho m. \qquad (56) \end{aligned}$$

We shall see hereafter that this is a general theorem of attractions.

205.] Of the values of X, Y, Z it is to be observed that each is proportional to the distance of the attracted particle from the principal plane which is perpendicular to the principal axis of the ellipsoid parallel to which it is the component of attraction. Consequently the particle is attracted by three components along the principal axes, each of which varies as the corresponding coordinate of the attracted particle.

It is also to be observed that the values of X, Y, Z in (51) &c. are not changed, if the quantities a, b, c are replaced by ka, kb, kc, where k is any number: the attractions therefore are not changed by the addition or subtraction of a shell contained between two ellipsoidal surfaces concentric and similar, provided that the attracted particle is within the interior. Hence we infer that a homogeneous shell contained between two similar and concentric ellipsoids attracts a particle within it equally in all directions. This theorem is generally known by the name of Newton's theorem on attractions, and is proved synthetically in the Principia. To it also the geometrical method of Art. 198 is immediately applicable.

In fig. 82 let O be the attracted particle, and let the shell, of which the section through O and the centre by the plane of the paper is drawn in the figure, be contained between two similar ellipsoidal surfaces concentric and similarly placed, and let us suppose the shell to be homogeneous. Consider O to be the vertex of solid angles which fill up the space around it; and to each one of these angles let the opposite and vertical angle be drawn as in the figure: let QPOP′Q′ be one of the lines of such angles made by the paper. Now of similar and similarly situated ellipsoids it is a property that $\text{PQ} = \text{P}'\text{Q}'$; let ω be the area of the spherical surface, described about O as a centre, with unity as the radius, which the cone intercepts: so that the volume, of which $\text{PQ}qp$ is the section made by the paper, consists of elements, each of which is equal to $\omega r^2 dr$, and the attraction of each of which on m placed at O is

$$\frac{m\rho}{r^2}\omega r^2 dr = m\rho\omega dr;$$

and of which the sum is $m\rho\omega \times \text{PQ}$; similarly the attraction of $\text{P}'\text{Q}'q'p'$ is $m\rho\omega \times \text{P}'\text{Q}'$, which is equal to the preceding; therefore the two attractions acting in opposite directions neutralize each other: and as the same result holds true for all the solid angles at O, so the resultant attraction of the shell on O vanishes.

As this proposition is independent of the thickness of the shell, it is also true for a shell of infinitesimal thickness; and therefore it is true also for a shell of any thickness, composed of homogeneous concentric, similar, and similarly-placed shells, the density of each of which varies according to any given law.

Hence also if an ellipsoid attracts a particle of its own mass, and a concentric and similar ellipsoidal surface is drawn through the place of the attracted particle, the ellipsoidal shell lying outside this latter surface will have no action on the particle; and the attraction of the ellipsoid will be reduced to that of the body lying within the latter surface.

Hence also it appears that if, as in fig. 83, m is without the shell, and from O a cone, intercepting a spherical area (ω) with a radius unity, is drawn, the attraction of the intercepted part of the shell at P is equal to that at Q; and as the same result is true for all similar cones, it follows that if O is considered to be a pole with reference to the exterior ellipsoid, the polar plane will divide the shell into two parts, the attractions of which on m at the pole are equal.

206.] In certain cases however the values given for X, Y, Z in Art. 203 can be integrated again.

Ex. 1. Let the bounding surface be an oblate spheroid; that is, let $a = b$; and let e be the eccentricity of the generating ellipse of the spheroid, so that

$$a^2 - c^2 = a^2 e^2; \qquad c^2 = a^2(1-e^2);$$

$$\therefore \quad \mathrm{X} = \frac{4\pi\rho m c\alpha}{a}\int_0^1 \frac{t^2\,dt}{\{1-e^2t^2\}^{\frac{1}{2}}}$$

$$= \frac{2\pi m\rho\alpha(1-e^2)^{\frac{1}{2}}}{e^3}\{\sin^{-1}e - e(1-e^2)^{\frac{1}{2}}\}; \qquad (57)$$

similarly
$$\mathrm{Y} = \frac{2\pi m\rho\beta(1-e^2)^{\frac{1}{2}}}{e^3}\{\sin^{-1}e - e(1-e^2)^{\frac{1}{2}}\}; \qquad (58)$$

$$\mathrm{Z} = 4\pi m\rho a^2\gamma\int_0^1 \frac{t^2\,dt}{c^2+a^2e^2t^2}$$

$$= \frac{4\pi m\rho\gamma}{e^2}\left\{1 - \frac{c}{ae}\tan^{-1}\frac{ae}{c}\right\}$$

$$= \frac{4\pi m\rho\gamma}{e^2}\left\{1 - \frac{(1-e^2)^{\frac{1}{2}}}{e}\sin^{-1}e\right\}. \qquad (59)$$

Hence it appears that the attraction depends solely on the eccentricity of the bounding spheroid, and is independent of its magnitude.

Thus if through the attracted particle a spheroidal surface is drawn similar to the given one, it will attract m with the same force as the given spheroid, and as any other similar concentric spheroid which includes m within its mass. Hence a spheroidal shell, the surfaces of which are similar and concentric, attracts a particle within it equally in all directions.

Ex. 2. Let the bounding surface be a prolate spheroid, so that $b = c$; also let the eccentricity of the generating ellipse be e, so that

$$a^2-b^2 = a^2e^2; \qquad b^2 = a^2(1-e^2);$$

then we have

$$\mathrm{X} = \frac{4\pi m\rho b^2\alpha}{a^2}\int_0^1 \frac{t^2\,dt}{1-e^2t^2}$$

$$= \frac{4\pi m\rho\alpha(1-e^2)}{e^2}\left\{\frac{1}{2e}\log\frac{1+e}{1-e} - 1\right\}; \qquad (60)$$

$$\mathrm{Y} = \frac{2\pi m\rho\beta}{e^2}\left\{1 - \frac{(1-e^2)^{\frac{1}{2}}}{2}\log\frac{1+e}{1-e}\right\}; \qquad (61)$$

$$\mathrm{Z} = \frac{2\pi m\rho\gamma}{e^2}\left\{1 - \frac{(1-e^2)^{\frac{1}{2}}}{2}\log\frac{1+e}{1-e}\right\}. \qquad (62)$$

As these expressions involve the eccentricity only, it follows that a spheroidal shell, the bounding surfaces of which are similar and concentric prolate spheroids, attracts a particle within it equally in all directions.

Ex. 3. If the bounding surface is a sphere: then $a = b = c$;

$$\therefore \quad \text{X} = \frac{4\pi m\rho\alpha}{3}; \qquad \text{Y} = \frac{4\pi m\rho\beta}{3}; \qquad \text{Z} = \frac{4\pi m\rho\gamma}{3}; \tag{63}$$

where α, β, γ are the coordinates of the attracted particle from the centre of the sphere as origin.

207.] Returning now to the expressions (44), (45) and (46), with the object of applying them to the case of the ellipsoid attracting an external particle, the limits of the r-integration will be r_1 and r_2, so that

$$\begin{aligned} \text{Z} &= \int_0^{\pi}\int_0^{\pi}\int_{r_2}^{r_1} m\rho \sin\theta\cos\theta \, dr\, d\theta\, d\phi \\ &= \int_0^{\pi}\int_0^{\pi} m\rho\sin\theta\cos\theta \frac{2(\text{B}^2-\text{AC})^{\frac{1}{2}}}{\text{A}}\, d\theta\, d\phi. \end{aligned} \tag{64}$$

Let A, B, C be replaced by their values, which are given in (41): then the element-function contains circular functions in an irrational form, and does not admit of further direct integration. We are therefore obliged to have recourse to an indirect process; and with a view to it, I will return to the symmetrical equation of the bounding ellipsoid.

Let the centre of the ellipsoid be the origin, and (α, β, γ) the place of m the attracted particle. Let a, b, c be the semi-axes of the ellipsoid, of which the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \tag{65}$$

then the axial-components of the attraction of this ellipsoid on m are the following:

$$\text{X} = \iiint \frac{\rho m(\alpha - x)\, dx\, dy\, dz}{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{3}{2}}}; \tag{66}$$

$$\text{Y} = \iiint \frac{\rho m(\beta - y)\, dx\, dy\, dz}{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{3}{2}}}; \tag{67}$$

$$\text{Z} = \iiint \frac{\rho m(\gamma - z)\, dx\, dy\, dz}{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{3}{2}}}; \tag{68}$$

the range of integration in each case being the space included within the surface of the ellipsoid.

From the symmetry of the formulae it is evident that the results of the others may be inferred from the integral of one of these expressions. We need therefore only consider one of them, and I will take the first, viz. (66).

Let $$a\left\{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right\}^{\frac{1}{2}} = \text{X}; \qquad \frac{b}{c}(c^2-z^2)^{\frac{1}{2}} = \text{Y}; \tag{69}$$
then
$$\text{X} = \rho m \int_{-c}^{c} \int_{-\text{Y}}^{\text{Y}} \int_{-\text{X}}^{\text{X}} \frac{(\alpha-x)\,dx\,dy\,dz}{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{3}{2}}}$$
$$= \rho m \int_{-c}^{c} \int_{-\text{Y}}^{\text{Y}} \left[\frac{dy\,dx}{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{1}{2}}}\right]_{-\text{X}}^{\text{X}}. \tag{70}$$
Now when x is replaced by its limiting values, the element-function has a form which does not admit of further direct integration, and we are obliged to have recourse to an indirect method. This has been discovered by Mr. Ivory, who has shewn that (70) expresses the corresponding axial-component of a certain ellipsoid on an internal particle; and as the latter has in the preceding Article been investigated and reduced to the form of a single integral, so may this also be expressed in the terms of a single integral; and consequently by Ivory's theorem we can effect one integration in the right-hand member of (70), and thus reduce it from a double integral to a single integral.

208.] It will be observed that when x is replaced by X, the point (x, y, z) in (70) is on the surface of the ellipsoid (65). Through the point (α, β, γ) let there be described an ellipsoid concentric with (65), and having moreover the foci of its principal sections coincident with the foci of the principal sections of the given ellipsoid. In this case the latter ellipsoid is said to be concentric and confocal with the former. Its equation is consequently
$$\frac{x^2}{a^2+\omega} + \frac{y^2}{b^2+\omega} + \frac{z^2}{c^2+\omega} = 1, \tag{71}$$
where ω is a quantity to be determined by the fact that (α, β, γ) lies in the surface of (71); so that we have
$$\frac{\alpha^2}{a^2+\omega} + \frac{\beta^2}{b^2+\omega} + \frac{\gamma^2}{c^2+\omega} = 1. \tag{72}$$

Now when the fractions are removed from this equation, and it is reduced to an integral form, it is a cubic in ω, and has three real roots.

Let us suppose $a^2 > b^2 > c^2$: then if in (72) we replace ω successively by ∞, $-c^2$, $-b^2$, $-a^2$, the results are $+$, $-$, $+$, $-$; so that of the roots one is greater than $-c^2$; another lies between $-c^2$ and $-b^2$; and the other lies between $-b^2$ and $-a^2$; and accordingly in the first case all the coefficients in (71) are positive, and the surface is an ellipsoid; in the second case the coefficients of the first two terms are positive, and that of the third term is negative, and the surface is a hyperboloid of one sheet; in the third case the coefficient of the first term is positive, and those of the other two terms are negative, and the surface is a hyperboloid of two sheets. Thus through a given point there can be drawn three surfaces of the second order concentric and confocal with a given ellipsoid; and indeed with any central surface of the second order. As however an ellipsoid is the surface which is required for the present problem, we must take the largest root of (72) as the corresponding value of ω; and this is a positive quantity, because the result is $+\infty$ when $\omega = \infty$, and is $-$ when $\omega = 0$.

209.] Let a', b', c' be the semi-principal axes of the concentric and confocal ellipsoid which has been thus described: so that

$$a'^2 = a^2+\omega, \qquad b'^2 = b^2+\omega, \qquad c'^2 = c^2+\omega; \tag{73}$$

and the equation to the ellipsoid is

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2} = 1; \tag{74}$$

and let $\mathrm{X}', \mathrm{Y}', \mathrm{Z}'$ be the axial-components of the attraction of this ellipsoid on a particle m, equal to the other particle, placed at a point $(\alpha', \beta', \gamma')$ within it. Then if

$$a'\left\{1-\frac{y'^2}{b'^2}-\frac{z'^2}{c'^2}\right\}^{\frac{1}{2}} = \mathrm{X}', \qquad \frac{b'}{c'}(c'^2-z'^2)^{\frac{1}{2}} = \mathrm{Y}', \tag{75}$$

$$\mathrm{X}' = \rho m \int_{-c'}^{c'}\int_{-\mathrm{Y}'}^{\mathrm{Y}'}\left[\frac{dy'\,dz'}{\{(\alpha'-x')^2+(\beta'-y')^2+(\gamma'-z')^2\}^{\frac{1}{2}}}\right]_{-\mathrm{X}'}^{\mathrm{X}'}; \tag{76}$$

with similar values for Y' and Z'. It will be observed that when x' is replaced by X', the point (x', y', z') is on the surface of (74).

Let us determine the points (x', y', z') $(\alpha', \beta', \gamma')$, which are thus far arbitrary, as follows: let $(\alpha', \beta', \gamma')$ be taken on the surface of the first ellipsoid; so that we have

$$\frac{\alpha'^2}{a^2}+\frac{\beta'^2}{b^2}+\frac{\gamma'^2}{c^2} = 1; \tag{77}$$

and let its place be related to that of (α, β, γ), so that

$$\frac{\alpha'}{a} = \frac{\alpha}{a'}, \qquad \frac{\beta'}{b} = \frac{\beta}{b'}, \qquad \frac{\gamma'}{c} = \frac{\gamma}{c'}; \tag{78}$$

also let (x', y', z') be so related to (x, y, z) that

$$\frac{x'}{a'} = \frac{x}{a}, \qquad \frac{y'}{b'} = \frac{y}{b}, \qquad \frac{z'}{c'} = \frac{z}{c}; \tag{79}$$

these relations being consistent, as it is evident, with the equations to the surfaces. Let (x, y, z) and $(\alpha', \beta', \gamma')$ which are on the first ellipsoid be P and Q′; and let (x', y', z') and (α, β, γ) which are on the second ellipsoid be P′ and Q; so that the denominators of the element-functions in (70) and (76) are respectively PQ and P′Q′.

Now P′Q′ = PQ; for

$$\begin{aligned}\mathrm{P'Q'}^2-\mathrm{PQ}^2 &= (x'-\alpha')^2+(y'-\beta')^2+(z'-\gamma')^2\\ &\qquad -(x-\alpha)^2-(y-\beta)^2-(z-\gamma)^2\\ &= \left(\frac{a'}{a}x-\alpha'\right)^2-\left(x-\frac{a'}{a}\alpha'\right)+\cdots+\cdots\\ &= \frac{a'^2-a^2}{a^2}(x^2-\alpha'^2)+\frac{b'^2-b^2}{b^2}(y^2-\beta'^2)+\frac{c'^2-c^2}{c^2}(z^2-\gamma'^2)\\ &= \omega^2\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-\frac{\alpha'^2}{a^2}-\frac{\beta'^2}{b^2}-\frac{\gamma'^2}{c^2}\right\}\\ &= 0;\end{aligned}$$

$$\therefore \quad \mathrm{P'Q'}=\mathrm{PQ}. \tag{80}$$

Two points, as P and P′, on two concentric and confocal ellipsoids, the coordinates of which are related by the conditions (79), are called *corresponding points*. Similarly Q and Q′ are also corresponding points.

210.] The preceding relations enable us to express the element-function of the double integral (70) in terms of that of the integral (76). Let us replace $y, z, \mathrm{X}, \mathrm{Y}$, in terms of $y', z', \mathrm{X}', \mathrm{Y}'$ by means of (79), and the denominator of the element-function by means of (80), and we have

$$\begin{aligned}\mathrm{X} &= \rho m\frac{bc}{b'c'}\int_{-c'}^{c'}\int_{-\mathrm{Y}'}^{\mathrm{Y}'}\left[\frac{dy'\,dz'}{\{(\alpha'-x')^2+(\beta'-y')^2+(\gamma'-z')^2\}^{\frac{1}{2}}}\right]_{-\mathrm{X}'}^{\mathrm{X}'}\\ &= \frac{bc}{b'c'}\mathrm{X}'.\end{aligned} \tag{81}$$

Similarly $\qquad \mathrm{Y}=\frac{ca}{c'a'}\mathrm{Y}', \qquad \mathrm{Z}=\frac{ab}{a'b'}\mathrm{Z}'. \qquad (82)$

And we have thus determined the axial-components of the attraction of an ellipsoid on an external particle in terms of the similar components of a concentric and confocal homogeneous ellipsoid on an equal particle within it, whose place is that on the interior ellipsoid which corresponds to that of the originally attracted particle.

Let X′, Y′, Z′ be replaced by their values given in (51), (52), and (50); and a', b', c' by their values given in (73), and we have

$$X = \frac{4\pi m\rho abca}{(a^2+\omega)^{\frac{1}{2}}} \int_0^1 \frac{t^2\,dt}{\{a^2+\omega+(b^2-a^2)t^2\}^{\frac{1}{2}}\{a^2+\omega+(c^2-a^2)t^2\}^{\frac{1}{2}}}; \quad (83)$$

and also similar values for Y and Z.

Thus the attraction depends on the evaluation of a single definite integral.

From (82) we have the following theorem:

The attractions with which two confocal homogeneous ellipsoids attract, parallel to each axis, equal particles placed at corresponding points on their surfaces, are as the products of the axes perpendicular to each component.

This theorem has also been extended by Poisson to the case in which the law of attraction is any function of the distance.

211.] By the preceding theorem we can determine the attraction of a spheroid on an external particle. Thus, for example, let us take the oblate spheroid; in which case $a = b$; and let e be the eccentricity of the generating ellipse, so that $a^2-c^2 = a^2e^2 = a'^2e'^2$, if e' is the eccentricity of the ellipse which generates the concentric and confocal spheroid which passes through the attracted particle. Then from (57),

$$\begin{aligned} X &= \frac{ac}{a'c'}X' \\ &= \frac{ac}{a'c'}\,\frac{2\pi m\rho a'}{e'^3}(1-e'^2)^{\frac{1}{2}}\{\sin^{-1}e'-e'(1-e'^2)^{\frac{1}{2}}\} \\ &= 2\pi m\rho\frac{ca}{ae^3}\{\sin^{-1}e'-e'(1-e'^2)^{\frac{1}{2}}\}, \qquad (84) \end{aligned}$$

where $e' = \dfrac{ae}{(a^2+\omega)^{\frac{1}{2}}}$, and ω is the positive root of the quadratic equation

$$\frac{\alpha^2+\beta^2}{a^2+\omega}+\frac{\gamma^2}{c^2+\omega} = 1.$$

Similarly

$$Y = \frac{2\pi m\rho c\beta}{ae^3}\{\sin^{-1}e'-e'(1-e'^2)^{\frac{1}{2}}\}; \qquad (85)$$

$$Z = \frac{4\pi m\rho(a^2+\omega)(c^2+\omega)^{\frac{1}{2}}\gamma}{a^2ce^3}\Big\{1-\frac{(1-e'^2)^{\frac{1}{2}}}{e'}\sin^{-1}e'\Big\}. \qquad (86)$$

Similarly may we determine the attraction of a prolate spheroid on an external particle.

212.] If the oblate spheroid is of small eccentricity, so that it differs little from a sphere, the right-hand members of (84), (85), (86) may be expanded in ascending powers of e', and approximate results may be obtained in an algebraical form. We shall neglect powers of e' above the second, and if we suppose the attracted particle to be close to the spheroid, $e' = e$, if e is the eccentricity of the generating ellipse of the spheroid. I will also suppose the attracted particle to lie in the plane of (x, z), so that $\beta = 0$, and $Y = 0$; for no loss of generality arises from this assumption. Thus

$$X = \frac{2\pi m\rho\alpha}{e^3}\{(1-e^2)^{\frac{1}{2}}\sin^{-1}e - e(1-e^2)\}$$

$$= \frac{4\pi m\rho\alpha}{3}\left\{1-\frac{e^2}{5}\right\}; \tag{87}$$

$$Z = \frac{4\pi m\rho\gamma}{3}\left\{1+\frac{2e^2}{5}\right\}. \tag{88}$$

Let R be the resultant attraction; and λ be the latitude of the place of m; that is, the angle at which the normal to the spheroid at the place of m is inclined to the equator. Let M = the mass of the spheroid; then

$$M = \frac{4\pi\rho a^2 c}{3} = \frac{4\pi\rho a^3}{3}\left(1-\frac{e^2}{2}\right).$$

$$\alpha^2 = \frac{a^4(\cos\lambda)^2}{a^2(\cos\lambda)^2 + c^2(\sin\lambda)^2} = c^2(\cos\lambda)^2(1+e^2)\{1+e^2(\sin\lambda)^2\}$$

$$\gamma^2 = c^2(\sin\lambda)^2(1-e^2)\{1+e^2(\sin\lambda)^2\};$$

$$\therefore \quad = \{X^2+Y^2\}^{\frac{1}{2}}$$

$$= \frac{mM}{a^2}\left\{1+\frac{3e^2}{10}+\frac{e^2(\sin\lambda)^2}{10}\right\}$$

$$= \frac{mM}{a^2}\left(1+\frac{3e^2}{10}\right)\left\{1+\frac{e^2}{10}(\sin\lambda)^2\right\}.$$

Now as R is the resultant attraction of the spheroid on m, it is what in Art. 123 has been called the weight of m; in reference of course to a homogeneous oblate spheroid of small eccentricity; and consequently R is identical with mg; hence

$$g = \frac{M}{a^2}\left(1+\frac{3e^2}{10}\right)\left\{1+\frac{e^2}{10}(\sin\lambda)^2\right\}. \tag{89}$$

Let G be the least value of g; that is, G is the value of g at the equator, when $\lambda = 0$; then

$$G = \frac{M}{a^2}\left(1+\frac{3e^2}{10}\right); \tag{90}$$

so that G varies directly as the mass and inversely as the square of the equatorial radius. Also

$$g = \mathrm{G}\left\{1 + \frac{e^2}{10}(\sin\lambda)^2\right\}; \tag{91}$$

that is, the increase of gravity of a homogeneous spheroid of small eccentricity varies as the square of the sine of the latitude as we pass from the equator to the pole.

In reference to this result, however, I would observe that the latest reductions of geodetical observations seem to shew that the figure of the earth is that of an ellipsoid with three unequal axes; and that the greatest and least equatorial axes are in the longitudes 15° 34′ E and 105° 34′ E respectively from Greenwich.

In an ellipse of small eccentricity, the ratio of the excess of the equatorial radius over the polar radius to the equatorial radius is called *the ellipticity*, and is denoted by ϵ; so that $a-c = a\epsilon$; but $\frac{a-c}{a} = 1-(1-\epsilon^2)^{\frac{1}{2}} = \frac{e^2}{2}$; therefore $\epsilon = \frac{e^2}{2}$; and

$$g = \mathrm{G}\left\{1 + \frac{\epsilon}{5}(\sin\lambda)^2\right\}\cdot \tag{92}$$

213.] The preceding formulae will also give the attraction of a homogeneous elliptical cylinder of an infinite length on an external particle on its surface.

In (83) and in the two similar values for Y and Z, let $a = \infty$, while b and c remain finite; then as the particle is on the surface, $\omega = 0$, and

$$\mathrm{X} = \frac{2\pi m\rho bc\alpha}{a^2}\int_{-1}^{1}\frac{t^2\,dt}{1-t^2} = 0,$$

as we may foresee. Let e be the eccentricity of the elliptic section; then

$$\begin{aligned}\mathrm{Y} &= \frac{4\pi m\rho c\beta}{b}\int_{0}^{1}\frac{t\,dt}{(1-e^2t^2)^{\frac{1}{2}}}\\ &= 4\pi m\rho\beta\,\frac{(1-e^2)^{\frac{1}{2}}-1+e^2}{e^2} = 4\pi m\rho\beta\,\frac{c}{b+c}; \qquad (93)\end{aligned}$$

$$\mathrm{Z} = 4\pi m\rho\gamma\,\frac{1-(1-e^2)^{\frac{1}{2}}}{e^2} = 4\pi m\rho\gamma\,\frac{b}{b+c}\cdot \tag{94}$$

214.] Hence also we may deduce the following theorem, which was stated first by Maclaurin in a particular form, and was demonstrated by him in that form:

Two confocal homogeneous ellipsoids attract an external particle along the same action-line, and with forces proportional to their masses.

Let the two confocal ellipsoids be E and E′, of which let the semi-axes be respectively a, b, c; a', b', c'; and the masses be M and M′. Let (α, β, γ) be the place of the attracted particle, and through it let a concentric and confocal ellipsoid E_0 be described. Let X, Y, Z, X′, Y′, Z′ be the axial-components of the attraction of E and E′ on m at (α, β, γ) respectively. Let a_0, b_0, c_0 be the semi-axes of E_0: also let X_1, Y_1, Z_1 be the components of the attraction of E_0 on a particle at the point on the surface of E which corresponds to (α, β, γ); then

$$\frac{X}{X_1} = \frac{bc}{b_0 c_0}; \qquad \frac{Y}{Y_1} = \frac{ca}{c_0 a_0}; \qquad \frac{Z}{Z_1} = \frac{ab}{a_0 b_0}.$$

Also let X_0, Y_0, Z_0 be the components of the attraction of E_0 on m at (α, β, γ) which is in its surface; then we have from (53), if $M_0 =$ the mass of E_0,

$$X_1 = \frac{3 M_0 m}{2} \frac{\alpha a}{a_0} \int_0^{\infty} \frac{du}{(u+a^2)\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{1}{2}}};$$

$$X_0 = \frac{3 M_0 m \alpha}{2} \int_0^{\infty} \frac{du}{(u+a^2)\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{1}{2}}}:$$

$$\therefore \quad \frac{X}{X_0} = \frac{X}{X_1} \frac{X_1}{X_0} = \frac{abc}{a_0 b_0 c_0} = \frac{M}{M_0};$$

$$\therefore \quad \frac{X}{X_0} = \frac{Y}{Y_0} = \frac{Z}{Z_0} = \frac{M}{M_0};$$

the last two members of the equality following from the symmetry of the formulae. Similarly for the attraction of E′ we have,

$$\frac{X'}{X_0} = \frac{Y'}{Y_0} = \frac{Z'}{Z_0} = \frac{M'}{M_0};$$

$$\therefore \quad \frac{X}{X'} = \frac{Y}{Y'} = \frac{Z}{Z'} = \frac{M}{M'};$$

which equality is the statement of the theorem.

215.] At the conclusion of this direct process of investigating attractions the two following observations are to be made:

(1) In the preceding investigations, whenever the law of attraction has been, as to distance, that of the distance directly, the attraction of the body on the attracted particle has been the same as if the body were condensed into its centre of gravity. Now this property admits of generalization, so that in all cases,

whatever is the form of the attracting body, the total attraction of it on a material particle is the same as if the body were condensed into its centre of gravity.

Let the centre of gravity be taken as the origin; (a, β, γ) the position of m the attracted particle; (x, y, z) a point of the attracting body, at which let the element $dx\,dy\,dz$ of its volume abut: let ρ be the density, and $r =$ the distance between m and (x, y, z): then if X, Y, Z are the axial-components of the attraction,

$$\begin{aligned} \mathrm{X} &= m\iiint \rho r\,dx\,dy\,dz\,\frac{a-x}{r} \\ &= m\iiint \rho\,(a-x)\,dx\,dy\,dz; \\ \mathrm{Y} &= m\iiint \rho\,(\beta-y)\,dx\,dy\,dz; \\ \mathrm{Z} &= m\iiint \rho\,(\gamma-z)\,dx\,dy\,dz; \end{aligned}$$

but because the origin is the centre of gravity,

$$\iiint \rho\,x\,dx\,dy\,dz = \iiint \rho\,y\,dx\,dy\,dz = \iiint \rho\,z\,dx\,dy\,dz = 0;$$

and if $\mathrm{M} =$ the mass of the attracting body,

$$\mathrm{M} = \iiint \rho\,dx\,dy\,dz;$$

$$\therefore \quad \mathrm{X} = m\mathrm{M}a, \qquad \mathrm{Y} = m\mathrm{M}\beta, \qquad \mathrm{Z} = m\mathrm{M}\gamma;$$

and these are the axial-components of the attraction of M, which is at the origin, on m placed at the point (a, β, γ); and therefore the proposition is proved.

(2) Some remarkable results arise, when a spherical shell or a sphere attracts a material particle, if the law of attraction is that of the inverse square of the distance; (1) the attraction of a spherical shell or of a sphere on an external particle is the same as if they were respectively condensed into their centres; and (2) the attraction of a spherical shell on an internal particle is zero. Also we shall hereafter shew that, Kepler's laws being assumed to be true, this law of attraction holds good in the motion of the celestial bodies; and also in electrical and magnetic phaenomena, results can be explained and accounted for by the same law. Now is there of this law any *a priori* probability? Can we assign any reasons why the attraction should vary directly as the product of the attracting masses, and inversely as the square of the distance? Suppose m to be

the mass of an attracting particle, the influence of the attraction of which on other matter radiates from it in all directions; and which is such that none of its quantity is lost by the process of propagation; let m be the vertex of a cone; and let us consider the parts of spherical surfaces which are described from m as a centre with different radii, and are intercepted by the cone. The areas of these spherical segments vary as the squares of their radii, and the same amount of attracting influence is spread over each one; therefore the intensity of the amount of attraction on an unit-surface varies inversely as the area of the spherical surface intercepted by the cone; and therefore varies inversely as the square of the distance of the unit-surface from the centre of the cone, that is, from the position of m; and as the same result is true for each unit which is at the same distance from m, therefore, if there are m' units, and r is the distance,

$$\text{the attraction} = \frac{mm'}{r^2};$$

and thus, on the hypothesis that none of the attracting influence is lost by means of or during the propagation, the law of attraction is that of the inverse square of the distance.

SECTION 2.—*Indirect investigation of the attraction of bodies.—The potential.*

216.] When one particle m attracts or repels another particle m', the two particles being at a distance r apart, the amount of force which is in action between them varies directly as the product of the masses, and is also a function of r; and the action-line of this force is the straight line which joins the particles. Now from these circumstances arise very remarkable relations between the components of the force of attraction. These components are expressed by quantities which are partial differential coefficients of the same function, so that when this function is determined, the components of attraction may be found by differentiation. Thus, let m' be the attracting and m the attracted particle; and let the law of attraction be represented by $f(r)$, where r is the distance between m' and m, so that the attraction or repulsion of m' on m is $mm'f(r)$. Let m be at (α, β, γ), and m' at (x', y', z'); then

$$r^2 = (\alpha - x')^2 + (\beta - y')^2 + (\gamma - z')^2. \qquad (95)$$

Let X, Y, Z be the axial-components of the attraction of m' on m; then

$$X = mm'f(r)\frac{a-x'}{r}, \quad Y = mm'f(r)\frac{\beta-y'}{r}, \quad Z = mm'f(r)\frac{\gamma-z'}{r}.$$

Let $f(r)$ be the derived function of $F(r)$, so that

$$d.F(r) = f(r)\,dr; \tag{96}$$

also $r\,dr = (a-x')\,da$; and consequently

$$X = mm'\frac{d.F(r)}{da}. \tag{97}$$

Similarly $$Y = mm'\frac{d.F(r)}{d\beta}; \qquad Z = mm'\frac{d.F(r)}{d\gamma}: \tag{98}$$

all these being partial differential coefficients with respect to the coordinates of the attracted point; and these can be found when $F(r)$ is known.

Now let us suppose m to be attracted by many particles m_1, $m_2, \ldots m_n$, whose places are respectively (x_1, y_1, z_1), (x_2, y_2, z_2), $\ldots (x_n, y_n, z_n)$, and whose distances from m are $r_1, r_2, \ldots r_n$ respectively; then

$$X = m\left\{m_1\frac{d.F(r_1)}{da} + m_2\frac{d.F(r_2)}{da} + \cdots\right\}$$

$$\left.\begin{aligned} &= m\frac{d}{da}\{m_1F(r_1) + m_2F(r_2) + \ldots\} \\ &= m\frac{d}{da}\Sigma.m'F(r); \\ \text{similarly} \quad Y &= m\frac{d}{d\beta}\Sigma.m'F(r); \qquad Z = m\frac{d}{d\gamma}\Sigma.m'F(r). \end{aligned}\right\} \tag{99}$$

Let $$\Sigma.m'F(r) = V; \tag{100}$$

then $$X = m\left(\frac{dV}{da}\right), \qquad Y = m\left(\frac{dV}{d\beta}\right), \qquad Z = m\left(\frac{dV}{d\gamma}\right); \tag{101}$$

that is, the axial-components of the attraction are partial differential coefficients of the function V, which is defined by (100).

In the formula (100) Σ expresses a sum; and becomes the symbol of a definite integral, when the attracting mass is a continuous body. In this latter case, if ρ is the density of the body at (x, y, z), the mass-element at that point is $\rho\,dx\,dy\,dz$, and

$$V = \iiint \rho\,F(r)\,dx\,dy\,dz. \tag{102}$$

Thus, if this function can be determined, the axial-components are expressed very simply as the partial derived functions of it with respect to the coordinates of the attracted particle.

V, as defined by (100) or by (102), is called *the potential of the mass* $\Sigma . m'$, in reference to m at (α, β, γ), which is attracted according to the law given by $f(r)$, where $F(r) = \int f(r)\, dr$.

217.] Let us now take the law of attraction to be that of the inverse square of the distance; so that

$$f(r) = -\frac{1}{r^2}; \qquad F(r) = \frac{1}{r}; \tag{103}$$

consequently

$$V = \Sigma . \frac{m'}{r} = \iiint \frac{\rho\, dx\, dy\, dz}{r}; \tag{104}$$

the latter or the former value of V being taken according as the attracting mass is continuous or discontinuous.

The value of r being that which is given in (95), we have

$$X = -m \left(\frac{dV}{d\alpha}\right) = m \iiint \frac{\rho\, (\alpha - x)\, dx\, dy\, dz}{r^3}, \tag{105}$$

$$Y = -m \left(\frac{dV}{d\beta}\right) = m \iiint \frac{\rho (\beta - y)\, dx\, dy\, dz}{r^3}, \tag{106}$$

$$Z = -m \left(\frac{dV}{d\gamma}\right) = m \iiint \frac{\rho (\gamma - z)\, dx\, dy\, dz}{r^3}; \tag{107}$$

and thus the axial-components of the resultant attraction depend on the definite integral which is given in the right-hand member of (104). Now this integral is evidently finite so long as the attracted particle is outside of, and not a part of, the attracting mass, even if the attracting mass is a closed shell having m within it: because in this case r would never vanish, and no value of the element-function would be infinite. If, however, the attracted particle were upon or were a particle of the attracting mass, it is not so evident but that the definite integral might be infinite, because for attracting particles contiguous to m, $r = 0$, and the corresponding value of the element-function is infinite. But if we take the origin at the place of the attracted particle, and refer to polar coordinates in space, we have

$$V = \iiint \rho\, r \sin\theta\, dr\, d\theta\, d\phi; \tag{108}$$

and this evidently does not take an infinite value when $r = 0$.

Similarly, if the attracted particle is at the origin, and X, Y, Z are expressed in terms of polar coordinates, it will be seen that these do not take infinite values, notwithstanding the attracted particle is contiguous to particles of the attracting mass.

Although the potential is a term by which the general integral given in (102) is called, yet the name is specially applied to the particular form in (104) and (108) where the law of attraction varies inversely as the square of the distance: for this is the law of gravitation, as also the law according to which electrical and magnetic forces act.

Thus *the potential* is the sum of every mass-element of the attracting mass divided by its distance from the attracted particle.

As the potential is expressed by a sum or by an integral, it is evident that the potential of a mass is the sum of the potentials of the parts of which that mass is composed.

The name of "Potential-function" was given to v by George Green, in his most remarkable "Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," which was published at Nottingham in 1828. But the abridged name "Potential" was given by Gauss in his memoir "On General Propositions relating to Attractive and Repulsive Forces varying inversely as the Square of the Distance," which is printed in "Resultate aus den Beobachtungen des Magnetischen Vereins im Jahre 1839."

218.] The potential is of so great importance not only in the theory of attractions but also in dynamics, hydromechanics, electricity, magnetism, and heat, that before we determine its value in particular cases it is desirable to consider its meaning from two points of view.

As it is the sum of every mass-element of the attracting mass divided by its distance from the attracted particle, it is a function independent of any particular system of coordinates to which the particle and the attracting mass may be referred. Its value depends indeed on the place of the attracted particle, and varies as that place varies; but it is independent of the mode of determining that place.

It has however the following important physical meaning: We estimate *work*, as we shall hereafter explain at length, by the product of pressure and the distance through which the pressure has acted estimated along the action-line of the pressure. Thus, for instance, if a weight, $= mg$, is raised through a vertical distance equal to h, the work which has been spent on the lifting that body, and is due, as we say, to its change of place, is mgh; and the work which would be recovered, if the body were replaced in its former position, is also mgh. Now potential is the particular form which the work of an unit-particle takes, when it is under the attraction of one or more other particles. For suppose an unit-particle to have come from an infinite distance under the attraction of a particle whose mass is m', and where the law of attraction is that of the inverse square of the distance, so that the attraction on the unit-particle at that distance is zero; then if w is the work when the particle is at the distance r, and $d\text{w}$ is the increment of work due to, or obtained by its passage over, the space dr,

$$d\text{w} = -\frac{m'}{r^2}\,dr;$$

$$\therefore \quad \text{w} = \frac{m'}{r}; \tag{109}$$

so that the mass of the attracting particle divided by the distance of the unit-particle from the place of the attracting particle is the work of the unit-particle acquired in moving from an infinite distance to a point at a distance r from the attracting particle. And as a similar result is true for an unit-particle under the attraction of many particles, and as the whole work of the unit particle is the sum of all the separate works, so

$$\text{the whole work} = \Sigma.\frac{m'}{r} = \iiint \frac{\rho\,dx\,dy\,dz}{r}; \tag{110}$$

the latter or former being taken according as the attracting mass is a continuous body, or is composed of separate particles. But as the right-hand member of (110) is the potential as defined by (104), the potential is the whole work obtained by an unit-particle in its passage from infinity to a point at a distance r, under the attraction of the several particles of the attracting body. This is the mechanical interpretation of the potential.

219.] The component along any line of the resultant attraction of the attracting mass on m at (a, β, γ) may thus be found.

Suppose (a, β, γ) to be a point on a curve or other line, of which ds is a length-element, so that the direction-cosines of ds are $\frac{da}{ds}, \frac{d\beta}{ds}, \frac{d\gamma}{ds}$; then the component along ds of the resultant attraction is

$$= \mathrm{X}\frac{d\alpha}{ds} + \mathrm{Y}\frac{d\beta}{ds} + \mathrm{Z}\frac{d\gamma}{ds}$$

$$= -\frac{m}{ds}\left\{\left(\frac{d\mathrm{V}}{d\alpha}\right)d\alpha + \left(\frac{d\mathrm{V}}{d\beta}\right)d\beta + \left(\frac{d\mathrm{V}}{d\gamma}\right)d\gamma\right\}$$

$$= -m\frac{d\mathrm{V}}{ds}; \qquad (111)$$

so that the force of attraction which acts in the line ds varies as $\frac{d\mathrm{V}}{ds}$.

Hence if R is the resultant attraction, and θ is the angle between its action-line and ds,

$$\mathrm{R}\cos\theta = -m\frac{d\mathrm{V}}{ds};$$

$$\therefore \int_0^1 \mathrm{R}\cos\theta\, ds = -m(\mathrm{V}_1 - \mathrm{V}_0).$$

And if the integration is carried through a closed ring, then $\mathrm{V}_1 = \mathrm{V}_0$, and

$$\int \mathrm{R}\cos\theta\, ds = 0.$$

The axial-components are evidently particular cases of (111). If the place of the attracted particle, in reference to a system of polar coordinates in space, is (r, θ, ϕ), then the components of attraction perpendicular to the meridian plane, along the radius vector, and perpendicular to the radius vector, the action-lines of both these last components being in the meridian plane, are respectively

$$\frac{-m}{r\sin\theta}\left(\frac{d\mathrm{V}}{d\phi}\right), \qquad -m\left(\frac{d\mathrm{V}}{dr}\right), \qquad \frac{-m}{r}\left(\frac{d\mathrm{V}}{d\theta}\right). \qquad (112)$$

If the attracting mass is a plane wire or a plane plate, and the attracted particle is in that plane, and the place of it is (r, θ) in reference to a system of polar coordinates, then the radial and transversal components of the attraction are respectively

$$-m\left(\frac{d\mathrm{V}}{dr}\right), \qquad -\frac{m}{r}\left(\frac{d\mathrm{V}}{d\theta}\right). \qquad (113)$$

Before I enter on the investigation of other general properties of the potential, and especially of the geometrical interpretation of it, I will determine its value in some particular cases, and derive from them the corresponding components of attraction.

220.] The potential of a thin straight rod on an external particle.

Let ω and ρ be respectively the area of the transverse section and the density of the rod. Let the place of the attracted particle m be (α, β); the extremity of the rod being the origin, and the axis of x lying along the rod, whose length $= a$. Then

$$\begin{aligned}
\text{V} &= \int_0^a \frac{\rho\,\omega\,dx}{\{\beta^2+(x-\alpha)^2\}^{\frac{1}{2}}} \\
&= \rho\,\omega \log \frac{a-\alpha+\{\beta^2+(a-\alpha)^2\}^{\frac{1}{2}}}{-\alpha+(\beta^2+\alpha^2)^{\frac{1}{2}}};
\end{aligned}$$

$$\therefore \quad \text{X} = -m\left(\frac{d\text{V}}{d\alpha}\right) = m\rho\,\omega\left\{\frac{1}{\{\beta^2+(a-\alpha)^2\}^{\frac{1}{2}}} - \frac{1}{(\beta^2+\alpha^2)^{\frac{1}{2}}}\right\};$$

$$\text{Y} = -m\left(\frac{d\text{V}}{d\beta}\right) = \frac{m\rho\,\omega}{\beta}\left\{\frac{a-\alpha}{\{\beta^2+(a-\alpha)^2\}^{\frac{1}{2}}} + \frac{\alpha}{(\beta^2+\alpha^2)^{\frac{1}{2}}}\right\};$$

which are the same results as those found in Art. 185.

221.] The potential of a thin homogeneous spherical shell.

Let $\rho =$ the density, $\tau =$ the thickness, $a =$ the radius of the shell; let $(0, 0, \gamma)$ be the place of the attracted particle. Then if the centre of the shell is the origin,

$$\begin{aligned}
\text{V} &= \rho\,\tau a^2 \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta\,d\theta\,d\phi}{(a^2-2a\gamma\cos\theta+\gamma^2)^{\frac{1}{2}}} \\
&= \frac{2\pi\rho\tau a}{\gamma}\Big[(a^2-2a\gamma\cos\theta+\gamma^2)^{\frac{1}{2}}\Big]_0^{\pi}. \qquad (114)
\end{aligned}$$

Now this radical takes different forms according as the attracted particle is within, or is external to, the shell.

If the particle is within the shell, γ is less than a; and

$$\begin{aligned}
\text{V} &= \frac{2\pi\rho\tau a}{\gamma}\{a+\gamma-(a-\gamma)\} \\
&= 4\pi\rho\tau a. \qquad (115)
\end{aligned}$$

If the particle is outside the shell, γ is greater than a; and

$$\begin{aligned}
\text{V} &= \frac{2\pi\rho\tau a}{\gamma}\{\gamma+a-\gamma+a\} \\
&= \frac{4\pi\rho\tau a^2}{\gamma}. \qquad (116)
\end{aligned}$$

In the former case $\frac{d\text{V}}{d\gamma} = 0$; and the shell exerts no attraction on a particle within it.

In the latter case

$$\begin{aligned}
\text{Z} &= \frac{4\pi\rho\tau a^2 m}{\gamma^2} \\
&= \frac{\text{M}m}{\gamma^2},
\end{aligned}$$

if the mass of the shell $= \text{M}$; and the attraction of the shell on the external particle is the same as if the mass of the shell were condensed into a particle at its centre.

If the shell is not thin, but contained between two concentric spherical shells whose radii are a and a' respectively, the potentials are thus found.

If the attracted particle is in the shell, and at a distance γ from the centre, V consists of two integrals; one of which corresponds to the matter outside the concentric sphere which passes through m, and the other to the matter within that sphere. Consequently, since $\text{V} = 0$, when $a = \gamma = a'$, that is, when there is no attracting matter,

$$
\begin{aligned}
\text{V} &= 4\pi\rho\int_{\gamma}^{a} a\,da + \frac{4\pi\rho}{\gamma}\int_{a'}^{\gamma} a^2\,da \\
&= 2\pi\rho(a^2-\gamma^2) + \frac{4\pi\rho}{3\gamma}(\gamma^3 - a'^3) \\
&= 2\pi\rho a^2 - \frac{2\pi\rho\gamma^2}{3} - \frac{4\pi\rho a'^3}{3\gamma}. \qquad (117)
\end{aligned}
$$

And if the particle is outside the shell,

$$
\begin{aligned}
\text{V} &= \frac{4\pi\rho}{\gamma}\int_{a'}^{a} a^2\,da \\
&= \frac{4\pi\rho}{3\gamma}(a^3 - a'^3) = \frac{\text{M}}{\gamma}, \qquad (118)
\end{aligned}
$$

since $\text{V} = 0$, when $a = a'$, because in that case there is no attracting matter.

Consequently if the attracting body be a full homogeneous sphere, then $a' = 0$; and for an internal particle

$$\text{V} = 2\pi\rho a^2 - \frac{2\pi\rho\gamma^2}{3};$$

and for an external particle,

$$\text{V} = \frac{4\pi\rho a^3}{3\gamma} = \frac{\text{M}}{\gamma}.$$

As both these expressions give the same value for V when $\gamma = a$, that is, when m is on the external surface of the shell, V undergoes no discontinuity at that point.

222.] The potential of a sphere composed of concentric spherical shells, the density of each being a function of its radius.

Of the shell of which the radius is r, and the thickness is dr, let the density be $f(r)$: then we have two cases according as the particle is outside the sphere, or is within it.

If the particle is outside the sphere, and at a distance from the centre $= \gamma$, and if a is the radius of the sphere,

$$\begin{aligned}\text{V} &= \int_0^a \int_0^\pi \int_0^{2\pi} \frac{r^2 f(r) \sin\theta \, d\phi \, d\theta \, dr}{(r^2 - 2r\gamma\cos\theta + \gamma^2)^{\frac{1}{2}}} \qquad (119) \\ &= \frac{4\pi}{\gamma} \int_0^a r^2 f(r) \, dr;\end{aligned}$$

whereby the potential may be found, when $f(r)$ is given. If M = the mass of the sphere,

$$4\pi \int_0^a r^2 f(r) \, dr = \text{M}; \quad \text{so that}$$

$$\text{V} = \frac{\text{M}}{\gamma};$$

and consequently the potential is the same as that of a particle, whose mass is M, placed at the centre of the sphere.

If the particle is inside the sphere, sò that γ is less than a, then the integral given in (119), and which expresses the potential, must be divided into two parts, the former of which will correspond to the shell on the interior surface of which the attracted particle is, and the latter to the sphere on the surface of which the attracted particle is: so that

$$\begin{aligned}\text{V} &= 2\pi \int_\gamma^a \int_0^\pi \frac{r^2 f(r) \sin\theta \, d\theta \, dr}{(\gamma^2 - 2\gamma r\cos\theta + r^2)^{\frac{1}{2}}} + 2\pi \int_0^\gamma \int_0^\pi \frac{r^2 f(r) \sin\theta \, d\theta \, dr}{(\gamma^2 - 2\gamma r\cos\theta + r^2)^{\frac{1}{2}}} \\ &= 4\pi \int_\gamma^a r f(r) \, dr + \frac{4\pi}{\gamma} \int_0^\gamma r^2 f(r) \, dr; \qquad (120)\end{aligned}$$

and this is the required potential.

If we take the γ-differential, according to the principles explained in Art. 96, Vol. II (Integral Calculus), we have

$$\left(\frac{d\text{V}}{d\gamma}\right) = -4\pi\gamma f(\gamma) + \frac{4\pi}{\gamma}\gamma^2 f(\gamma) - \frac{4\pi}{\gamma^2} \int_0^\gamma r^2 f(r) \, dr;$$

$$\therefore \quad \text{Z} = -m\left(\frac{d\text{V}}{d\gamma}\right) = \frac{4\pi m}{\gamma^2} \int_0^\gamma r^2 f(r) \, dr; \qquad (121)$$

and because $4\pi \int_0^\gamma r^2 f(r) \, dr$ is the mass of the sphere whose radius is γ, it follows that the shell lying outside that sphere exercises no effect on the attracted particle.

223.] The potential of a body of finite dimensions on a particle at a very great distance.

Let us suppose the attracting mass to be a continuous body, and M to be its mass. Let the origin be taken at its centre of mass; so that

$$\iiint \rho \, x \, dx \, dy \, dz = \iiint \rho \, y \, dx \, dy \, dz = \iiint \rho \, z \, dx \, dy \, dz = 0. \quad (122)$$

Also let us so far anticipate what will be proved hereafter as to assume the possibility of a system of coordinate-axes such that with reference to them

$$\iiint \rho\, y z\, dx\, dy\, dz = \iiint \rho\, z x\, dx\, dy\, dz = \iiint \rho\, x y\, dx\, dy\, dz = 0, \quad (123)$$

and let the attracting body be referred to this system; also let (a, β, γ) be the place of m the attracted particle. Let

$$\sigma^2 = a^2 + \beta^2 + \gamma^2,$$

so that σ is very large in comparison of the distance of any mass-element of the attracting body from the origin. Then

$$\mathrm{V} = \iiint \frac{\rho\, dx\, dy\, dz}{r},$$

where

$$\begin{aligned} r^2 &= (a-x)^2 + (\beta-y)^2 + (\gamma-z)^2 \\ &= a^2 + \beta^2 + \gamma^2 - 2(ax + \beta y + \gamma z) + x^2 + y^2 + z^2 \\ &= \sigma^2 - 2(ax + \beta y + \gamma z) + x^2 + y^2 + z^2. \qquad (124) \end{aligned}$$

Therefore expanding by the binomial theorem,

$$\frac{1}{r} = \frac{1}{\sigma} + \frac{ax + \beta y + \gamma z}{\sigma^3} + \frac{3(ax + \beta y + \gamma z)^2 - (x^2 + y^2 + z^2)\sigma^2}{2\sigma^5}, \quad (125)$$

omitting terms involving in the denominator higher powers of σ; so that

$$\mathrm{V} = \iiint \rho \left\{ \frac{1}{\sigma} + \frac{ax + \beta y + \gamma z}{\sigma^3} + \frac{3(ax + \beta y + \gamma z)^2 - (x^2 + y^2 + z^2)\sigma^2}{2\sigma^5} \right\} dx\, dy\, dz. \quad (126)$$

Now of the several definite integrals which this expression contains, the first $= \frac{\mathrm{M}}{\sigma}$, because $\iiint \rho\, dx\, dy\, dz = \mathrm{M}$; those corresponding to the second term in the expression vanish by reason of (122); of those in the third term some vanish by reason of (123), and we have ultimately,

$$\mathrm{V} = \frac{\mathrm{M}}{\sigma} + \frac{1}{2s^5} \iiint \rho \{ (2a^2 - \beta^2 - \gamma^2) x^2 + (-a^2 + 2\beta^2 - \gamma^2) y^2 + (-a^2 - \beta^2 + 2\gamma^2) z^2 \} dx\, dy\, dz. \quad (127)$$

To abridge this expression let

$$\left. \begin{aligned} \mathrm{A} &= \iiint \rho (y^2 + z^2)\, dx\, dy\, dz, \\ \mathrm{B} &= \iiint \rho (z^2 + x^2)\, dx\, dy\, dz, \\ \mathrm{C} &= \iiint \rho (x^2 + y^2)\, dx\, dy\, dz; \end{aligned} \right\} \quad (128)$$

where A, B, C express the sums of the products of each mass-element of the attracting body and the square of its distance from the axes of x, y, z respectively; and substituting these in (127) we have

$$V = \frac{M}{\sigma} + \frac{1}{2\sigma^3}\{(B+C-2A)\alpha^2 + (C+A-2B)\beta^2 + (A+B-2C)\gamma^2\}; \quad (129)$$

and this is the required value to the stated degree of approximation.

If we neglect the whole second term,

$$V = \frac{M}{\sigma}; \qquad (130)$$

and the potential is the same as it would be if the whole attracting body were condensed into a particle at its centre of mass; and thus if R is the resultant attraction, by (111),

$$\begin{aligned} R &= -m\frac{dV}{d\sigma} \\ &= \frac{Mm}{\sigma^2}. \qquad (131) \end{aligned}$$

Hence we have the following very important theorem:

The attraction of a body or of a material system on a particle at a distance from its centre of mass, which is very great in comparison of the linear dimensions of the body, is the same as it would be if the whole attracting mass were condensed into a particle at its mass-centre.

Also as m may be a mass-element of a finite body M′, and as this theorem will be true for every mass-element, so we have the following theorem:

Two bodies or material systems, of which the linear dimensions are very small in comparison of the distance between their mass-centres, attract each other in the same manner as if the mass of each were condensed into a particle at its mass-centre.

Thus as each heavenly body consists of attracting particles, and the linear dimensions of these bodies are small in comparison of their distances from each other, they attract each other approximately as if the mass of each were condensed into a particle at its mass-centre. Consequently in the investigation of the motion of any one of these bodies, we may consider it to be under the attraction of many particles, each attracting as its mass and inversely as the square of its distance from the mass-centre of the attracted body.

Theorems similar to these have already been demonstrated of spheres, whatever is the distance between their centres; so that what is approximately true of bodies of any form, the mass-centres of which are at a very great distance apart, is true of two spheres, whatever is the distance between their centres. This result may also be inferred from the preceding process of investigation; for in spheres, by reason of the symmetrical distribution of their matter, $A = B = C$, and all the terms in the expansion of V, except the first, vanish.

224.] Returning to the general value in (129), and taking in all the terms, we may, with the object of determining the components of attraction, express it as follows:

$$V = \frac{M}{\sigma} + \frac{B+C-2A}{2\sigma^3} + \frac{3}{2\sigma^5}\{(A-B)\beta^2 + (A-C)\gamma^2\}; \quad (132)$$

$$\therefore \quad X = -m\left(\frac{dV}{d\alpha}\right)$$

$$= \frac{Mm\alpha}{\sigma^3} + \frac{3m\alpha}{2\sigma^5}(B+C-2A) + \frac{15m\alpha}{2\sigma^7}\{(A-B)\beta^2 + (A-C)\gamma^2\}; \quad (133)$$

$$Y = \frac{Mm\beta}{\sigma^3} + \frac{3m\beta}{2\sigma^5}(C+A-2B) + \frac{15m\beta}{2\sigma^7}\{(B-C)\gamma^2 + (B-A)\alpha^2\}; \quad (134)$$

$$Z = \frac{Mm\gamma}{\sigma^3} + \frac{3m\gamma}{2\sigma^5}(A+B-2C) + \frac{15m\gamma}{2\sigma^7}\{(C-A)\alpha^2 + (C-B)\beta^2\}; \quad (135)$$

which are the axial-components of the attraction of the mass M on m. If we are investigating the action of M on m, these must be taken with negative signs, as M tends to draw m towards the origin.

In reference to these expressions, the first terms are evidently the axial-components of a central force, whose centre, that is, the point in which the force originates, is the origin; and consequently, as regards rotatory motion, this force has no effect: but this is not the case as to the other terms of the expressions, inasmuch as the corresponding coefficients are generally not equal: and if L, M, N are the axial-components of the resulting couple, then by (104), Art. 68,

$$\left.\begin{aligned} L &= Z\beta - Y\gamma = -\frac{3m\beta\gamma}{\sigma^5}(B-C); \\ M &= X\gamma - Z\alpha = -\frac{3m\gamma\alpha}{\sigma^5}(C-A); \\ N &= Y\alpha - X\beta = -\frac{3m\alpha\beta}{\sigma^5}(A-B). \end{aligned}\right\} \quad (136)$$

These results will be of considerable use hereafter, when we come to the investigation of the motion of a body about an axis passing through its mass-centre.

225.] The potential of a homogeneous ellipsoid.

Let the centre of the ellipsoid be the origin, and (a, β, γ) the place of m the attracted particle. Then

$$\text{v} = \iiint \frac{\rho\, dx\, dy\, dz}{\{(a-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{1}{2}}}; \tag{137}$$

the range of integration being the space included within the ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1. \tag{138}$$

For the evaluation of this triple integral, I propose to take the method explained in Art. 283, Vol. II (Integral Calculus), and to determine it, as far as it is possible, by Dirichlet's process of a discontinuous function.

Taking the discontinuous function given in (68) Art. 283, Vol. II (Integral Calculus), we have

$$\left.\begin{aligned} \frac{2}{\pi}\int_0^\infty \frac{\sin t \cos kt}{t}\, dt &= 1, \text{ when } k \text{ is less than } 1; \\ &= 0, \text{ when } k \text{ is greater than } 1. \end{aligned}\right\} \tag{139}$$

Also since $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$ is less than, or greater than 1, according as (x, y, z) is within or without the ellipsoid,

$$\frac{2}{\pi}\int_0^\infty \frac{\sin t}{t} \cos\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) t\, dt = 1, \text{ or } = 0,$$

according as (x, y, z) is within or without the ellipsoid which is given in (138).

To abridge the expression let

$$(a-x)^2+(\beta-y)^2+(\gamma-z)^2 = u^2; \tag{140}$$

then we may express v in the following form:

$$\text{v} = \frac{2\rho}{\pi}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\sin t}{t} \cos\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) t\, \frac{dx\, dy\, dz\, dt}{u}; \tag{141}$$

the limits of integration having been extended to ∞ and to $-\infty$, and thus being constant, so that the order, in which the several integrations may be effected, is indifferent.

And v can be expressed in another form which is more convenient for our present purpose;

v = the real part

$$\text{of } \frac{2\rho}{\pi}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{\sin t}{t}\, e^{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)t\sqrt{-1}}\,\frac{dx\,dy\,dz\,dt}{u}. \tag{142}$$

We must however replace $\frac{1}{u}$ by a definite integral, which will give an expression of a more convenient form.

By (280) Art. 129, Vol. II (Integral Calculus),

$$\int_0^\infty e^{-x^2}dx = \frac{\pi^{\frac{1}{2}}}{2}.$$

Let x^2 be replaced by $-\sqrt{-1}\,a^2 s$; then we have

$$\frac{1}{a} = \frac{(-1)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}}\int_0^\infty e^{a^2 s\sqrt{-1}}\,\frac{ds}{s^{\frac{1}{2}}};$$

$$\therefore\quad \frac{1}{u} = \frac{(-1)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}}\int_0^\infty e^{u^2 s\sqrt{-1}}\,\frac{ds}{s^{\frac{1}{2}}};$$

∴ v = the real part

$$\text{of } \frac{2\rho}{\pi^{\frac{3}{2}}}(-1)^{\frac{1}{4}}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{\sin t}{t}\, e^{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)t\sqrt{-1}}\, e^{u^2 s\sqrt{-1}}\,\frac{dx\,dy\,dz\,dt\,ds}{s^{\frac{1}{2}}};$$

$$= \frac{2\rho}{\pi^{\frac{3}{2}}}(-1)^{\frac{1}{4}}\int_0^\infty\int_0^\infty \textsc{i}\,\frac{\sin t}{t}\, e^{(\alpha^2+\beta^2+\gamma^2)s\sqrt{-1}}\,\frac{dt\,ds}{s^{\frac{1}{2}}},$$

where I = the triple integral which includes the *x*- *y*- *z*-integrations.

Now for the *x*-integral we have by Art. 138, Vol. II (Integral Calculus),

$$\int_{-\infty}^\infty e^{\left\{\left(s+\frac{1}{a^2}\right)x^2-2\alpha s x\right\}\sqrt{-1}}\,dx = \frac{\pi^{\frac{1}{2}}a}{(-1)^{\frac{1}{4}}(a^2 s+t)^{\frac{1}{2}}}\,e^{-\sqrt{-1}\,\frac{a^2\alpha^2 s^2}{a^2 s+t}}:$$

and similar values are true for the *y*- and the *z*-integrals; so that

$$\textsc{i} = \frac{\pi^{\frac{3}{2}}}{(-1)^{\frac{3}{4}}}\,\frac{abc\,e^{-\sqrt{-1}\left(\frac{a^2\alpha^2}{a^2 s+t}+\frac{b^2\beta^2}{b^2 s+t}+\frac{c^2\gamma^2}{c^2 s+t}\right)s^2}}{\{(a^2 s+t)(b^2 s+t)(c^2 s+t)\}^{\frac{1}{2}}}.$$

Consequently v is the real part of

$$\frac{2\rho abc}{(-1)^{\frac{1}{2}}}\int_0^\infty\int_0^\infty \frac{\sin t}{t}\,\frac{e^{-\sqrt{-1}\left(\frac{a^2\alpha^2}{a^2 s+t}+\frac{b^2\beta^2}{b^2 s+t}+\frac{c^2\gamma^2}{c^2 s+t}\right)s^2}}{\{(a^2 s+t)(b^2 s+t)(c^2 s+t)\}^{\frac{1}{2}}}\,e^{(\alpha^2+\beta^2+\gamma^2)s\sqrt{-1}}\,\frac{dt\,ds}{s^{\frac{1}{2}}}; \tag{143}$$

The variables *s* and *t* are thus far unconnected; let us suppose $s\omega = t$, where ω is a new variable dependent on *u* and independent of *t*: then

$$ds = -\frac{t\,d\omega}{\omega^2}; \quad \text{and}$$

$$s^2\left\{\frac{a^2\alpha^2}{a^2s+t}+\frac{b^2\beta^2}{b^2s+t}+\frac{c^2\gamma^2}{c^2s+t}\right\}=\frac{t}{\omega}\left\{\frac{a^2\alpha^2}{a^2+\omega}+\frac{b^2\beta^2}{b^2+\omega}+\frac{c^2\gamma^2}{c^2+\omega}\right\}$$

$$=\frac{t}{\omega}\{\alpha^2+\beta^2+\gamma^2\}-t\left\{\frac{\alpha^2}{a^2+\omega}+\frac{\beta^2}{b^2+\omega}+\frac{\gamma^2}{c^2+\omega}\right\}$$

$$=\frac{t}{\omega}(\alpha^2+\beta^2+\gamma^2)-tk,$$

$$\text{if } k=\frac{\alpha^2}{a^2+\omega}+\frac{\beta^2}{b^2+\omega}+\frac{\gamma^2}{c^2+\omega}; \qquad (144)$$

so that V is the real part

$$\text{of} \quad \frac{2\rho abc}{(-1)^{\frac{1}{2}}}\int_0^\infty\int_0^\infty\frac{\sin t}{t^2}\,\frac{e^{kt\sqrt{-1}}\,dt\,d\omega}{\{(a^2+\omega)(b^2+\omega)(c^2+\omega)\}^{\frac{1}{2}}}, \qquad (145)$$

$$\text{of } \sqrt{-1}\;2\rho abc\int_0^\infty\int_0^\infty\frac{\sin t}{t^2}\,\frac{(\cos kt+\sqrt{-1}\sin kt)\,dt\,d\omega}{\{(a^2+\omega)(b^2+\omega)(c^2+\omega)\}^{\frac{1}{2}}}; \qquad (146)$$

$$\therefore\quad \text{V}=-2\rho abc\int_0^\infty\int_0^\infty\frac{\sin t}{t^2}\,\frac{\sin kt\,dt\,d\omega}{\{(a^2+\omega)(b^2+\omega)(c^2+\omega)\}^{\frac{1}{2}}}; \qquad (147)$$

where k is a function of α, β, γ and ω, by reason of (144).

This is the potential of a homogeneous ellipsoid on a particle at (α, β, γ) whether the place of it is within or without the ellipsoid.

For the components of attraction, as k is a function of α, we have

$$\left(\frac{d\text{V}}{d\alpha}\right)=-4\rho\alpha abc\int_0^\infty\int_0^\infty\frac{\sin t}{t}\,\frac{\cos kt\,dt\,d\omega}{(a^2+\omega)\{(a^2+\omega)(b^2+\omega)(c^2+\omega)\}^{\frac{1}{2}}}; \qquad (148)$$

with similar values of $\left(\frac{d\text{V}}{d\beta}\right)$ and $\left(\frac{d\text{V}}{d\gamma}\right)$.

Now if (α, β, γ) is within the ellipsoid, k is evidently less than 1; and consequently,

$$\int_0^\infty\frac{\sin t}{t}\cos kt\,dt=\frac{\pi}{2};$$

therefore

$$\text{X}=-m\left(\frac{d\text{V}}{d\alpha}\right)=2\pi\rho m\alpha abc\int_0^\infty\frac{d\omega}{(a^2+\omega)\{(a^2+\omega)(b^2+\omega)(c^2+\omega)\}^{\frac{1}{2}}}; \qquad (149)$$

$$\text{Y}=-m\left(\frac{d\text{V}}{d\beta}\right)=2\pi\rho m\beta abc\int_0^\infty\frac{d\omega}{(b^2+\omega)\{(a^2+\omega)(b^2+\omega)(c^2+\omega)\}^{\frac{1}{2}}}; \qquad (150)$$

$$\text{Z}=-m\left(\frac{d\text{V}}{d\gamma}\right)=2\pi\rho m\gamma abc\int_0^\infty\frac{d\omega}{(c^2+\omega)\{(a^2+\omega)(b^2+\omega)(c^2+\omega)\}^{\frac{1}{2}}}; \qquad (151)$$

which are the values already found by the direct process.

If (α, β, γ) is without the ellipsoid, so that $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}$ is greater than 1, let Ω be the greatest root of the cubic

$$\frac{\alpha^2}{a^2+\omega}+\frac{\beta^2}{b^2+\omega}+\frac{\gamma^2}{c^2+\omega}=1; \tag{152}$$

then as the left-hand member of this equation decreases as ω increases, so is the quantity less than 1 for all values of ω from ∞ to Ω; and consequently for all these values of ω, k is less than 1, and for other values k is greater than 1. Hence if we confine ω within the limits for which k is less than 1, we have, as before,

$$\int_0^\infty \frac{\sin t \cos kt}{t}\,dt=\frac{\pi}{2};$$

and accordingly for an external particle,

$$\left(\frac{dV}{d\alpha}\right)=-2\pi\rho\alpha abc\int_\Omega^\infty \frac{d\omega}{(a^2+\omega)\{(a^2+\omega)(b^2+\omega)(c^2+\omega)\}^{\frac{1}{2}}};$$

$$X=2\pi\rho m\alpha abc\int_\Omega^\infty \frac{d\omega}{(a^2+\omega)\{(a^2+\omega)(b^2+\omega)(c^2+\omega)\}^{\frac{1}{2}}};$$

and if in this quantity we replace ω by $\omega+\Omega$, and put

$$a^2+\Omega=a'^2, \qquad b^2+\Omega=b'^2, \qquad c^2+\Omega=c'^2, \tag{153}$$

$$X=2\pi\rho m\alpha abc\int_0^\infty \frac{d\omega}{(a'^2+\omega)\{(a'^2+\omega)(b'^2+\omega)(c'^2+\omega)\}^{\frac{1}{2}}}. \tag{154}$$

Similarly,

$$Y=2\pi\rho m\beta abc\int_0^\infty \frac{d\omega}{(b'^2+\omega)\{(a'^2+\omega)(b'^2+\omega)(c'^2+\omega)\}^{\frac{1}{2}}}; \tag{155}$$

$$Z=2\pi\rho m\gamma abc\int_0^\infty \frac{d\omega}{(c'^2+\omega)\{(a'^2+\omega)(b'^2+\omega)(c'^2+\omega)\}^{\frac{1}{2}}}. \tag{156}$$

These quantities which contain only single definite integrals give the axial-components of the attraction of a homogeneous ellipsoid on an external particle; they are of the same form as (53), (54), (55) which assign the components of attraction of a homogeneous ellipsoid on an internal particle.

From (152) and (153) it is evident that a', b', c' are the principal semi-axes of the surface concentric and confocal with the given ellipsoid, and the value of ω which we have chosen shews that they are the principal semi-axes of a concentric and confocal ellipsoid which passes through (α, β, γ).

226.] From these values of the components of attraction the following theorems are deduced:

I. If in (149), (150), (151) a, b, c are replaced by ka, kb, kc respectively, and ω is replaced by $k^2\omega$, X, Y, Z are unchanged. Thus the homogeneous shell contained between two similar concentric and similarly-placed ellipsoids has no attraction on a particle placed within the smaller surface. This is Newton's theorem.

For such a shell $\left(\frac{dV}{da}\right) = \left(\frac{dV}{d\beta}\right) = \left(\frac{dV}{d\gamma}\right) = 0$; consequently the potential has the same value for all points within the inner surface of the shell.

II. Let X′, Y′, Z′ be the components of attraction of a homogeneous ellipsoid whose semi-axes are a', b', c' on a particle m situated at (a', β', γ') within the ellipsoid; then by (149),

$$\text{X}' = 2\pi\rho m a' a' b' c' \int_0^\infty \frac{d\omega}{(a'^2+\omega)\{(a'^2+\omega)(b'^2+\omega)(c'^2+\omega)\}^{\frac{1}{2}}};$$

but by (154) if X is the x-component of attraction of a homogeneous concentric and confocal ellipsoid on an equal particle at (a, β, γ),

$$\text{X} = 2\pi\rho m a abc \int_0^\infty \frac{d\omega}{(a'^2+\omega)\{(a'^2+\omega)(b'^2+\omega)(c'^2+\omega)\}^{\frac{1}{2}}};$$

$$\therefore \quad \frac{\text{X}'}{\text{X}} = \frac{a' a' b' c'}{a\, abc}; \tag{157}$$

Let (a', β', γ') on the ellipsoid (a, b, c) be the point corresponding to (a, β, γ) on the ellipsoid (a', b', c'); so that

$$\frac{a'}{a} = \frac{a}{a'};$$

then
$$\frac{\text{X}'}{\text{X}} = \frac{b'c'}{bc}; \tag{158}$$

similarly
$$\frac{\text{Y}'}{\text{Y}} = \frac{c'a'}{ca}; \qquad \frac{\text{Z}'}{\text{Z}} = \frac{a'b'}{ab}; \tag{159}$$

and these three equations constitute Ivory's theorem.

III. Maclaurin's theorem follows immediately from the expressions given in (154), (155), (156).

Let E and E′ be the two homogeneous concentric and confocal ellipsoids which attract m at (a, β, γ); and let X, Y, Z, X′, Y′, Z′ be the respective components of attraction; let a_1, b_1, c_1 be the principal semi-axes of the concentric and confocal ellipsoid passing through (a, β, γ): then

$$\text{X} = 2\pi\rho m a abc \int_0^\infty \frac{d\omega}{(a_1^2+\omega)\{(a_1^2+\omega)(b_1^2+\omega)(c_1^2+\omega)\}^{\frac{1}{2}}};$$

$$\text{X}' = 2\pi\rho m a a' b' c' \int_0^\infty \frac{d\omega}{(a_1^2+\omega)\{(a_1^2+\omega)(b_1^2+\omega)(c_1^2+\omega)\}^{\frac{1}{2}}};$$

$$\therefore \quad \frac{\text{X}}{\text{X}'} = \frac{abc}{a'b'c'} = \frac{\text{M}}{\text{M}'} = \frac{\text{Y}}{\text{Y}'} = \frac{\text{Z}}{\text{Z}'}, \tag{160}$$

if M and M′ are the masses of the two ellipsoids. Thus the ellipsoids attract the external particle with forces proportional to their masses, and along the same line of action.

IV. Hence also the attractions on an external particle of two homogeneous ellipsoidal shells, the external bounding surfaces of which are concentric and confocal ellipsoids, and the internal surfaces of which are ellipsoids concentric, similar and similarly-placed to the external surfaces respectively, and the thicknesses along the same axis are as the axes of the external surfaces, are as the masses of the shells, and have the same action-line.

As this theorem is true for shells of any thickness, it is also true when the shells are infinitesimally thin.

V. If V and V′ are the potentials of E and E′ with respect to an external particle,

$$\frac{\text{V}}{\text{M}} = \frac{\text{V}'}{\text{M}'}. \tag{161}$$

227.] The potential and attraction of ellipsoidal shells.

By an ellipsoidal shell we mean a shell of which the thickness is infinitesimally small, and the bounding surfaces are two similar, similarly placed, and concentric ellipsoidal surfaces.

Consequently if a, b, c are the principal semi-axes of the interior surface, and ka, kb, kc, of the exterior surface, the thickness of the shell at the extremities of the principal axes are severally $(k-1)a$, $(k-1)b$, $(k-1)c$, and are proportional to the corresponding axes respectively;

$$\text{also the volume of the shell} = \frac{4\pi}{3}(k^3-1)abc. \tag{162}$$

Now the potentials of two thin ellipsoidal shells on an external particle are to one another as the masses.

In proof of this it is to be observed that we may deduce it from theorem IV. of the preceding Article; for as the attractions of such shells on an external particle in all directions are as the masses of the shells, so must also the potentials of the two be equal.

The following however is a proof, independent of the preceding calculation of the potential of the homogeneous ellipsoid in an external particle.

Let (α, β, γ) be the place of the attracted particle m; and through it let an ellipsoidal surface E_0 be described concentric and confocal with the exterior surface E of the shell: and also let a similar, similarly situated and concentric ellipsoid be described within the former, and infinitesimally near to it, so that a shell is formed on the exterior surface of which m is.

Let a_0, b_0, c_0 be the principal semi-axes of E_0; and let P_0 (x_0, y_0, z_0) be any point on E_0. To (x_0, y_0, z_0) let the corresponding point $\text{P}(x, y, z)$ on E be taken; so that

$$\frac{x}{a} = \frac{x_0}{a_0}; \qquad \frac{y}{b} = \frac{y_0}{b_0}; \qquad \frac{z}{c} = \frac{z_0}{c_0}:$$

$$\therefore \quad \frac{dx\,dy\,dz}{abc} = \frac{dx_0\,dy_0\,dz_0}{a_0 b_0 c_0}. \tag{163}$$

To the place $\text{Q}\,(\alpha, \beta, \gamma)$ of the attracted particle let the corresponding point $\text{Q}'(\alpha', \beta', \gamma')$ be taken on E; then by reason of (80) Art. 209, $\text{PQ} = \text{P}_0\text{Q}'$;

$$\therefore \quad \frac{1}{abc}\,\frac{dx\,dy\,dz}{\text{PQ}} = \frac{1}{a_0 b_0 c_0}\,\frac{dx_0\,dy_0\,dz_0}{\text{P}_0\,\text{Q}'}; \tag{164}$$

and summing these expressions so as to include the whole shells, for these are corresponding spaces,

$$\iiint \frac{\rho\,dx\,dy\,dz}{\text{PQ}} = \frac{abc}{a_0 b_0 c_0} \iiint \frac{\rho\,dx_0\,dy_0\,dz_0}{\text{P}_0\,\text{Q}'}; \tag{165}$$

and as the volumes of the shells vary as the product of the principal semi-axes, this equation shews that the potential of the inner shell on the particle m at Q which is exterior to it, has to the potential of the exterior shell on the particle m at Q′ which is interior to it, the ratio of the masses of the shells.

Now the potential of an ellipsoidal shell on a particle within it is the same for all positions of the particle, and is consequently constant. Let V_0 be the potential of E_0 on an internal particle m; and let V be the potential of E on the external particle m at (α, β, γ): then from (165) if M and M_0 are the masses of the shells,

$$\frac{\text{V}}{\text{M}} = \frac{\text{V}_0}{\text{M}_0}. \tag{166}$$

Also let there be another ellipsoidal shell E′, of which the outer surface is concentric and confocal with the outer surface of E, and to which m at (a, β, γ) is also external; let M′ be its mass, and V′ its potential; then

$$\frac{V'}{M'} = \frac{V_0}{M_0}; \tag{167}$$

$$\therefore \quad \frac{V}{M} = \frac{V'}{M'}; \tag{168}$$

that is, the potentials of two concentric and confocal ellipsoidal shells on an external particle are to one another as their masses.

Hence also if two concentric and confocal ellipsoidal shells attract a particle m, which is external to both, the components of the attraction along any line vary as the masses of the shells; and the shells also attract the particle along the same action-line.

228.] The action-line of the attraction may thus be determined. Since V_0 and M_0 are the same for all positions of m on the surface of E_0, V by reason of (166) is also constant for all these positions of m; consequently as m is shifted from one place to another on this surface there is no change in the value of V; that is, for such a displacement $\frac{dV}{ds} = 0$; and therefore there is no action of attraction along the surface, and the action is wholly normal to it. And therefore if through the place of the attracted particle an ellipsoid is described concentric and confocal with the exterior surface of the attracting shell, the action-line of the attraction is normal to this ellipsoid.

Hence also if a conical surface is described having its vertex at the place of the attracted particle and enveloping the ellipsoid, the internal axis of the cone is the action-line of the resultant attraction of the shell. Steiner has given a geometrical proof of this theorem. See Crelle, Vol. XII. p. 141.

229.] The amount of attraction of an ellipsoidal shell on an external particle m may thus be found.

Let the ellipsoid E′ in Art. 227 be that on the external surface of which m lies. Then if we can determine the attraction of the shell corresponding to E′ on m, we can by means of (168) deduce that of the shell corresponding to E.

Let O be the position of m on the exterior surface of the shell, a section of which by a plane through the normal OG and the centre C is delineated in fig. 84: then the line of action of the resultant attraction of the shell is, as just now shewn, the line OG. Let a series of very small solid angles originate at O:

and let that one of which the section is $\text{O}\text{P}p'q'\text{Q}$ intercept an area ω of a spherical surface described from O as a centre with the radius = unity: so that the area intercepted at a distance $r = \omega r^2$: now the volume of each of the mass-elements of the shell thus intercepted and at a distance r from $\text{O} = \rho\omega r^2 dr$; and therefore its attraction on $m = m\rho\omega dr$; and therefore the attraction of OPQ on $m = m\rho\omega \times \text{OP}$, and the attraction of $pp'q'q$ on $\text{O} = m\rho\omega \times pp'$: but by reason of the similarity of the surfaces $\text{OP} = pp'$: therefore the attraction on m of the part of the shell intercepted by the cone ω in the direction $\text{OP} = 2m\rho\omega \times \text{OP}$; and therefore the attraction in the direction OG

$$= 2m\rho\omega\,\text{OP}\cos\text{POG}$$

$$= 2m\rho\omega\,\text{O}n = 2m\rho\omega\tau, \tag{169}$$

if τ is the normal thickness of the shell. And since the surface of a hemisphere, whose radius = unity, is 2π, therefore

$$\text{the attraction of the whole shell} = 4\pi m\rho\tau, \tag{170}$$

and thus varies as the thickness of the shell.

This theorem deserves a passing remark. If m is inside the shell, the resultant attraction is zero; if it is outside, and on or very near to the surface of the shell, the attraction $= 4\pi m\rho\tau$; so that on passing through the attracting matter of the shell, the attraction suddenly and discontinuously changes its value. It will be observed that this value of the attraction is the same as that of a plate of infinite area, whether that area be square or circular, see Arts. 192, 200, and whatever is the distance of the particle from the plate.

If in fig. 84 $\text{C}m = r$, $m\text{O} = dr$; and p is the perpendicular from the centre on the tangent plane at O,

$$\frac{\tau}{dr} = \frac{p}{r}; \tag{171}$$

but by reason of the similarity of the bounding surfaces,

$$\frac{dr}{r} = \frac{da}{a} = \frac{db}{b} = \frac{dc}{c}; \tag{172}$$

therefore the attraction of the shell, in the direction of the normal OG,

$$= \frac{4\pi m\rho p}{a}\,da. \tag{173}$$

Now if the equation to the exterior surface is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

then, $$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4};$$

and the direction-cosines of the normal are

$$\frac{px}{a^2}, \quad \frac{py}{b^2}, \quad \frac{pz}{c^2};$$

therefore if X′, Y′, Z′ are the components of the attraction of the shell on m, placed at the point (x, y, z), along the three coordinate axes,

$$\text{X}' = \frac{4\pi m\rho p^2 x\,da}{a^3}; \quad \text{Y}' = \frac{4\pi m\rho p^2 y\,db}{b^3}; \quad \text{Z}' = \frac{4\pi m\rho p^2 z\,dc}{c^3}; \tag{174}$$

and if R is the resultant attraction,

$$\text{R} = \frac{4\pi\rho m p\,da}{a}. \tag{175}$$

Thus the attraction of an ellipsoidal shell on a particle on its surface varies as the perpendicular from the centre of the shell on its tangent plane at the place of the attracted particle. And the attraction is the greatest when the particle is at the extremity of the longest principal semi-axis, and is least when the particle is at the extremity of the shortest principal semi-axis.

Also the shell equally attracts all equal particles placed at points on the exterior surface the tangent planes at which are equally distant from the centre. Thus at all points on the curve of double curvature which is the intersection of the two ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and $$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{1}{k^2},$$

m is equally attracted by the ellipsoidal shell.

Hereby we are led to the determination of the attraction of an ellipsoidal shell on an external particle m.

Through (α, β, γ) the place of m let there be described an ellipsoid E_0, whose principal semi-axes are a_0, b_0, c_0, concentric and confocal with the exterior surface E of the ellipsoidal shell; and let there be described within this ellipsoid a concentric, similar, and similarly situated ellipsoid, of such a thickness da_0 at the extremity of the semi-axis a_0, that

$$\frac{da_0}{a_0} = \frac{da}{a}; \tag{176}$$

then m is on the surface of an ellipsoidal shell. And by the concluding paragraph of Art. 227, if X_0 and X are the x-components of the attractions on m of the shells whose external surfaces are respectively E_0 and E,

$$\text{X} = \frac{abc}{a_0\,b_0\,c_0}\,\text{X}_0. \tag{177}$$

Consequently replacing X_0 by its value which is given in (174),

$$\begin{aligned}\text{X} &= \frac{abc}{a_0\,b_0\,c_0}\,\frac{4\pi m\rho p_0{}^2 a\,da_0}{a_0{}^3}\\ &= \frac{\text{M}m}{a_0\,b_0\,c_0}\,\frac{p_0{}^2 a}{a_0{}^2},\end{aligned} \tag{178}$$

where M, which $= \frac{4\pi\rho abc}{3}\left(\frac{da}{a}+\frac{db}{b}+\frac{dc}{c}\right) = 4\pi\rho bc\,da$, is the mass of the attracting shell. Hence also

$$\text{Y} = \frac{\text{M}m}{a_0\,b_0\,c_0}\,\frac{p_0{}^2\beta}{b_0{}^2}; \qquad \text{Z} = \frac{\text{M}m}{a_0\,b_0\,c_0}\,\frac{p_0{}^2\gamma}{c_0{}^2}; \tag{179}$$

where the letters with the subscript 0 refer to the ellipsoid passing through (α, β, γ) which is the place of the attracted particle m.

Hence also if R is the resultant of this attraction,

$$\text{R} = \frac{\text{M}m p_0}{a_0\,b_0\,c_0}. \tag{180}$$

230.] This result leads to a remarkable theorem.

Let R be the attraction of M on an unit-particle on E_0 corresponding to the perpendicular p_0; then if $d\text{S}$ is a surface-element of E_0 at that point, $\text{R}\,d\text{S}$ is the attraction which acts on $d\text{S}$; and consequently the attraction of M which acts on the whole surface of E_0

$$= \int \text{R}\,d\text{S} = \frac{\text{M}}{a_0\,b_0\,c_0}\int p_0\,d\text{S};$$

but $\int p_0\,d\text{S}$ = three times the volume of the ellipsoid E_0, and consequently $= 4\pi\,a_0\,b_0\,c_0$;

$$\therefore \int \text{R}\,d\text{S} = 4\pi\text{M}; \tag{181}$$

and as the right-hand member is independent of the position of the attracted particle, it follows that the sum of the actions with which an ellipsoidal shell attracts all the elements of a concentric and confocal ellipsoidal shell is constant.

Hereafter we shall have a general theorem which includes this.

231.] The results which have been demonstrated for the attraction of a homogeneous ellipsoidal shell on an external particle are of course the differentials of the attraction of a full ellipsoid, either homogeneous or heterogeneous according to certain laws of varying density; and the attraction of these latter can be inferred from the preceding values by means of integration in the following manner:

Let a, b, c be the principal semi-axes of an ellipsoid attracting a particle m at (α, β, γ) which is a point external to the ellipsoid. Let the ellipsoid be resolved into a series of similar, similarly placed, and concentric ellipsoidal shells, the density of all being the same in a homogeneous ellipsoid, and in the heterogeneous ellipsoid the law of density being such that it is uniform throughout each shell. Let $a\theta$, $b\theta$, $c\theta$ be the principal semi-axes of the exterior surface of one of these shells, θ being a proper fraction; then the thickness is such that

$$\frac{da}{a} = \frac{db}{b} = \frac{dc}{c} = \frac{d\theta}{\theta}.$$

Through (α, β, γ) let an ellipsoid be described concentric and confocal with the exterior surface of the elementary ellipsoidal shell whose semi-axes are $a\theta, b\theta, c\theta$: then the equation of the ellipsoid thus described will be of the form

$$\frac{\alpha^2}{a^2+\omega} + \frac{\beta^2}{b^2+\omega} + \frac{\gamma^2}{c^2+\omega} = \theta^2; \qquad (182)$$

of which cubic equation we take that root which makes all the denominators positive; and let this ellipsoid be the exterior surface of a thin shell, the interior surface of which is a similar and similarly placed ellipsoid, and of which the thickness is such that

$$\frac{da_0}{a_0} = \frac{db_0}{b_0} = \frac{dc_0}{c_0} = \frac{da}{a} = \cdots = \frac{d\theta}{\theta}.$$

Now by Article 229 the x-component of the attraction of the elementary shell on m is

$$\begin{aligned} &= 4\pi m \rho \alpha abc \frac{\theta^3 p_0^{\,2}\, da_0}{a_0^{\,4} b_0 c_0} \\ &= 4\pi m \rho \alpha abc \frac{p_0^{\,2} \theta^2\, d\theta}{a_0^{\,3} b_0 c_0}, \qquad (183) \end{aligned}$$

where a_0, b_0, c_0 are the principal semi-axes of the ellipsoid (182), and p_0 is the perpendicular from the centre on its tangent plane at (α, β, γ); and consequently the attraction of the whole ellipsoid is the θ-integral of (183) as θ varies from 0 to 1. So that if X is the x-component of the attraction,

$$\mathrm{X} = 4\pi m\alpha abc \int_0^1 \frac{\rho p_0^2 \theta^2 d\theta}{a_0^2 b_0 c_0}; \tag{184}$$

and $a_0^2 = (a^2+\omega)\theta^2$, $b_0^2 = (b^2+\omega)\theta^2$, $c_0^2 = (c^2+\omega)\theta^2$; and

$$\frac{\theta^4}{p_0^2} = \frac{\alpha^2}{(a^2+\omega)^2} + \frac{\beta^2}{(b^2+\omega)^2} + \frac{\gamma^2}{(c^2+\omega)^2}. \tag{185}$$

The definite integral will take a simpler form if we make ω the variable; for differentiating (182) we have

$$-\frac{\theta^4}{p_0^2} d\omega = 2\theta d\theta,$$

and corresponding to $\theta = 0$, and $\theta = 1$, $\omega = \infty$ and $\omega = \Omega$, where Ω is the positive root of

$$\frac{\alpha^2}{a^2+\omega} + \frac{\beta^2}{b^2+\omega} + \frac{\gamma^2}{c^2+\omega} = 1. \tag{186}$$

And substituting these we have

$$\mathrm{X} = 2\pi m\alpha abc \int_{\Omega}^{\infty} \frac{\rho d\omega}{(a^2+\omega)\{(a^2+\omega)(b^2+\omega)(c^2+\omega)\}^{\frac{1}{2}}}; \tag{187}$$

with similar expressions for Y and Z.

If ρ is constant, that is, if the ellipsoid is homogeneous, these expressions are the same as those already determined in Art. 225.

If the ellipsoid is heterogeneous, and the variation of the density is such that when the ellipsoid is resolved into concentric, similar, and similarly placed shells, ρ is a function of θ, that is, of the ratio of any one of its central radii to the coincident radius of the bounding ellipsoid, then the element-function in (184) is a function of θ only, and the problem requires the evaluation of a single definite integral.

If ρ varies inversely as θ, that is, if the density of each shell varies inversely as the length of its principal semi-axes, which is a hypothesis made by some investigators of the figure of the earth, the definite integral can be expressed in finite terms.

SECTION 3.—*General theorems in attractions.*

232.] Our investigations in attractions have thus far been limited to the action of matter contained within and filling closed surfaces of very special forms; we proceed now to certain general theorems which are applicable to matter distributed in a much more general manner.

The equipotential surface is the locus-surface of all those points (α, β, γ) at which the potential of a given mass has a given value. Thus it is the surface whose equation is

$$\mathrm{V} = f(\alpha, \beta, \gamma) = \mathrm{C}, \tag{188}$$

V being otherwise defined as in (104) or (108).

Since $\left(\frac{d\mathrm{V}}{d\alpha}\right), \left(\frac{d\mathrm{V}}{d\beta}\right), \left(\frac{d\mathrm{V}}{d\gamma}\right)$ are proportional to the direction-cosines of the normal of (188) at the point (α, β, γ), and since these are also severally proportional to the axial-components of the attraction on m, it follows that the action-line of that attraction is normal to the equipotential surfaces; and as a similar result is true for all points on the surface, the surface cuts orthogonally the action-lines of the attraction on all particles at its surface; and consequently if the surface were a smooth shell capable of resisting pressure in its normal direction only, the attracted particle would be at rest at every point on the surface. For this reason the equipotential surface is called *a surface of equilibrium*, or, as the French mathematicians term it, *surface de niveau* (a level surface.) In reference to the mathematical theory of heat, it is called *an isothermal surface*.

As C in the right-hand member of (188) is arbitrary, so as it varies, a series of equipotential surfaces is formed; and these all cut at right angles the lines of action of the resultant attraction. We thus obtain a system of lines which are cut orthogonally by the system of equipotential surfaces, and the tangent at every point of each of these lines is the action-line of the resultant attraction at that point. For this reason these lines are called *lines of force*. We have already had instances of them and of equipotential surfaces. In Art. 187, all ellipses of which A and B are the foci are equipotential lines or lines of equilibrium, and confocal and concentric hyperbolae which intersect these ellipses at right angles are the lines of force in the plane of the paper; and in space, prolate spheroids, of which A and B are the foci, are the surfaces of equilibrium, the lines of force being a series of confocal hyperbolae. Also from Art. 228 the equilibrium surfaces of an ellipsoidal shell are concentric and confocal ellipsoids, and the lines of force are the intersections of the concentric and confocal hyperboloids of one and two sheets respectively which pass through the place of an attracted particle.

The corresponding points on a series of concentric and confocal ellipsoids are all on the same line of force.

If X, Y, Z are the axial-components of the attraction, and $R^2 = X^2+Y^2+Z^2$, the direction-cosines of the normal of the equipotential surface are severally $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$.

If two equipotential surfaces with reference to the same attracting mass have a common point, they are coincident in all their points; for $V = c$ and $V = c'$ cannot be satisfied by the same values of (α, β, γ), unless $c = c'$, in which case they are identical; and thus they coincide in all their points.

An equipotential surface, $V = c$, is a closed surface; for it is evidently continuous; and it cannot go off to infinity; for in that case $V = 0$, and this result is inconsistent with $V = c$.

Of two equipotential surfaces the interior is that to which the greater attraction corresponds.

For two successive equipotential surfaces the force of attraction on any point varies inversely as the distance between the surfaces. This is evident from the theorem given in (111) Art. 219; for if dn = the distance between their surfaces, and R is the resultant attraction,

$$R = -m\frac{dV}{dn}; \tag{189}$$

that is, R varies inversely as dn; and the magnitude of the attraction is given by (189).

233.] The potential is subject to the following theorem, which is largely applied in subsequent physical investigations. Since

$$V = \iiint \frac{\rho\, dx\, dy\, dz}{r}, \tag{190}$$

and

$$r^2 = (\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2; \tag{191}$$

$$\therefore \left(\frac{dV}{d\alpha}\right) = -\iiint \frac{\rho\,(\alpha-x)\, dx\, dy\, dz}{r^3};$$

$$\left.\begin{aligned} \left(\frac{d^2V}{d\alpha^2}\right) &= \iiint \left\{\frac{3\,(\alpha-x)^2}{r^5} - \frac{1}{r^3}\right\} \rho\, dx\, dy\, dz; \\ \text{similarly}\quad \left(\frac{d^2V}{d\beta^2}\right) &= \iiint \left\{\frac{3\,(\beta-y)^2}{r^5} - \frac{1}{r^3}\right\} \rho\, dx\, dy\, dz; \\ \left(\frac{d^2V}{d\gamma^2}\right) &= \iiint \left\{\frac{3\,(\gamma-z)^2}{r^5} - \frac{1}{r^3}\right\} \rho\, dx\, dy\, dz; \end{aligned}\right\} \tag{192}$$

therefore by addition

$$\left(\frac{d^2 \text{v}}{d\alpha^2}\right)+\left(\frac{d^2 \text{v}}{d\beta^2}\right)+\left(\frac{d^2 \text{v}}{d\gamma^2}\right)=0. \tag{193}$$

This theorem was discovered by Laplace. It does not however hold true when the place of the attracted particle lies within the space occupied by the attracting matter, this matter being continuous; that is, for the attraction of a body acting on one of its own particles; because for particles immediately contiguous to the attracted particle r is infinite, so that v takes *the form* of infinity, and its first partial differentials take indeterminate forms; thus the preceding process is apparently incorrect, and we must consider the subject with greater exactness. Let us consider it first in reference to the case of a homogeneous sphere; let the centre of the sphere be the origin; and let $\alpha^2+\beta^2+\gamma^2=\sigma^2$. Then if v and v′ are the potentials of the sphere according as m is internal or external to it, we have from Art. 221,

$$\text{v}=2\pi\rho a^2-\frac{2\pi\rho\sigma^2}{3}; \qquad \text{v}'=\frac{4\pi\rho a^3}{3\sigma};$$

$$\left(\frac{d\text{v}}{d\alpha}\right)=-\frac{4\pi\rho\alpha}{3}; \quad \left(\frac{d\text{v}}{d\beta}\right)=-\frac{4\pi\rho\beta}{3}; \quad \left(\frac{d\text{v}}{d\gamma}\right)=-\frac{4\pi\rho\gamma}{3};$$

$$\left(\frac{d\text{v}'}{d\alpha}\right)=-\frac{4\pi\rho a^3\alpha}{3\sigma^3}; \quad \left(\frac{d\text{v}'}{d\beta}\right)=-\frac{4\pi\rho a^3\beta}{3\sigma^3}; \quad \left(\frac{d\text{v}'}{d\gamma}\right)=-\frac{4\pi\rho a^3\gamma}{3\sigma^3};$$

$$\left(\frac{d^2\text{v}}{d\alpha^2}\right)=\left(\frac{d^2\text{v}}{d\beta^2}\right)=\left(\frac{d^2\text{v}}{d\gamma^2}\right)=-\frac{4\pi\rho}{3};$$

$$\left(\frac{d^2\text{v}'}{d\alpha^2}\right)=\frac{4\pi\rho a^3(3\alpha^2-\sigma^2)}{3\sigma^5}; \quad \left(\frac{d^2\text{v}'}{d\beta^2}\right)=\frac{4\pi\rho a^3(3\beta^2-\sigma^2)}{3\sigma^5};$$

$$\left(\frac{d^2\text{v}'}{d\gamma^2}\right)=\frac{4\pi\rho a^3(3\gamma^2-\sigma^2)}{3\sigma^5}.$$

If the particle m is on the surface of the sphere, $\sigma=a$; and $\text{v}=\text{v}'$; also the first partial differential coefficients of v and v′ become identical, so that all these quantities are continuous, although the law of variation changes abruptly at the surface. But the second partial differential coefficients of v and v′ are not the same at the surface of the sphere; thus,

$$\left(\frac{d^2\text{v}}{d\alpha^2}\right)=-\frac{4\pi\rho}{3}; \quad \left(\frac{d^2\text{v}'}{d\alpha^2}\right)=-\frac{4\pi\rho}{3}+\frac{4\pi\rho\alpha^2}{a^2};$$

with similar values for the other second partial differential coefficients. Thus these vary discontinuously; and the bounding surface of the sphere is the locus of the points of discontinuity. Hence we have

$$\left(\frac{d^2 v}{d\alpha^2}\right) + \left(\frac{d^2 v}{d\beta^2}\right) + \left(\frac{d^2 v}{d\gamma^2}\right) = -4\pi\rho;$$

$$\left(\frac{d^2 v'}{d\alpha^2}\right) + \left(\frac{d^2 v'}{d\beta^2}\right) + \left(\frac{d^2 v'}{d\gamma^2}\right) = 0.$$

We need not be surprised at these cases of discontinuity; they are of frequent occurrence in the application of analysis to physical enquiries, and evidently arise from the discontinuous distribution of matter; we shall demonstrate hereafter, see Art. 237, the integral equivalent of (193) by another process; and the cause of the discontinuity will at once become evident; and so also will the meaning of the right-hand members of the two preceding expressions.

This illustration also enables us to determine the value of the left-hand member of (193) when m is a particle of the attracting body. For let us suppose a small sphere inclosing the attracted particle to be taken out of the attracting mass, and the radius of it to be so small, that the density within the sphere may be considered constant: let v_0 be the potential of this small sphere, and v' the potential of the whole excess of the attracting mass over the sphere: then by reason of (193),

$$\left(\frac{d^2 v'}{d\alpha^2}\right) + \left(\frac{d^2 v'}{d\beta^2}\right) + \left(\frac{d^2 v'}{d\gamma^2}\right) = 0.$$

And for the small sphere, as just now proved,

$$\left(\frac{d^2 v_0}{d\alpha^2}\right) + \left(\frac{d^2 v_0}{d\beta^2}\right) + \left(\frac{d^2 v_0}{d\gamma^2}\right) = -4\pi\rho.$$

And thus if the attracted particle is a part of the attracting mass, since $v = v_0 + v'$,

$$\left(\frac{d^2 v}{d\alpha^2}\right) + \left(\frac{d^2 v}{d\beta^2}\right) + \left(\frac{d^2 v}{d\gamma^2}\right) = -4\pi\rho. \qquad (194)$$

This correction of Laplace's theorem was made by M. Poisson.

(193) and (194) are evidently invariants, whatever is the system of rectangular axes to which the bodies are referred. This theorem can be proved easily also from the formulae for transformation of axes.

234.] The following is another proof of this theorem.*

Let the place of the attracted particle be the origin, and let the system of reference be that of polar coordinates in space; so that

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* See Cours de Mécanique, par M. Sturm; No. 127. Paris, 1861.

$$\mathrm{V} = \iiint \rho r \sin\theta\, dr\, d\theta\, d\phi;$$

and if f, g, h are the direction-angles of r,

$$-\frac{\mathrm{X}}{m} = \left(\frac{d\mathrm{V}}{d\alpha}\right) = \iiint \rho \sin\theta \cos f\, dr\, d\theta\, d\phi;$$

$$-\frac{\mathrm{Y}}{m} = \left(\frac{d\mathrm{V}}{d\beta}\right) = \iiint \rho \sin\theta \cos g\, dr\, d\theta\, d\phi;$$

$$-\frac{\mathrm{Z}}{m} = \left(\frac{d\mathrm{V}}{d\gamma}\right) = \iiint \rho \sin\theta \cos h\, dr\, d\theta\, d\phi;$$

the limits of integration being in all these integrals such as to include the entire mass of the attracting body.

When α, β, and γ vary, the variation of them causes displacement of the origin; but the only quantities which are dependent on this displacement are ρ and the limits of the r-integration. Consequently

$$\left(\frac{d^2\mathrm{V}}{d\alpha^2}\right) + \left(\frac{d^2\mathrm{V}}{d\beta^2}\right) + \left(\frac{d^2\mathrm{V}}{d\gamma^2}\right)$$

$$= \iiint \left\{\left(\frac{d\rho}{d\alpha}\right)\cos f + \left(\frac{d\rho}{d\beta}\right)\cos g + \left(\frac{d\rho}{d\gamma}\right)\cos h\right\} \sin\theta\, dr\, d\theta\, d\phi$$

$$+ \iint \left\{\left(\frac{dr_1}{d\alpha}\right)\cos f + \left(\frac{dr_1}{d\beta}\right)\cos g + \left(\frac{dr_1}{d\gamma}\right)\cos h\right\} \rho_1 \sin\theta\, d\theta\, d\phi,$$

where r_1 is the superior limit of r, and ρ_1 is the corresponding density.

But
$$\left(\frac{d\rho}{d\alpha}\right)\cos f + \left(\frac{d\rho}{d\beta}\right)\cos g + \left(\frac{d\rho}{d\gamma}\right)\cos h = \frac{d\rho}{dr},$$

$$\left(\frac{dr_1}{d\alpha}\right)\cos f + \left(\frac{dr_1}{d\beta}\right)\cos g + \left(\frac{dr_1}{d\gamma}\right)\cos h = -1;$$

$$\therefore \quad \left(\frac{d^2\mathrm{V}}{d\alpha^2}\right) + \left(\frac{d^2\mathrm{V}}{d\beta^2}\right) + \left(\frac{d^2\mathrm{V}}{d\gamma^2}\right)$$

$$= \iiint \frac{d\rho}{dr} \sin\theta\, dr\, d\theta\, d\phi - \iint \rho_1 \sin\theta\, d\theta\, d\phi.$$

Also

$$\iiint \frac{d\rho}{dr} \sin\theta\, dr\, d\theta\, d\phi = \iint (\rho_1 - \rho_0) \sin\theta\, d\theta\, d\phi$$

$$= \iint \rho_1 \sin\theta\, d\theta\, d\phi - \rho_0 \iint \sin\theta\, d\theta\, d\phi$$

$$= \iint \rho_1 \sin\theta\, d\theta\, d\phi - 4\pi\rho_0$$

when ρ_0 is the density at the origin. Consequently

$$\left(\frac{d^2 V}{d\alpha^2}\right)+\left(\frac{d^2 V}{d\beta^2}\right)+\left(\frac{d^2 V}{d\gamma^2}\right) = -4\pi\rho_0.$$

235.] Now the equivalents of the expressions (193) and (194) are often more convenient when expressed in terms of polar coordinates. Let the positions of the attracting and the attracted particles respectively be (r', θ', ϕ'), (r, θ, ϕ): then

$$V = \iiint \frac{\rho r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi'}{\{r^2 - 2rr'(\sin\theta \sin\theta' \cos(\phi-\phi') + \cos\theta\cos\theta') + r'^2\}^{\frac{1}{2}}}; \quad (195)$$

and transforming (193) and (194) into their equivalents in terms of partial differential coefficients of V with respect to r, θ, and ϕ, as in Ex. 2, Art. 108, Vol. I (Differential Calculus), we have

$$\left(\frac{d^2 V}{dr^2}\right)+\frac{2}{r}\left(\frac{dV}{dr}\right)+\frac{1}{r^2}\left(\frac{d^2 V}{d\theta^2}\right)+\frac{\cos\theta}{r^2\sin\theta}\left(\frac{dV}{d\theta}\right)+\frac{1}{r^2(\sin\theta)^2}\left(\frac{d^2 V}{d\phi^2}\right)$$

$$= 0, \quad \text{or} \quad = -4\pi\rho, \quad (196)$$

according as the attracted particle is not or is part of the attracting mass.

The left-hand member of (196) may be put into another form which is in many cases more convenient. Let $\cos\theta = \mu$; then, as in the last part of Ex. 2, Art. 108, Vol. I (Differential Calculus), the expression becomes

$$\frac{1}{r}\left(\frac{d^2 . rV}{dr^2}\right)+\frac{1}{1-\mu^2}\left(\frac{d^2 V}{d\phi^2}\right)+\frac{d}{d\mu}\left\{(1-\mu^2)\left(\frac{dV}{d\mu}\right)\right\} = 0,$$

$$\text{or} \quad = -4\pi\rho, \quad (197)$$

according as the attracted particle is not or is part of the attracting mass.

236.] Before I present these theorems in another form, and from another point of view, I will shew their use in determining the potentials of certain given masses. Hereby we shall verify results already obtained, and exhibit them in reference to a different process.

Ex. 1. To determine the potential of a shell composed of a series of concentric spherical shells, each of which is homogeneous, and the density is a function of the radius of the shell.

Let the origin be at the common centre of all the shells; let r' be the radius of any shell, and $\rho = f(r')$, the density of that shell; r_1 and r_0 being the radii of the external and internal bounding spheres. Let r be the distance from the origin of the attracted particle; then V is evidently a function of r only, and is independent of θ and ϕ.

(1) Let m be external to the shell; then (196) becomes

$$\frac{d^2\mathrm{V}}{dr^2}+\frac{2}{r}\frac{d\mathrm{V}}{dr}=0;$$

$$\therefore \quad r^2\frac{d\mathrm{V}}{dr}=-c; \tag{198}$$

where c is an arbitrary constant; but c evidently is equal to the mass of the shells, because the attraction of a system of concentric shells on an external particle varies as the mass of the shells. Hence

$$\frac{d\mathrm{V}}{dr}=-\frac{\mathrm{M}}{r^2};$$

$$\therefore \quad \mathrm{V}=\frac{\mathrm{M}}{r}, \tag{199}$$

no constant having been introduced, because $\mathrm{V}=0$ when $r=\infty$.

If the sphere is not full, but there is a hollow cavity within the interior shell, and m is in that cavity, then in (198) $c=0$, because the resultant attraction vanishes for all positions of m within that cavity; consequently for all such positions V is a constant.

(2) Let m be within the shell occupied by the attracting mass; then (196) becomes

$$\frac{d^2\mathrm{V}}{dr^2}+\frac{2}{r}\frac{d\mathrm{V}}{dr}+4\pi\rho=0; \tag{200}$$

whence we have

$$r^2\frac{d\mathrm{V}}{dr}+4\pi\int_{r_0}^{r}r^2\rho\,dr=0; \tag{201}$$

the lower limit of integration having been brought in, because $\frac{d\mathrm{V}}{dr}=0$, when $r=r_0$. Thus

$$\frac{d\mathrm{V}}{dr}=-\frac{4\pi}{r^2}\int_{r_0}^{r}r^2\rho\,dr; \tag{202}$$

but $4\pi\int_{r_0}^{r}r^2\rho\,dr=$ the mass of the shell, the external and internal radii of which are respectively r and r_0; consequently

$$\frac{d\mathrm{V}}{dr}=-\frac{\mathrm{M}}{r^2}; \tag{203}$$

and thus the attraction of the shell varies as the mass contained within the concentric shell bounded by the internal bounding surface and the concentric spherical surface which passes through the attracted particle, and inversely as the square of the radius of the latter shell. Consequently the matter lying outside that surface produces no attractive effect on the particle.

Again, from (202),

$$\frac{dV}{dr} = -\frac{4\pi}{r^2}\int_{r_0}^{r} r^2 \rho\, dr$$

$$= -4\pi\rho r + 4\pi\rho r - \frac{4\pi}{r^2}\int_{r_0}^{r} r^2\rho\, dr;$$

$$\therefore \quad V = 4\pi\int_{r}^{r_1} \rho r\, dr + \frac{4\pi}{r}\int_{r_0}^{r} r^2\rho\, dr, \tag{204}$$

the lower limits having been introduced in accordance with the remark made at the end of Art. 221.

Ex. 2. To find the potential of a system of concentric cylindrical shells of infinite length, and coaxal, of each of which the density is uniform.

If the origin is taken at a definite point on the axis, and the axis of the cylinder is taken to be the z-axis, it is evident that $\frac{dV}{dz} = 0$, and that consequently the resultant attraction is perpendicular to the z-axis. Hence also $\frac{d^2V}{dz^2} = 0$; whence, as V is evidently a function of r only, we have

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0, \text{ or } = -4\pi\rho, \tag{205}$$

according as the attracted particle is without or within the space occupied by the attracting matter.

If m is outside the cylinder, by integration of (205) we have

$$r\frac{dV}{dr} = c, \tag{206}$$

where c is an arbitrary constant. If the cylinder is hollow, for all places of m within that cavity the attraction vanishes, and consequently $c = 0$; for all points external to the cylinder, the attraction varies inversely as the distance from the axis of the cylinder, and

$$V = c\log\frac{r}{r_0}, \tag{207}$$

where r_0 is the radius of the internal surface.

If m is in the space occupied by the attracting matter,

$$r\frac{dV}{dr} = -4\pi\int_{r_0}^{r}\rho r\, dr, \tag{208}$$

where r_0 is the radius of the interior surface, and $\frac{dV}{dr} = 0$ when $r = r_0$.

Ex. 3. The potential of matter arranged in parallel plates, each of which is of uniform density and of infinite extent, may thus be found:

Let the x-axis be perpendicular to the plates; so that the resultant attraction must be parallel to it, the attractions parallel to the plates vanishing; consequently $\left(\frac{d\text{v}}{d\beta}\right) = \left(\frac{d\text{v}}{d\gamma}\right) = 0$; and therefore

$$\left(\frac{d^2\text{v}}{da^2}\right) = 0, \text{ or } = -4\pi\rho, \tag{209}$$

according as the attracted particle m is without or within the space occupied by the attracting matter.

For positions of m outside the plates

$$\frac{d\text{v}}{da} = c, \tag{210}$$

where c is an arbitrary constant; and consequently the attraction of the plates is the same, whatever is the distance of m from the surface of the plates.

If the particle is within the plate,

$$\frac{d^2\text{v}}{da^2} = -4\pi\rho;$$

$$\therefore \quad \frac{d\text{v}}{da} = -4\pi\int\rho\, da;$$

$$= -4\pi\rho a, \tag{211}$$

if the density of the plates is constant, and the origin is placed midway between the bounding surfaces of the plates, so that the attraction vanishes when $a = 0$. Thus if the whole thickness of the system of plates is t, when $a = \frac{t}{2}$, $\frac{d\text{v}}{da} = -2\pi\rho t$; and when $a = -\frac{t}{2}$, $\frac{d\text{v}}{da} = 2\pi\rho t$; and the whole attraction of the plates on a particle at the surface $= 2\pi\rho m t$.

237.] The theorems concerning the partial-differentials of the potential which have been demonstrated in Arts. 233 and 234, and have been applied in Art. 236, admit of statement in an integral form: and in that form have been demonstrated by Gauss, Sir William Thomson, and Professor Stokes. The following is the most elementary proof:

Let s be any closed surface, and let o be a point either within it or without it, at which is a particle m' of the attracting matter. Then if ds is a surface-element of s, and dn is an element of the normal to the surface drawn outwards at the element ds, so that

$\frac{dV}{dn}$ is the force which the attracting matter exerts at ds in a line normal to s,

$$\iint \frac{dV}{dn} ds = -4\pi M_0, \text{ or } = 0, \tag{212}$$

where M_0 is the whole attracting matter within s, according as O is within or without s.

Firstly, let O be within s. From O draw a straight line OL cutting the surface, and produce it in one direction so far that it cannot cut the surface again. OL will cut s in one point P_1 at least, and may cut it in more points if s is a re-entrant surface: but the number of points of section must be uneven. Let these points be $P_1, P_2, P_3, \dots$; and let $OP_1 = r_1$, $OP_2 = r_2, \dots$; about the line OL and including it, from O as a vertex let a cone be described, at the vertex of which is the small solid angle ω. Then if from O as a centre, spheres are described with radii $r_1, r_2, \dots$, the areas of the spherical surfaces intercepted within the cone are severally ωr_1^2, $\omega r_2^2, \dots$; let $ds_1, ds_2, \dots$ be the surface-elements of s at $P_1, P_2 \dots$ intercepted by this cone, and let $\theta_1, \theta_2, \dots$ be the angles at which dn, the normal-element, drawn outwards, at each of these points is inclined to the line OL: then

$$\omega r_1^2 = -ds_1 \cos\theta_1; \quad \omega r_2^2 = ds_2 \cos\theta_2; \quad \omega r_3^2 = -ds_3 \cos\theta_3; \dots \tag{213}$$

Let $N_1, N_2, \dots$ be the components along the normals to s at $P_1, P_2, \dots$, of the attraction of these surface-elements on m'; so that

$$N_1 = \frac{m'}{r_1^2}\cos\theta_1; \quad N_2 = \frac{m'}{r_2^2}\cos\theta_2; \dots\dots\dots \tag{214}$$

consequently

$$N_1 ds_1 + N_2 ds_2 + \dots = -m'\omega + m'\omega - m'\omega + \dots, \tag{215}$$

the number of terms of which is uneven: and

$$\therefore \quad \Sigma . N ds = -m'\omega.$$

Let this process be repeated for all angular space about O; this is effected by the integration of the left-hand member through the whole of s, and by the corresponding integral of ω, which is 4π; and thus for m',

$$\iint N ds = -4\pi m'.$$

But a similar result will be true for every particle of the attracting matter M_0 which is contained within s; in which case

N must be replaced by its value $\frac{dV}{dn}$ given in (189), where V is the potential of M_0; so that

$$\iint \frac{dV}{dn} dS = -4\pi M_0. \tag{216}$$

Secondly, let O be without S. From O draw a straight line OL as in the former case, cutting the surface in P_1, P_2, ...; the number of these points of intersection is at least two, and may be greater; but the number is necessarily even. The same construction and the same symbols being used as in the former case,

$$\omega r_1^2 = dS_1 \cos\theta_1; \quad \omega r_2^2 = -dS_2 \cos\theta_2; \quad \omega r_3^2 = dS_3 \cos\theta_3; \ldots \tag{217}$$

$$N_1 = \frac{m'}{r_1^2} \cos\theta_1; \quad N_2 = \frac{m'}{r_2^2} \cos\theta_2; \quad N_3 = \frac{m'}{r_3^2} \cos\theta_3; \ldots \tag{218}$$

$$\begin{aligned} \therefore \quad N_1 dS_1 + N_2 dS_2 + \ldots &= m'\omega - m'\omega + m'\omega - m'\omega + \ldots \\ &= 0, \end{aligned}$$

as the number of terms in the series is even;

$$\therefore \quad \Sigma . N dS = 0.$$

Let the process be repeated for all that angular space about O, which is necessary to include the whole surface of S; then for all that surface, and for m', we have

$$\iint N dS = 0.$$

Also a similar result will be true for every other particle of the attracting mass which is outside S; and if V is the potential of all that matter, then replacing N by its equivalent,

$$\iint \frac{dV}{dn} dS = 0. \tag{219}$$

Hence we have the theorem:

If V is the potential of any mass M, part of which, viz. M_0, is within, and the rest is without a closed surface S, of which dn is an element of the normal drawn outwards, then, the whole surface of S being the range of integration,

$$\iint \frac{dV}{dn} dS = -4\pi M_0. \tag{220}$$

Or, in other words, If we consider the attraction of a mass of matter on the surface-elements of a closed surface, the sum of the attractions estimated along the normal to the surface at its several points drawn outwards is equal to $-4\pi M_0$, where M_0 is the attracting matter within the surface. Equation (181) in Art. 230 contains a particular case of this last theorem.

The proof here given of course includes the case in which the closed surface is an equipotential surface, and in which the resultant attraction on the surface-element acts along the normal.

238.] Laplace's equation (193) and Poisson's extension of it (194) may be deduced from (220) by the following process, due to Professor Stokes*:

Let us take the more general case of the attracting particle being within the surface s, and let us take this surface to be the surface of the small elementary parallelepipedon $d\alpha\, d\beta\, d\gamma$. Then for the face $dy\, dz$ the value of $\iint \frac{d\text{v}}{dn}\, ds$ is $-\left(\frac{d\text{v}}{d\alpha}\right) d\beta\, d\gamma$; and for the opposite face it is $\left\{\left(\frac{d\text{v}}{d\alpha}\right) + \left(\frac{d^2\text{v}}{d\alpha^2}\right) d\alpha\right\} d\beta\, d\gamma$; and therefore for this pair, the value of the integral is $\left(\frac{d^2\text{v}}{d\alpha^2}\right) d\alpha\, d\beta\, d\gamma$; similar results are true for the other two pairs of faces; so that the left-hand member of (220) becomes

$$\left\{\left(\frac{d^2\text{v}}{d\alpha^2}\right) + \left(\frac{d^2\text{v}}{d\beta^2}\right) + \left(\frac{d^2\text{v}}{d\gamma^2}\right)\right\} d\alpha\, d\beta\, d\gamma. \tag{221}$$

Now the density in the elementary parallelepipedon being constant, and being ρ, say, $\text{M}_0 = \rho\, d\alpha\, d\beta\, d\gamma$; therefore from (220)

$$\left(\frac{d^2\text{v}}{d\alpha^2}\right) + \left(\frac{d^2\text{v}}{d\beta^2}\right) + \left(\frac{d^2\text{v}}{d\gamma^2}\right) = -4\pi\rho. \tag{222}$$

If the particle is outside the parallelepipedon, the right-hand member vanishes.

Also (220) may be deduced from (222) by integration in the following way:

Let (222) be multiplied by $d\alpha\, d\beta\, d\gamma$, and be the subject of integration through a given space within a closed surface s, which contains the attracting matter M_0; then

$$\iiint\left\{\left(\frac{d^2\text{v}}{d\alpha^2}\right) + \left(\frac{d^2\text{v}}{d\beta^2}\right) + \left(\frac{d^2\text{v}}{d\gamma^2}\right)\right\} d\alpha\, d\beta\, d\gamma = -4\pi \iiint \rho\, d\alpha\, d\beta\, d\gamma. \tag{223}$$

Now $\iiint \left(\frac{d^2\text{v}}{d\alpha^2}\right) d\alpha\, d\beta\, d\gamma = \iint \left[\left(\frac{d\text{v}}{d\alpha}\right)\right]_{\alpha_0}^{\alpha_1} d\beta\, d\gamma$, where α_1 and α_0 are the abscissae to the points where a line parallel to the axis of α, at distances β and γ from the planes of (γ, α) and (α, β) respectively, cuts the surface; and if ds_1 is the surface-element at (α, β, γ) and λ is the angle between the normal at that point drawn outwards and the α-axis, $d\beta\, d\gamma = ds_1 \cos\lambda_1 = -ds_0 \cos\lambda_0$; and if a cylinder is described with the generating lines parallel to the axis of x, and circumscribing s, the line of contact divides s into two parts; and the range of

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* Cambridge and Dublin Mathematical Journal, Vol. IV, p. 215. In the same memoir the reader will find references to the works of Gauss and Sir William Thomson.

$$\iint \left(\frac{d\text{v}}{d\alpha_1}\right) \cos\lambda_1\, d\text{s}_1$$

is the part of the surface which is farthest from the origin; and the range of $\iint \left(\frac{d\text{v}}{d\alpha_0}\right) \cos\lambda_0\, d\text{s}_0$ is the part nearest to the origin. Hence if the range is the whole surface of s,

$$\iiint \left(\frac{d^2\text{v}}{d\alpha^2}\right) d\alpha\, d\beta\, d\gamma = \iint \left(\frac{d\text{v}}{d\alpha}\right) \cos\lambda\, d\text{s}; \qquad (224)$$

and the other two parts of (223) will take similar forms; also $\iiint \rho\, dx\, dy\, dz = \text{M}_0$; therefore (223) becomes

$$\iint \left\{ \left(\frac{d\text{v}}{d\alpha}\right) \cos\lambda + \left(\frac{d\text{v}}{d\beta}\right) \cos\mu + \left(\frac{d\text{v}}{d\gamma}\right) \cos\nu \right\} d\text{s} = -4\pi \text{M}_0; \qquad (225)$$

but since λ, μ, ν are the direction-angles of dn, which is an element of the normal of s drawn outwards, by (111), Art. 219,

$$\cos\lambda \left(\frac{d\text{v}}{d\alpha}\right) + \cos\mu \left(\frac{d\text{v}}{d\beta}\right) + \cos\nu \left(\frac{d\text{v}}{d\gamma}\right) = \frac{d\text{v}}{dn}; \qquad (226)$$

$$\therefore \quad \iint \frac{d\text{v}}{dn}\, d\text{s} = -4\pi \text{M}_0; \qquad (227)$$

and this is the integral equation of the normal attraction through a closed surface.

239.] The potential does not admit of a maximum or minimum value at any point in free space, where there is no attracting matter. For in this problem v is a function of α, β, γ; and is subject to the condition given in (193); and thus in (39), Art. 163, Vol. I (Differential Calculus), the coefficient of θ^2 vanishes, and consequently the three roots of that cubic equation cannot be all of the same sign; but this condition is necessary when v has a critical value, and accordingly v does not admit of a maximum or minimum value.

This fact may also be inferred from the statements of Art. 237. For did the potential admit of a maximum or minimum value at a point in free space, a closed surface could be described about that point, and so near to it, that at every point within it the potential would be less or greater than that at the point; so that $\frac{d\text{v}}{dn}$ would be negative or positive all through the surface, and consequently $\iint \frac{d\text{v}}{dn}\, d\text{s}$ would be finite; and this is impossible, as such a surface contains none of the attracting matter.

Hence if the potential is constant at all points on the closed surface, S, which includes none of the attracting matter, it has also the same value for all points within S. Because if this is not so, there must be one or more critical values within it, and this has been shewn to be impossible.

As V does not admit of a maximum or minimum value in free space it increases in some directions and decreases in others, remaining constant for all points on an equipotential surface; consequently a material particle under the action of attracting bodies cannot be in a position of stable equilibrium. This theorem is due to Mr. Earnshaw, and is given by him in the "Philosophical Transactions of Cambridge," Vol. VII, March, 1839.

240.] The theorem given in Art. 237 leads to the following very remarkable results in reference to equipotential surfaces:

Let an infinitesimal area be taken on an equipotential surface, $V = c$; and let the normals to the surface be drawn all round the contour of the infinitesimal area; these evidently form a tubular surface of small section. Also let another equipotential surface, $V = c'$, be drawn intersecting the tubular surface: let ω and ω' be the areas of the sections of the tubular surface made by these equipotential surfaces. Let R and R' be the forces of attraction on ω and ω' respectively, the lines of action of them being normal to the areas of ω and ω'; then as there is no force of attraction perpendicular to the sides of the tube, as the generating lines are lines of force, and as no attracting matter is supposed to be within the tube, by (219) we have

$$R\omega - R'\omega' = 0; \qquad (228)$$

that is, the attracting force is inversely proportional to the section of the tube. This theorem gives the variation of the attraction along a line of force so long as it does not pass through attracting matter.

If the tube contains attracting matter and the volume between ω and ω' is full of matter, then the difference of the whole attractive forces on the two ends of the tube varies as the quantity of matter contained in the tube between these two areas.

The following are particular cases of this theorem*:

When the attracting body is symmetrical about a point, the lines of force are obviously straight lines drawn from this point. So that in this case the tube becomes a cone, and ω is proportional to the square of the distance from the vertex; consequently the attraction varies inversely as the square of the distance.

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* See Thomson and Tait, Natural Philosophy, Vol. I, p. 365. Oxford, 1867.

If the attracting matter is distributed symmetrically about an axis in cylindrical shells of infinite length, the lines of force are perpendicular to this axis, and the tube becomes a wedge, the section of which is proportional to the distance from the axis; and the attraction therefore varies inversely as the distance from the axis.

If the attracting matter is composed of a system of parallel plates, the lines of force are all parallel, and the tube becomes a cylinder, the area of the section of which is the same at all distances, and consequently the attraction is the same at all distances.

241.] The following theorem in the Integral Calculus, discovered by George Green, and contained in his "Essay on the Theories of Electricity and Magnetism," is more general than the preceding; and is fundamental in many subsequent investigations.

If s is a closed surface containing a given quantity of matter, and ds is a surface-element of it, and dn is the element of the normal drawn outwards; and if U and V are two functions of α, β, γ which do not become infinite at any point within s, then

$$\iiint \mathrm{U}\left\{\left(\frac{d^2\mathrm{V}}{d\alpha^2}\right)+\left(\frac{d^2\mathrm{V}}{d\beta^2}\right)+\left(\frac{d^2\mathrm{V}}{d\gamma^2}\right)\right\} d\alpha\, d\beta\, d\gamma-\iint \mathrm{U}\,\frac{d\mathrm{V}}{dn}\,d\mathrm{s}$$

$$=\iiint \mathrm{V}\left\{\left(\frac{d^2\mathrm{U}}{d\alpha^2}\right)+\left(\frac{d^2\mathrm{U}}{d\beta^2}\right)+\left(\frac{d^2\mathrm{U}}{d\gamma^2}\right)\right\} d\alpha\, d\beta\, d\gamma-\int \mathrm{V}\,\frac{d\mathrm{U}}{dn}\,d\mathrm{s}, \quad (229)$$

in which the triple integrals comprise all the matter contained within the surface s, and the range of the double integral is the whole closed surface s.

Let us take the integrals in the left-hand member of the equality; then

$$\iiint \mathrm{U}\left(\frac{d^2\mathrm{V}}{d\alpha^2}\right) d\alpha\, d\beta\, d\gamma$$

$$=\iint \Big[\mathrm{U}\left(\frac{d\mathrm{V}}{d\alpha}\right) d\beta\, d\gamma\Big]_0^1-\iiint \left(\frac{d\mathrm{U}}{d\alpha}\right)\left(\frac{d\mathrm{V}}{d\alpha}\right) d\alpha\, d\beta\, d\gamma$$

$$=\iint \mathrm{U}\left(\frac{d\mathrm{V}}{d\alpha}\right)\cos\lambda\, d\mathrm{s}-\iiint \left(\frac{d\mathrm{U}}{d\alpha}\right)\left(\frac{d\mathrm{V}}{d\alpha}\right) d\alpha\, d\beta\, d\gamma,$$

replacing the first definite integral by its value as given in Art. 238; so that the whole left-hand member of (229) becomes

$$\iint \mathrm{U}\left\{\left(\frac{d\mathrm{V}}{d\alpha}\right)\cos\lambda+\left(\frac{d\mathrm{V}}{d\beta}\right)\cos\mu+\left(\frac{d\mathrm{V}}{d\gamma}\right)\cos\nu\right\}ds$$

$$-\iiint\left\{\left(\frac{d\mathrm{U}}{d\alpha}\right)\left(\frac{d\mathrm{V}}{d\alpha}\right)+\left(\frac{d\mathrm{U}}{d\beta}\right)\left(\frac{d\mathrm{V}}{d\beta}\right)+\left(\frac{d\mathrm{U}}{d\gamma}\right)\left(\frac{d\mathrm{V}}{d\gamma}\right)\right\}d\alpha\,d\beta\,d\gamma-\iint \mathrm{U}\,\frac{d\mathrm{V}}{dn}ds;$$

but the third of these integrals is equal to the first, as we have explained in Art. 238; so that omitting these terms, the quantity becomes

$$-\iiint\left\{\left(\frac{d\mathrm{U}}{d\alpha}\right)\left(\frac{d\mathrm{V}}{d\alpha}\right)+\left(\frac{d\mathrm{U}}{d\beta}\right)\left(\frac{d\mathrm{V}}{d\beta}\right)+\left(\frac{d\mathrm{U}}{d\gamma}\right)\left(\frac{d\mathrm{V}}{d\gamma}\right)\right\}d\alpha\,d\beta\,d\gamma. \quad (230)$$

Now as this expression is symmetrical with regard to U and V, it is likewise the value of the second member of (229): so that the theorem is hereby established, whatever is the form of the functions U and V.

If, however, one of the functions U and V becomes infinite at any value of α, β, γ within S, certain corrections must be made. Let us suppose U to be infinite at a point $\mathrm{P}_0(\alpha_0, \beta_0, \gamma_0)$ within S; and let us suppose V to become $\frac{1}{r}$ at that point. Let a sphere be described from that point as centre with an infinitesimal radius $= a$; then the preceding theorem is manifestly true for all the attracting matter external to this sphere. And with regard to the sphere, since $\mathrm{V}=\frac{1}{r}$, $\left(\frac{d^2\mathrm{V}}{d\alpha^2}\right)+\left(\frac{d^2\mathrm{V}}{d\beta^2}\right)+\left(\frac{d^2\mathrm{V}}{d\gamma^2}\right)=0$; so that $\iiint \mathrm{U}\left\{\left(\frac{d^2\mathrm{V}}{d\alpha^2}\right)+\left(\frac{d^2\mathrm{V}}{d\beta^2}\right)+\left(\frac{d^2\mathrm{V}}{d\gamma^2}\right)\right\}d\alpha\,d\beta\,d\gamma$ has the same value whether the district of integration is diminished by the volume of the sphere or not. Also

$$\iiint \mathrm{V}\left\{\left(\frac{d^2\mathrm{U}}{d\alpha^2}\right)+\left(\frac{d^2\mathrm{U}}{d\beta^2}\right)+\left(\frac{d^2\mathrm{U}}{d\gamma^2}\right)\right\}d\alpha\,d\beta\,d\gamma$$

is by the principles of definite integration equal to $\iiint d\alpha\,d\beta\,d\gamma$, which is the volume of the sphere, multiplied by some mean value of $\frac{1}{r}\left\{\left(\frac{d^2\mathrm{U}}{d\alpha^2}\right)+\left(\frac{d^2\mathrm{U}}{d\beta^2}\right)+\left(\frac{d^2\mathrm{U}}{d\gamma^2}\right)\right\}$ which is in magnitude of the same order as $\frac{1}{r}$, since $\left(\frac{d^2\mathrm{V}}{d\alpha^2}\right)+\dots$ is finite for all points within the sphere; and consequently when the radius of the sphere is infinitesimal this must be omitted. Also $\iint \mathrm{V}\frac{d\mathrm{U}}{dn}\,ds$ $=$ a mean value of $\frac{d\mathrm{U}}{dn}$ multiplied by $\iint \mathrm{V}\,ds = \frac{1}{a}\,4\pi a^2 = 4\pi a$; and this vanishes when $a = 0$. Also $\iint \mathrm{U}\frac{d\mathrm{V}}{dn}\,ds = \mathrm{U}_0\iint \frac{d\mathrm{V}}{dn}\,ds$, where U_0 is the value of U at $(\alpha_0, \beta_0, \gamma_0)$; and since $\mathrm{V} = \frac{1}{r}$, $\frac{d\mathrm{V}}{dn} = -\frac{d\mathrm{V}}{dr} = \frac{1}{r^2}$; consequently $\iint \frac{d\mathrm{V}}{dn}\,ds = \frac{1}{a^2}\,4\pi a^2 = 4\pi$; so that in the limit, for the sphere,

$$\iint \mathrm{U}\frac{d\mathrm{V}}{dn}\,ds = 4\pi\,\mathrm{U}_0.$$

Hence when the whole ranges of integration are considered,

$$\iiint \mathrm{U}\left\{\left(\frac{d^2\mathrm{V}}{d\alpha^2}\right)+\left(\frac{d^2\mathrm{V}}{d\beta^2}\right)+\left(\frac{d^2\mathrm{V}}{d\gamma^2}\right)\right\}d\alpha\,d\beta\,d\gamma-\iint \mathrm{U}\frac{d\mathrm{V}}{dn}\,ds+4\pi\mathrm{U}_0$$

$$=\iiint \mathrm{V}\left\{\left(\frac{d^2\mathrm{U}}{d\alpha^2}\right)+\left(\frac{d^2\mathrm{U}}{d\beta^2}\right)+\left(\frac{d^2\mathrm{U}}{d\gamma^2}\right)\right\}d\alpha\,d\beta\,d\gamma-\iint \mathrm{V}\left(\frac{d\mathrm{U}}{dn}\right)ds. \quad (231)$$

Similarly, if V becomes infinite at any point within the range of integration, an analogous correction must be made for it; and also similar corrections for any other points, whatever be their number, at which such values take place.

242.] One or two remarks have to be made in conclusion:

(1) Throughout this Chapter I have spoken of attracting *masses*, and have denoted mass by the symbol m, and mass-element of the attracting body by the value given in Art. 121; viz. $\rho\,dx\,dy\,dz$; and I have retained this conception, to give consistency to the imagined action. But the preceding theorems are of much wider application than to gravitating matter in the ordinary meaning of the word: they apply to electrical and magnetical action; and thus the meaning of m must be enlarged; and must be taken to denote quantities of attracting action or of influence, whether of free electricity or of magnetism, whatever these may be. It is indeed with reference to these latter subjects that the theorems are so important.

(2) We have spoken always of attraction. But the theorems are also true for repellent action when the repulsion varies directly as the products of the repelling masses and inversely as the square of the distance between them. This extension will

be made in the mathematical expressions of the theorems, if the masses are affected with negative signs; and such a change is necessary in applications to electricity, where two influences, positive and negative electricity, are in operation.

(3) If a good conducting body is charged with electricity, and then placed in a good non-conducting medium, such as dry air, there is equilibrium in the interior, and the remaining free electricity passes to the surface of the body, and there forms a shell of varying density or power, which is kept at rest by the pressure of the external air. For since the interior is at rest, the potential throughout is constant: and consequently

$$\left(\frac{d^2 \mathrm{v}}{d\alpha^2}\right) = \left(\frac{d^2 \mathrm{v}}{d\beta^2}\right) = \left(\frac{d^2 \mathrm{v}}{d\gamma^2}\right) = 0:$$

and thus, by (194), $-4\pi\rho = 0$; and $\rho = 0$; whence it follows that there is no electricity in the interior, and the free electricity is carried to the surface.

(4) And the free electricity forms a shell which is in equilibrium, of which the thickness may be considered constant, and the density variable. This shell has no action on the interior parts of the body, and consequently its interior surface is at rest. The exterior surface is open to the pressure of the air, and as this acts only normally to the surface, the exterior surface of the shell is an equipotential surface.

(5) If the electrised body is an ellipsoid, the shell of electricity on its surface will be ellipsoidal, and we may consider it to be of constant density, and of variable thickness, and to be contained between two similar and concentric ellipsoids, so that the thickness at any point varies as the central radius vector to that point.

(6) The repulsive action on any particle in its external surface is normal to the surface at the point, and proportional to the thickness at the point.

(7) The repulsive action on a particle at different points on the external surface is proportional to the perpendicular distance from the centre on the tangent plane to the ellipsoid at the point; so that at the extremities of the principal axes the repulsive action varies as the length of the axis.

ANALYTICAL MECHANICS.

PART II.

DYNAMICS; THE MOTION OF MATERIAL PARTICLES.

CHAPTER VII.

MOTION, ITS AFFECTIONS, ITS LAWS, AND ITS EQUATIONS.

SECTION 1.—*Introductory; on motion, matter, time, space.*

243.] On resuming the course of our treatise of the science of motion which was interrupted at the end of Article 11, it is necessary to make some preliminary observations.

Mechanics is the science which treats of the action and effects of forces on material particles and bodies at rest and in motion; that part of it which relates to bodies at rest, that is, under the action of many forces in equilibrium, is called *Statics*, and has been discussed in the preceding part: and that part of it which relates to motion is called *Dynamics*, and has lately been termed *Kinetics*, and will be developed in the following parts of the work: the passage from the latter to the former, and the process by which the principles of the latter include those of the former, as the general science includes its particular subordinate, will be investigated hereafter.

Dynamics, as it is intended to unfold the subject in the following pages, will be presented to the student in a twofold aspect: primarily and chiefly it will be considered as a positive and exact science, such as I have attempted to sketch it in the first Chapter; and of that nature of which the pure sciences of number and geometrical space are supposed to be. Motion is the fundamental idea of it; that, viz., out of which spring all the truths of the science, and from axiomatic statements of which they are deductively inferred. Dynamics, as such, is a science of speculation and thought; doubtless in the construction of it experience may have *suggested* much, but the so-called necessity of its principles is derived from another source. Secondarily, it is my purpose to shew that the science is useful to explain phaenomena of the world external to us: hence arises the necessity of proving that the axioms and the first statements of the pure science are true in the subject-matter of cosmical observation, and that the laws of natural phaenomena are included within the range of the pure science. Now for this end large experience, in the way of observation and experiment, is frequently required. The operations of nature are complex, and it is only with deep searching that they allow themselves to be so far unravelled as to exhibit the laws they are subject to. In this respect then it is necessary to apply a limit to our inquiry; and I propose only to shew, and that concisely, that the axioms of the pure science, or the laws of motion, are true in cosmical matter; so that, thus far at least, it is likely that we are on the right road of *natural* philosophy. And it will also be desirable, here and there, to point out certain salient laws, such as the law of gravitation and Kepler's laws of planetary motion, that our attention may be directed to them rather than to others. As in the preceding Chapter it was beside our object to enter on the complete discussion of attractions as applied to the determination of the figures of the earth and of the planets, to the theory of heat, and to magnetism and electricity, because such applications can be made only on certain hypotheses, and with the development of functions in series involving infinitesimal terms, the knowledge of which belongs to the special subject: so in the following treatise I shall not enter on the planetary or lunar theories, because such subjects require special knowledge, and belong to physical astronomy: but the general equations of dynamics will be investigated in all their breadth, and will be brought down to that stage where these special sciences commence; and will not, except in very simple instances, be applied to cases or under circumstances wherein such special knowledge is required. Thus our science is a principal and normal one; *normal,* I say, because it is that to the rules of which each special subordinate science conforms: and the greater or less that conformity is, the more or less complete is that special science; and *principal,* because the laws of dynamics are those which the special science takes and applies, each in its form and degree; and they are so large, that many forms of them are included, which observation has not yet shewn to exist in the material universe. The applied part also serves a moral purpose, insomuch that it enables man to fathom the depths of the laws of Cosmos, to express them in a concise form, and thus to study the works of God. It is for these reasons that the science of motion is the most perfect of the physical sciences.

Although philosophically perhaps it might be more correct separately to investigate these two branches of the subject, yet, as the treatise is didactic, it is more convenient to consider parts of one or the other, as they arise in the course of it.

The nature of the symbols which will be employed requires a remark; we shall have to speak of *time, space, velocity, matter;* these are heterogeneous quantities, and cannot be operated on so as to multiply time into space or mass into velocity; this is self-evident. But these quantities will be represented by symbols such as t, dt, s, ds, v, dv, m, dm; and these are *numbers,* and not the concrete things. Thus t expresses the t times an unit of time is taken; dv the dv times an unit of velocity is taken; and the numbers, of course, can be multiplied together, and the resultant of the operation is number of that kind which the symbols express before the operation. The unit of concretion however, which is to be introduced after the operation, may be different to that previous to the operation: see Art. 124. The concrete units are of course arbitrary, but remain unaltered during the whole of an operation. Sometimes a second, sometimes a year is taken as the unit of time; sometimes a foot, sometimes the earth's radius, sometimes the mean distance of the earth from the sun, is taken as the line-unit; these units vary according to the problem; and the circumstances of it will generally guide us to a judicious choice.

244.] *Motion* is the fundamental idea of mechanics; motion, that is, either real or virtual, either in act or in power; and therefore the science is more correctly termed *the science of motion.* Motion need not be defined: it is too general to be capable of useful expression by means of a more general term; it is a quality or a state: one result of it is change of position of the thing moving: I say, *thing* moving; for a necessary element in an adequate conception of motion as the fundamental idea of mechanics is that something moves: motion exists in, and is of, something; and that something, in which it is, and of which it is a state, is *matter*.

Now although we do not know matter as free from force and consequently from motion of which force is the cause, and we do not know force except as affecting matter, yet the conceptions of motion and matter are distinctly separate, and will be advantageously considered separate from each other. The body of doctrine concerning abstract motion, that is, concerning motion in itself and free from all consideration of its subject and of its causes, is called *Kinematics*, and will be treated of in the following section, and in other parts of the work as the necessity for it arises. It is a geometrical subject, and the limits of it are those which geometry imposes. When complete it embraces the whole theory of pure mechanism, for it teaches all possible kinds of motion, and the modes of transmitting and converting them*. We shall give only those elements which are required for the purposes of this treatise.

When however motion is treated of in connection with matter as its subject, and as the result of force acting on matter, the body of doctrine concerning it is called *Mechanics;* and *Dynamics* is that part of mechanics where the force produces active motion in matter.

It will obviously be convenient to treat these two subjects separately from each other; and the former of course is antecedent to the latter. I shall consider them in their most simple forms in the two following sections of this Chapter; but I must first make some other general observations on motion and matter.

245.] When I speak of matter as the subject of mechanics, the term is not limited to the matter of the members of the solar system; to that which has sensible properties, and which gravitates; but it embraces everything that moves or is capable of motion; the particles of air of course are included; and they gravitate, and they are the subject-matter by the vibrations of which sound is propagated; the particles of light which the emission-theory of light assumes, and the aethereal molecules of the undulatory theory, are also included. Matter is treated as the subject of motion; and when it is spoken of, it is supposed to have one essential property, and that is *mobility*.

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* Many excellent treatises exist on this special subject. Let me mention but two of marked excellence: (1) Willis' Principles of Mechanism; London, 1841: (2) Cour de Mécanique et Machines, par Edm. Bour (premiere fascicule, Cinématique); Paris, 1865.

Matter also admits of divisibility without limit: a very large quantity of it may have motion, or a very small, nay, an infinitesimal part of it; and this is called *a particle;* such as is analogous to a geometrical point: and its other properties, mobility and such like, are independent of the quantity of it. This remark is important; because it will be necessary to divide the subject, according as we consider the motion of a finite quantity of matter, which is supposed to consist of an infinite number of particles, and which is called *a material system* or *a material body:* or according as we consider that of a material particle. The quantity of matter which a body or a particle contains is, as already stated, called its *mass*.

Of motion, and consequently of matter with reference to its property of mobility, there are two other affections, which, by reason of their abstract nature, need not be defined: viz. *time* and *space:* it is sufficient for us to be able to form a notion of them, and to enunciate of them such properties as are required for our purpose. Space and time, like matter, are continuous and divisible; and these affections are without limit. Space may be very large, nay, infinite; we cannot fix the boundaries of that space in which the heavenly bodies are; and it may be very small, such as that occupied by a chemical atom or a material particle. Time also admits of degrees as to quantity; it may be an instant; such an infinitesimal, that the aggregate of an infinity will make only finite time: or it may reach through the present moment from ages bygone to ages to come. Motion, matter, time and space, stand to each other in the following relations. Matter exists in space and time; all matter, even the minutest particle, occupies space. No two particles of matter and also no two bodies can occupy the same space at the same time; this property of matter is called its *impenetrability*. The same matter cannot be in two different places at the same time: hence a particle of matter or a body cannot pass from one position to another without lapse of time: time is consumed in the passage; and therefore a change of place requires time. And as a longer or a shorter time may be spent in the passage, so do we conceive of the rate or speed at which a particle or a body moves. And hence arises the quality of matter which is called *velocity;* velocity being the degree of swiftness or slowness with which matter moves. From these relations arises the necessity of measuring space and time, and of determining equal spaces and equal times. As material bodies exist in space, they have *volume* and *form;* volume depending on the quantity of space which they occupy, and form on the bounding terms of that space; but the knowledge of equal spaces must be found in an adequate knowledge of space. The method of measuring volume is founded on the geometrical principle of superposition, and two volumes are equal which occupy the same or equal spaces. The notion of equal times and also the measure of equal times arises out of the idea of time, and an idea of time is not adequate unless it has these notions; it is true that the passage of time is marked by events which take place in it; and equal times are marked by the regular recurrence of similar events; that is, by uniform motion; but equal times* are in themselves altogether indepèndent of any particular kind of motion; they exist before it and they enable us to apprehend and to measure such a motion: equal times therefore must be deduced from the notion of time.

SECTION 2.—*The Kinematics of a particle in a straight path.*

246.] The most simple motion of a material particle is that in which it describes along a straight line equal linear spaces in equal times; the motion of it is then said to be uniform, and the velocity to be constant; these two expressions in fact being equivalent. But when equal spaces are not described in equal times, the velocity is said to be variable; such a velocity may vary continuously or discontinuously; but it will be necessary for us to consider only a continuously-varying velocity; because a discontinuous variation will be a succession of constant velocities, changing abruptly, and, as it were, by impulses.

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* M. Poisson, Traité de Mécanique, 2de Ed. Tome I, p. 205, writes: "La notion des temps égaux, et la mesure du temps ne sont fondées nécessairement sur aucune loi particulière de mouvement, et l'on peut, en conséquence, les supposer dans la définition du mouvement uniforme et de toute autre sorte de mouvemens." Dr. Whewell, in his Treatise on Mechanics, Ed. 5, Art. 102, says: "Those intervals of time, in which there is no discoverable reason why they should be unequal, are supposed equal."

In the case of constant velocity, equal linear spaces are described in equal times; now although the velocity of a moving material particle is a quality or state of the particle itself and resides in it, and is that by which it differs from a particle at rest, and although no account more exact can be given of it, yet the velocity can be measured; and the measure is taken to be *the number of units of linear space passed through in an unit of time*. If therefore a material particle describes uniformly v units of linear space in one unit of time, v is the measure of the velocity: and if s represents the space passed through by the particle in t units of time, then, bearing in mind the last clause in Art. 243,

$$s = vt; \tag{1}$$

and

$$v = \frac{s}{t}. \tag{2}$$

Thus velocity is linear space, and is *the linear space described in one unit of time*.

I may observe that, if a particle describes a path with uniform velocity, this result is true whatever is the form of the path, be it straight or curved, a curve plane or of double curvature.

If the velocity continuously changes, equal spaces are not described in equal times, and the velocity becomes a function of the time. Let the time be resolved into infinitesimal elements; and let us suppose the particle at the end of the time t to be at a distance s from an origin fixed on the line, and to be at that time moving with a velocity v: that is, if the particle were to move for one unit of time with the velocity which it has at s, it would describe v units of space in that unit of time; and suppose ds to be the space described in dt, the next element of t; then, if v is the velocity at the beginning, and $v+dv$ is the velocity at the end, of dt, the *mean* velocity with which ds has been described may be expressed by $v+\theta dv$, where θ is a proper fraction, and is positive or negative according as the velocity is increasing or decreasing: therefore by reason of (1),

$$ds = (v+\theta dv)\,dt;$$

and neglecting the infinitesimal of the second order, as by the principles of infinitesimal calculus we are obliged to do, we have

$$ds = v\,dt; \tag{3}$$

that is, ds units of space are described in dt units of time by the particle moving with the velocity v at the beginning of dt; and therefore dividing through by dt, we have $\frac{ds}{dt}$ equal to the space described in one unit of time; and this is velocity; and thus we have

$$\frac{ds}{dt} = v. \tag{4}$$

In the cases therefore, both of constant and of varying velocity, velocity is the space described in an unit of time; and is, by reason of (2) and (4), the ratio of the space described to the time during which it is described; and is in the latter case the ratio of two infinitesimals.

The unit of velocity is evidently the velocity with which a particle describes uniformly an unit of space in an unit of time.

It will be observed that θ has disappeared: now as it is upon the sign of θ that an increasing or decreasing velocity depends, so are the results (3) and (4) true in both cases.

The following are examples of the preceding theory:

Ex. 1. If a particle describes uniformly 100 feet in 10 seconds, and a foot and a second are respectively the line-unit and the time-unit, the velocity of the particle is 10. But if the time-unit is half a second the velocity is 5.

Ex. 2. Find the position of a particle at a given time when the velocity varies as the distance from a given point on the rectilinear path.

$$\text{Here} \quad \frac{ds}{dt} = ks; \qquad \therefore \quad \frac{ds}{s} = k\,dt;$$

$$\therefore \quad \log\frac{s}{s_0} = kt; \qquad s = s_0 e^{kt};$$

if s_0 is the distance of the particle from the origin, when $t = 0$.

Ex. 3. Find the position of the particle when the velocity varies as the time.

$$\text{Here} \qquad \frac{ds}{dt} = kt;$$

$$\therefore \quad s = s_0 + \frac{1}{2}kt^2;$$

where s_0 is the value of s, when $t = 0$.

247.] Let us now suppose the particle to be moving along its path with an increasing (or decreasing) velocity; and to fix our thoughts let us suppose the velocity to be increasing; then this increase may take place either uniformly, or at varying rate.

Firstly, let us suppose the increase of velocity to take place at an uniform rate. Let us suppose f to be the increment of velocity in an unit of time; then if the velocity is zero at the beginning of t, and the velocity is v at the end of t,

$$v = ft; \tag{5}$$

but if v_0 is the velocity at the beginning of t, and v is the velocity at the end of t,

$$v - v_0 = ft; \tag{6}$$

so that the velocity-increment in the time t varies as f and also varies as t.

f, which is the velocity-increment in an unit of time, is often called *the acceleration;* but as the former term is more suggestive, I shall generally employ it; it evidently takes a negative sign if the velocity decreases as the time increases. When the velocity increases uniformly, f is constant, and the acceleration is constant; these two expressions being equivalent.

If the particle at the beginning of t is moving with a velocity v_0, and the velocity-increment is negative, then if v is the velocity at t,

$$v = v_0 - ft. \tag{7}$$

Also since the particle is at rest when $v = 0$, that rest takes place when

$$t = \frac{v_0}{f}. \tag{8}$$

Secondly, let us suppose the increase of velocity to take place at a varying rate, so that there are not equal increments of velocity in equal times; then the increase of velocity is a function, either explicit or implicit, of the time.

Let the time be resolved into equal elements; and let us suppose the particle at the time t to be moving with a velocity v, and at the time $t+dt$ to be with a velocity $v+ds$; then if f is the velocity-increment at the time t, and $f+df$ at the time $t+dt$, $f+\theta\,df$, where θ is a proper fraction, is the *mean* velocity-increment during the time dt; and consequently by reason of (5),

$$dv = (f+\theta\,df)\,dt;$$

$$\therefore \quad dv = f\,dt, \tag{9}$$

if we omit the infinitesimal of the second order; that is, dv units of velocity are added in the time dt. Hence, dividing by dt,

$$\frac{dv}{dt} = f. \tag{10}$$

In this latter case the velocity-increment or acceleration is said to vary. And thus whether it is uniform or varying, it is the increase of velocity in an unit of time; and is also the ratio of the increase of the velocity to the time in which that increase takes place, and is in the latter case the ratio of two infinitesimals.

Thus the unit of acceleration or the unit of velocity-increment is, when the increase of velocity is an unit in an unit of time.

If the velocity decreases, f is negative; and from (10) we have

$$\frac{dv}{dt} = -f. \tag{11}$$

These expressions shew that an unit of acceleration is that which corresponds to an uniform increase of an unit of velocity in an unit of time.

248.] Taking these results in combination with those of the preceding Article, we have the following values:

In the general case of varying velocity and of varying positive acceleration, from (4) and (10),

$$\begin{aligned} f = \frac{dv}{dt} &= \frac{d}{dt} \cdot \frac{ds}{dt} \\ &= \frac{d^2s\,dt - d^2t\,ds}{dt^3}; \end{aligned} \tag{12}$$

and therefore if s is equicrescent,

$$f = -\frac{ds\,d^2t}{dt^3}; \tag{13}$$

and if t is equicrescent,

$$f = \frac{d^2s}{dt^2}. \tag{14}$$

We shall suppose t to be an equicrescent-variable throughout the whole treatise, unless it is stated expressly that it is not so.

These values suggest the following remarks:

Let a particle be moving, and let it describe the space s in the time t:

(1) Let us suppose the space and the time to be resolved into corresponding infinitesimal increments, so that neither all the dt's nor all the ds's are equal: in which case neither t nor s is equicrescent; and thus (12) correctly represents the velocity-increment due to one unit of time: but the expression is unnecessarily complicated, and is therefore of little practical use.

(2) Let the time be resolved into equal elements, that is, let t be equicrescent: then $d^2t = 0$, and (14) expresses the velocity-increment. Now if the velocity is constant, all the corresponding elements of space will be equal: that is, all the ds's will be equal and $d^2s = 0$: there will, in this case, be no velocity-increment.

If the velocity is not constant, the ds's corresponding to equal dt's will not be equal; there will be an excess of one ds over the preceding or succeeding ds, and thus there will be a d^2s: as dt is constant, let us assume it to be the unit of time: then ds is the velocity; and d^2s is the velocity-increment; and therefore measures the acceleration force. It is also to be observed, that if the velocity-increment is constant, d^2s is constant, and therefore $d^3s = 0$: but if on the other hand the velocity-increment is variable, the d^2s's vary, and d^3s is not equal to zero: similarly we might proceed, and shew under what circumstances d^3s would be constant, and therefore $d^4s = 0$.

(3) Let the space be the equicrescent variable; in which case, if the velocity is constant, the dt's corresponding to the ds's are equal, and $d^2t = 0$; but if the velocity is not constant, equal dt's do not correspond to equal ds's, and therefore d^2t will not be equal to zero: in this case (13) is the expression for the velocity-increment, f being affected with a negative sign because the velocity-increment becomes greater, as the time to which it is due becomes less; and therefore the dt's, to which equal successive ds's are due, are decreasing, and therefore d^2t is negative.

249.] The following are simple illustrations of the preceding formulae; it is unnecessary to add others as the subject will be amply applied in the succeeding Chapter.

Ex. 1. If there is no velocity-increment, $\frac{d^2s}{dt^2} = 0$; so that if v_0 is the constant velocity,

$$\frac{ds}{dt} = v_0;$$

$$\therefore \quad s = s_0 + v_0 t.$$

Ex. 2. If the acceleration or velocity-increment is constant,

$$s = s_0 + v_0 t + \frac{1}{2} f t^2;$$

where s_0 and v_0 are the values of the space and the velocity respectively, when $t = 0$.

Hence if a particle moves from rest from the origin with a constant velocity-increment,

$$s = \frac{1}{2} f t^2;$$

and thus the space described varies as the square of the time.

Ex. 3. If the velocity-increment varies as the time from rest,

$$\frac{d^2s}{dt^2} = kt;$$

$$\therefore \quad \frac{ds}{dt} = v_0 + \frac{1}{2}kt^2,$$

$$s = v_0 t + \frac{k}{6}t^3.$$

Ex. 4. If the velocity-increment varies as the distance from a given point in the line of motion, and is negative,

$$\frac{d^2s}{dt^2} = -ks;$$

$$\therefore \quad \frac{2\,ds\,d^2s}{dt^2} = -2\,ks\,ds,$$

$$\frac{ds^2}{dt^2} = k(a^2-s^2),$$

if a is the value of s, when the particle is at rest;

$$\therefore \quad \frac{-ds}{(a^2-s^2)^{\frac{1}{2}}} = k^{\frac{1}{2}}dt;$$

the negative sign being taken, as I will suppose the particle to move towards the origin;

$$\therefore \quad \cos^{-1}\frac{s}{a} = k^{\frac{1}{2}}t,$$

if $t = 0$, when $s = a$, and the particle is at rest;

$$\therefore \quad s = a\cos k^{\frac{1}{2}}t.$$

SECTION 3.—*The dynamics of a particle moving in a straight line.*

250.] The preceding observations on the kinematics of a particle are all that we require at present. We shall return to the subject in Chapter IX; and must now enter on the further consideration of matter as the subject of motion; and we shall state explicitly certain properties of matter, beyond those which have been stated, in Part I, of it as the subject of pressure or statical force. An important question meets us at the outset; according to our conception of matter, as the subject of motion, has it any power of changing its state;

has it when at rest a power of putting itself into motion? has it when in motion a power of itself either of increasing or of diminishing its velocity? An adequate conception of matter involves a reply to these questions in the negative. *Matter is inert;* it has no power of acting on itself or of changing its own state as to rest or motion. If it is at rest, it will remain at rest: if it is moving with a given velocity along a rectilineal path, it will continue to move with that velocity along that path: there is no more reason why it should change its course towards one side of that line than towards the other: this is equivalent to saying that lapse of time does not affect matter's state as to rest or motion. And not only does matter remain as it is, unless acted on by some source of velocity external to itself, but it also passively submits to external influence: whatever effect is communicated to it, that is also developed in it. Now I am not saying that matter does not act on other matter, for the matter of our physical system does so act: thus leaden balls attract each other: particles of air repel each other: but it does not change its own state. Whenever therefore—and this is most important—matter's state is changed either from rest to motion, or *vice versâ,* or when its velocity is increased or diminished, that change is due to some adequate cause, and velocity is communicated to it from some source external to itself. This source is called *force;* and force is either accelerating or retarding according as the velocity of matter is, by its action, increased or diminished: a more exact definition by means of its measure will be given hereafter. From the fact that matter is inert, or, in other words, from the principle of inertia, will be inferred the first equations, or propositions, of the science. The principle may be stated in the following form, and is then commonly called the first Law of Motion:

Matter at rest remains at rest, and matter in motion continues to move in the same line and direction, and with unvaried velocity, unless acted on by some force external to itself.

This principle of inertia is axiomatic, and is the first axiom in the construction of the science; it rules that when a change of state takes place in matter, that change is due to the action of some cause external to the matter.

251.] As we shall apply our theoretical investigations largely to the matter of the earth, and of other bodies of the solar system, it is worth while shortly to inquire how far the properties of matter which have been axiomatically stated are fulfilled in that particular matter of which they consist.

As to Mobility; the fact is shewn by daily observation: bodies falling towards the earth, particles of matter constantly moving in the air and as seen in a sunbeam, the waters of the sea never at rest, the motion of the moon and of the planets, the motion of particles of air in the wind, all bear evidence to this property: nothing is seen quiescent; everything is in motion.

As to Inertia: terrestrial matter seldom changes its state without our being able to assign the cause; and hence we inductively infer, that the cause could always be assigned, if our knowledge of the moving matter and its circumstances was perfect. Consider a particle of iron, placed on a smooth table; relatively to the table it is at rest: but let a magnet be placed so that the particle of iron is within its influence; the particle will begin immediately to move towards it; and the longer the space is through which the particle moves, the greater will be its velocity; thus the magnet is the cause of the motion of the particle at first, and also of its subsequently increasing velocity. Now let another magnet be introduced of the same power as the former, and acting along the same line of action, and in an opposite direction, so that the action of the former magnet on the particle of iron is neutralized: then it is found that the iron-particle will continue to move with the velocity which it has at the time when the neutralizing magnet is introduced: that is, the velocity which it has at that instant is a quality residing in it, and which it has of itself no power to annihilate: its velocity will, it is true, during the subsequent motion become less and less; yet it appears that such a loss of velocity is caused by the friction against the table, the resistance of the air, and so on: for if these impediments are diminished, the particle continues to move with a velocity less rapidly decreasing: and hence we infer that if they were entirely removed, there would be no diminution of the iron's velocity.

So again if a ball is projected along a level surface, such as a bowling-green, the rougher the surface is the more impediment does it offer to the ball's motion, and the sooner is the ball reduced to rest: but if the surface is smooth, as a pavement, or smoother still, as a plate of glass, or as ice, the longer will the ball continue to move; eventually, however, it will be reduced to rest, because it is impossible to remove all the impediments which are continually acting on it as retarding forces, and are thereby withdrawing velocity from it.

Again, if a suspended pendulum oscillates, the time ere its motion ceases will be longer if it vibrates on a knife-edge than if it is suspended by a spring, because the resistance of the former is less than that of the latter; and if it oscillates in the exhausted receiver of an air-pump, the time ere its motion ceases will be longer than if the oscillations take place in air. From experiments such as these, it is inductively inferred, that if all the hinderances are removed, and if the moving matter does not receive velocity from any other source, it has in itself no power either to increase or to diminish its own velocity.

The nearly uniform periods of the planets, and the almost constant length of the mean sidereal day, in a similar manner tend to shew that the same law is true in the matter of which the bodies of the solar system consist.

The principle of inertia was first recognised by Galileo: mechanicians had before his time failed to give a correct exposition of the principles of mechanics because they knew not this fact.

252.] Matter therefore can neither generate velocity for itself out of its own resources, neither can it absorb into itself velocity which it has, or velocity which is communicated to it: it is alike "natural" to it to be at rest and in motion; whenever therefore its state changes, some cause external to itself is the origin of the change; if the velocity is increased, some velocity has been communicated to it; if it is diminished, velocity has been abstracted from it: whatever causes a change of velocity is called *force*, and the word "force" will be used in Dynamics in this meaning only.

The word "force," as thus stated, has not the exactness which an exact science requires. Such terms are not precise enough unless the quantities which they express are measurable: and as force is an active cause, it will be measured by its effects. Now the effect of a force is velocity, and consequently the velocity communicated to or impressed upon the moving matter in a given time, say, in an unit of time, is the measure of the force. But the velocity impressed on a particle is equal to the velocity expressed in its actual motion, inasmuch as matter has no power to absorb or to produce any of such impressed velocity; and consequently the velocity-increment, or the acceleration, in an unit of time is the measure of the force. Hence if a force causes in a moving material particle an increase of velocity f in an unit of time, the force may be correctly denoted by f, because that signifies its effect, viz. the velocity-increment of which the force is the cause.

Hence the unit of force is that which impresses an unit of velocity in an unit of time.

Also the varieties of force are in this respect as many as are the velocity-increments which they produce. Thus if a force communicates equal velocities in equal successive time-elements, the force is said to be constant; and according as it increases or diminishes the velocity, it is called *an accelerating* or *a retarding force*. If a force, on the other hand, communicates unequal velocities in equal successive elements of time, it is called *a variable force*, and an accelerating or a retarding variable force according as the velocity is increased or diminished by its action. The law according to which the velocity is communicated is called *the law of the force*. The velocity which a force transfers to a body is called *the impressed velocity;* and the velocity which is developed by the action of the force in the moving body is called *the expressed velocity*. In the case of a single particle the velocity expressed in its motion is equal to that impressed by the force on it; but if that particle is a member of a material system or of a body, for reasons which will be given hereafter, it will appear that this is not the case. Thus in a moving particle the impressed and expressed velocities are indeed the same thing viewed from different points.

From these explanations of force and its varieties the following results arise. Firstly, let us suppose a force to be constant and to act on a material particle, which is moving in a straight path, along the line of its motion; and let f be the velocity which is communicated by this force in an unit of time; then if the force acts for t units of time, the velocity communicated is ft; and if the particle was moving with a velocity u when the force began to act, and with a velocity v at the time t, and the force is accelerating,

$$v = u + ft; \tag{15}$$

and if the force is retarding,

$$v = u - ft. \tag{16}$$

Secondly, let us suppose the force to be variable, and suppose it to be such that at the time t a velocity f would be impressed by it in an unit of time, if it were constant during that unit:

and to be such that at the time $t+dt$ a velocity $f+df$ would be impressed by it in an unit of time if it were constant during that unit. Then, if θ is a proper fraction, $f+\theta\, df$ would be the *average* or *mean* velocity impressed in an unit of time during the time dt: and consequently if dv is the velocity actually impressed in dt,

$$dv = (f+\theta\, df)\, dt;$$

and omitting the infinitesimal of the second order,

$$dv = f\, dt; \tag{17}$$

and this assigns the increase of velocity which takes place in the time dt by the action of the force f. This force is the increase of velocity in an unit of time.

If we require the amount of the velocity which is impressed by a finite accelerating force in a finite time, this must be deduced from (17) by integration; and the process can be effected immediately if f is constant or is a function of t, since in this case

$$v = \int f dt;$$

but it must be done indirectly if f is a function of v or of s.

We have however brought our investigation of the effects of force to this point: viz. that its effect and its measure is the acceleration or velocity-increment which has been discussed in the previous section on Kinematics, and we have

$$f = \frac{dv}{dt} = \frac{d}{dt}\frac{ds}{dt}$$
$$= \frac{d^2 s}{dt^2}; \tag{18}$$

and consequently all the results of this equation, its various forms, and the remarks which have been hitherto made on it, are applicable to it, when f is the accelerating force. We shall have so many applications of this equation hereafter that it is unnecessary now to insert any.

253.] When a force acts on a particle at rest, *the action-line of the force* is of course *the line of motion of the particle;* but when a force acts on a particle in motion the action-line of the force may be, or may not be, the line of motion of the particle. In the preceding article we have supposed the former case. The latter case, which is more general and also more important, will be fully discussed in following Chapters.

If two or more forces act simultaneously on a particle in the line of its motion, the resultant effect will evidently be the sum of their separate effects. Thus suppose a material particle to be moving with a constant velocity v, and two constant forces f and f' to act on it, the effects of which are severally to produce velocities f and f' in one unit of time; and suppose each of these forces to act for t units of time: then the velocity of the particle at the end of t units of time will be $v+ft+f't$. If one of the forces, say f', act in a direction contrary to that of the particle's motion, it will abstract velocity, and the velocity of the particle will, at the end of t units of time, be $u+ft-f't$. A similar result is of course true when the forces are variable.

Hence if two forces are capable of communicating equal velocities to the same body in equal infinitesimal elements of time, the two forces are said to be equal, and are such, that when applied to the same body in opposite directions along the same line of action, they neutralize each other, and do not change the body's velocity. This is the definition of equal forces. Similarly, forces which in equal infinitesimal elements of time will produce in a given body, twice, thrice, &c. the velocity which another force will, are estimated as double, triple, &c. of this latter force.

254.] Force, such as we have considered it, impresses finite velocity in a finite time; and the effects of it have been resolved into elements corresponding to infinitesimal elements of time. Thus if a force acts for a finite time, and if the law of the force is given, the total velocity impressed by it during the whole time may be found by integration, and the whole velocity will be the measure of the force's action. A force of this kind is commonly called *a finite* accelerating, or retarding, *force*. But suppose a force to act, and to communicate a very great velocity in a very short time, such as the explosive force of gunpowder, which will impress a very great velocity on a cannon-ball in the very short time during which the ball is passing along the bore of the gun, then doubtless if the law of the communication of the velocity is known, the whole velocity may be found as in the former case, and will be the measure of the action of the force; but if the law of the force is not known, and the force acts for a short time and then ceases, the whole velocity which is impressed by it may be taken as the measure of its action. A force of this kind is called *an impulsive* or *instantaneous force*. This force does not, it is to be observed, differ in kind from finite accelerating force; the communication and the development is as gradual in one case as in the other; the difference consists in the mode of measurement of its effect: in the former case the law of force is given, and the total action of the force is determined by integration: in the latter case, whether the law of force is known or not, the action of the force is measured by the whole velocity which has been communicated by it.

255.] Hitherto motion and velocity have been considered independently of the quantity of matter of which they are. Velocity and its properties have been discussed as being of a mass, and, to fix our thoughts, we have assumed a material particle to be the matter moving; but it has been unnecessary to introduce any reference to the quantity of matter, because the velocity of a material particle and of a mass of large dimensions may be the same: and inertia as a property of matter does not require any conditions as to the quantity of matter: it is true equally of a particle and of a large body. But now it is necessary to consider velocity in reference to quantity of matter or mass: because the equations of motion of moving matter, from which all the theorems of dynamics will be deduced, are formed by comparing *the velocity impressed with the velocity expressed;* and thus a question arises, whether two bodies having equal velocities impressed on them will move with equal velocities, whatever are their masses? No doubt, by the principle of sufficient reason, if their masses are equal, the expressed velocities will also be equal: but what will be their expressed velocities, if the masses are unequal? In reply to this question we must strictly define equality of mass; and be on our guard against an argument in a circle: equal masses must not be defined to be those on which, when equal forces act, equal velocities are impressed; when equal forces are defined to be those which impress equal velocities on equal masses. We have already spoken of mass, and of its mode of measurement by means of weight, in Section 1, Chapter IV; but the following process of determining it is that which is most appropriate to our present purpose. If two masses having the form of spheres, and moving with their centres along a straight line, and in opposite directions, impinge on each other, and if each is by the collision brought to rest, these masses are said to be equal: so that equal masses are defined in the following terms:

Two masses are equal which moving with equal velocities along the same straight line, in opposite directions, and impinging on each other, are reduced to rest by the collision.

When many masses have by this process been determined to be equal to each other, we may collect two or more into one mass, and thus obtain masses which shall be any multiple of a given mass: and by a reverse process we may obtain masses which are submultiples of another mass; and thus we may obtain masses which bear any ratio to each other. Thus if m equal masses are collected into one mass, and m' into another, the ratio of these collected masses will be to each other as m to m'.

In the present volume I propose to consider the motion of a material particle only. It is much more simple than that of a body, and for this reason: if a body moves, its particles may all of them describe equal and parallel paths, in which case the body is said to have only motion of translation; or the particles of the body may revolve one about another, without the relative positions of them being changed, in which case the body has the motion of rotation: a full investigation of these kinematical circumstances will be found in Part III of the Treatise: whereas as a particle occupies space equal to an infinitesimal geometrical point, the motion of rotation may be neglected, and we have to consider motion of translation only.

Experience teaches us, in the case of terrestrial matter, that if two particles are at rest, and if it is required to make them move with equal velocities, a greater force may be required to act on one than on the other; and the reason assigned is, that one has much greater *force of inertia* than the other. Now this is an inaccurate expression. *Inertia** has no force: it neither destroys nor generates velocity; motion and rest are equally natural to matter. The true reason is, one mass is greater than another, and therefore has a greater quantity of matter for velocity to be communicated to. These circumstances however must be thoroughly examined, as they lie at the very foundation of our subject.

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* Many writers on Mechanics use the expression "force of inertia;" and lately $\frac{d^2s}{dt^2}$ has been called "force of inertia;" the expression is surely inaccurate and unphilosophical, if the words are used in the senses which I have assigned to them: and I cannot but refer to M. Poisson, who is no mean authority on such questions, for a corroboration of the view of the subject here taken. Traité de Mécanique, 2de Ed. Tome I, Art. 120.

256.] Let two material particles be in motion; of which I will suppose one to be the unit-mass, for the unit is arbitrary, and the other to contain *m* unit-masses; and let them move with equal constant velocities *v*: if the *m* unit-masses of the larger mass are separate, each would move with the same velocity *v*, and therefore the sum of the velocities of all the particles moving separately would be *mv*; and by the principle of inertia the sum of the velocities is not changed when all the particles are collected into one common mass, and therefore the quantity of velocity which is expressed in the moving mass *m* is *mv*: that is, is in quantity *m* times the velocity of the unit-mass. Although therefore the velocity of both the masses is the same as to intensity, yet in *quantity or amount of velocity*, that of the mass *m* is *m* times that of the unit-mass.

As we shall frequently speak of this quantity of velocity, it is convenient to assign to it a distinctive name: it is, as explained above, the product of the numbers expressing the mass and the velocity, and has been ordinarily called *momentum* or quantity of motion; although the term is somewhat inaccurate, yet, to avoid the inconvenience of new nomenclature, I shall use it, and shall signify by it the quantity of velocity which exists in moving matter: and shall henceforth signify by the term "velocity," velocity as to intensity.

The increments of these will be called respectively *the momentum-increment* and *the velocity-increment;* and of these, when expressed, the mathematical equivalents will be $m\frac{d^2s}{dt^2}$ and $\frac{d^2s}{dt^2}$, if t is an equicrescent variable.

Thus the momentum-unit is the product of the mass-unit into the velocity-unit. It is evident also that the momentum of a body is equal to the sum of the momenta of its several parts.

The distinction which is drawn between velocity as to intensity and velocity as to quantity or momentum, may be illustrated by the following analogies. Suppose two masses of the same substance, one of which is ten times as large as the other, to be in the same state of *temperature*, and suppose both to be heated so as to be of the same higher temperature: then to these masses heat has been transferred from some external source; and to the larger mass ten times as much as that to the smaller one; and thus, although both are of the same heat as to intensity, yet the quantity of transferred heat in one is ten times as great as that in the other. A thermometer measures temperature, that is, heat as to intensity; in the same manner does the space described in one unit of time measure velocity as to intensity.

Again, suppose a certain quantity of light from a given source to be received by a table or a given area; the smaller the area is, if it receives the whole light, so much more intense will the illumination be. But if a given area is illuminated equally throughout its surface, the greater the surface is, the greater also will be the quantity of light received by it. Thus light does as to intensity vary inversely as the area over which it is spread: but the quantity of light received by a surface illuminated equally throughout varies directly as the surface.

257.] It appears then that when force acts on matter, and communicates velocity to it, the effect is momentum, and not only velocity as to intensity. And this subject has to be considered both with reference to velocity and to velocity-increment; that is, with reference to impulsive and to finite accelerating force. Now the law of inertia rules that in both these cases, the momentum expressed is equal to the momentum impressed.

Firstly then if the force is impulsive; let m be the mass of a particle, which I will assume to be at rest: and let a force act upon it in such a way that it instantly moves with a velocity v, then the expressed momentum is mv. Let Q be the momentum impressed by the action of the force, which is like a blow; then by the preceding principle,

$$\mathrm{Q} = mv; \tag{19}$$

$$\therefore \quad v = \frac{\mathrm{Q}}{m}: \tag{20}$$

which assigns the velocity communicated to the particle by the blow.

If the particle was previously moving with the velocity u, and the force acted on it in the line of its motion, then if v is the velocity after the action of the force,

$$v = u + \frac{\mathrm{Q}}{m}. \tag{21}$$

The following are illustrative of this theorem:

If a particle of m mass-units moves with a velocity v its momentum is mv; and if all its momentum is transferred to a particle m', and v' is the consequent velocity of m',

$$m'v' = mv;$$

$$\therefore \quad v' = \frac{mv}{m'};$$

which determines the velocity of m'.

Thus if a particle, as a small ball, of mass = 3, moves with a velocity = 4, its momentum is 12; and if it impinges directly on another particle of mass = 2, and is reduced to rest by the impact, the whole of the momentum will have been transferred to m', and m' will move with a velocity = 6. 6 therefore will be the expressed velocity, and 12 will be the expressed momentum, of this latter particle.

Hence also momentum is the measure of the pressure of percussion of a moving mass.

So if two particles, moving along the same line and in opposite directions with velocities which are inversely proportional to the masses, impinge directly on each other, they will be reduced to rest by the collision.

Again, if a cannon-ball of mass = 10 is fired from a gun, and emerges from the bore with a velocity = 250, the momentum of the ball is 2500, and this will be the measure of the explosive force of the gunpowder.

Hence also it appears that whenever momentum is impressed on a mass by means of matter acting upon it, it is withdrawn from some other source, or an equivalent momentum is simultaneously produced in an opposite direction. Hence also we infer that the whole amount of momentum is always the same. Momentum cannot be created: it can only be transferred. It may perhaps be thought that momentum can be generated by muscular action, say, that a stone may be thrown, and thus receive momentum, by the muscular action of the arm: we must not however be deceived by appearances: let a person stand in a frame suspended as the scale of a balance, and which is capable of moving freely: if he impresses momentum on any body, as, for instance, if he throws a stone, it will be found that he moves in a direction directly opposite: and the product of his mass and the velocity with which he moves in the scale will be equal and opposite to that which he has given to the stone: the apparent creation then of momentum in one direction is accompanied by the creation of an equal quantity in the opposite direction. A similar effect takes place when momentum is imparted to a mass by means of a pressure against the earth.

258.] The same principles apply, and lead to similar results when the force is finite accelerating. If a particle of mass m receives a velocity-increment $= \frac{d^2 s}{dt^2}$, by the action of a force, the expressed momentum-increment due to the force is $m\,\dfrac{d^2s}{dt^2}$; and consequently if the force is such as to impress a velocity-increment f on an unit-mass, then

$$mf = m\frac{d^2s}{dt^2}; \tag{22}$$

and
$$f = \frac{d^2s}{dt^2}. \tag{23}$$

And if S is the whole impressed momentum, which may or may not vary with m,

$$S = m\frac{d^2s}{dt^2}. \tag{24}$$

The source of the impressed-momentum has been usually called *moving force;* and as it is equal to the expressed momentum-increment, we take this latter to be its measure; and as the accelerating force is measured by the velocity-increment, that is, by $\dfrac{d^2s}{dt^2}$, so the measure of the accelerating force is that of the moving force acting on a mass-unit. Hence also the *moving force-unit is that which impresses an unit of velocity on a mass-unit* in an *unit of time.*

Equations (19) and (24) are called *equations of motion;* they define all possible kinds of motion of a particle in a rectilinear path.

Let us exemplify this result: suppose a particle m to be falling towards the earth: it is found by experiment that the earth's attraction is an uniformly accelerating force, which impresses on the falling particle a velocity-increment of 32 feet (approximately) in one second of time; let a second therefore be the time-unit, and let us represent the number 32 by g (= gravity): then mg is the impressed momentum-increment, and $m\,\dfrac{d^2s}{dt^2}$ is the expressed momentum-increment: therefore, by reason of (22),

$$mg = m\frac{d^2s}{dt^2}; \tag{25}$$

$$\therefore\quad g = \frac{d^2s}{dt^2}.$$

The illustration may be more correctly represented when we take account of the mass of the earth. For since the attraction between the earth and the particle is mutual, the particle attracts the earth while the earth attracts the particle: if therefore m and M are the masses of the particle and of the earth respectively, the velocity-increments of the particle and earth in an infinitesimal element of time are inversely as the masses.

Similar too is the mutual attraction of the earth and moon: the expressed velocity-increments with which they move towards each other are inversely as their masses. Hence it follows that their centre of mass would remain at rest, if the earth and moon had no other motion than that which is due to their mutual attraction: but owing to the action of the sun, and the motion of each in space, the centre of gravity describes an ellipse with the sun in one of the foci.

Hence then it follows that (1) moving forces do not impress equal velocities on different masses, unless they are proportional to the masses; (2) the velocities expressed in equal masses are proportional to the moving forces; (3) the velocities expressed in unequal masses by equal moving forces are inversely proportional to the masses. Hence also we infer that when a moving force impresses velocity on a mass, the velocity expressed varies directly as the moving force and inversely as the mass. This last proposition has been commonly called the third Law of Motion, and is enuntiated in a form such as,

When moving force produces velocity in a given mass, the velocity produced is inversely proportional to the mass.

Sir Isaac Newton calls the following proposition the third law of motion: "Action and reaction are equal and opposite." This however is no more than a statement in plain language of (19) and (24); and it is necessary to explain the meaning of the terms *action* and *reaction*, and how they are measured.

And here we have come to the second axiomatic principle which is necessary to the construction of our science: when the state of matter, as to motion, changes, a measure of the change is hereby given: the product of the mass and of the velocity which is expressed in a given time is the measure of the force which has caused the change, and is by the principle of inertia equal to the impressed momentum. From this equation all the results of dynamics will be deduced.

259.] As equation (22) is a differential expression of the second order in terms of s and t, and as in the complete solution of a dynamical problem it is required that s should be expressed in integral terms of t, it is evident that this differential equation must undergo two integrations before the required solution is obtained. Now this process will take different intermediate forms according to the form of f.

If f is constant, or if it is a function of t, say $f = \phi(t)$, then (24) becomes

$$m\frac{d^2s}{dt^2} = m\phi(t);$$

$$\therefore \quad m\left(\frac{ds}{dt} - v_0\right) = \int_{t_0}^{t} m\phi(t)\,dt, \tag{26}$$

when v_0 is the velocity, when $t = t_0$;

$$\therefore \quad m\frac{ds}{dt} = mv_0 + \int_{t_0}^{t} m\phi(t)\,dt; \tag{27}$$

and thus the result gives momentum; (26) giving the momentum which accrues in the time $t - t_0$, and (27) giving the momentum at the time t. In both cases however the result is momentum. The space may be found in terms of the time from (27) by another t-integration. Both these are cases of time-integration.

If f is a function of s; say $f = \phi(s)$, then

$$m\frac{d^2s}{dt^2} = m\phi(s).$$

Let us multiply both sides by ds; then

$$m\,ds\frac{d^2s}{dt^2} = m\phi(s)\,ds;$$

$$\therefore \quad \frac{mv^2 - mv_0^2}{2} = \int_{s_0}^{s} m\phi(s)\,ds; \tag{28}$$

the left-hand member of this equation is called the *vis viva** of the particle m; that is, vis viva is a quantity which varies as the product of the mass of a particle and the square of its velocity, and the form of the left-hand member shews that it is convenient to take ½ as the coefficient of variation. The right-hand member is the sum of the products of the mass of the particle into the product of the impressed velocity-increment and the distance over which it acts. This is called the *work done by the force* through the distance $s - s_0$; consequently the work done by a force exerting action through a given distance is equal to the increase of vis viva which has accrued to the particle in its motion through that distance.

Thus the unit of work is that which is done by an unit of accelerating force acting on an unit of matter through an unit of space. And if the earth's attraction at a given place is the unit-accelerating force, and the mass of a pound is the unit-mass, and a foot is the unit-space, the unit of work is that which is required to raise the weight of one pound through a vertical space of one foot. This is called *a foot-pound;* and is the unit of work generally adopted by British engineers.

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* It is called by Sir W. Thomson and Professor Tait "Kinetic Energy."

Thus the work done in raising a weight through a vertical distance is proportional to the weight raised and to the vertical distance through which it is raised.

Generally for any force, the work, as thus defined, done during an infinitesimal displacement of the particle on which it acts, is the virtual moment of the force, which has been described in Article 108.

If in the motion vis viva is lost, negative work is done by the force; that is, the work is stored up as potential work in the particle or mass on which the force has acted. Thus if work is spent on winding up a watch, that work is stored in the coiled spring, and is thus potential and ready to be restored under adapted circumstances. Similarly, if a weight is raised through a vertical distance, work is spent in raising it, and that work may be recovered by lowering the weight through the same vertical distance.

This theorem, stated in the most general form, is the modern principle of conservation of energy or of work; and is made the fundamental theorem of abstract dynamics as applied to natural philosophy.

In this case we have an instance of space-integrals. And as forces in nature are functions of the distance, this is the form which dynamical problems take in physics; we shall hereafter have many examples in the solution of problems which are capable of such application.

There is still another form which f may have: it may be a function of the velocity; that is, f may be of the form $\phi\left(\frac{ds}{dt}\right)$; in which case

$$m\frac{d^2 s}{dt^2} = m\phi\left(\frac{ds}{dt}\right); \tag{29}$$

and of this equation we may generally take either the time-integral or the space-integral. Thus if we take the time-integral, replacing $\frac{ds}{dt}$ by v, we have

$$\frac{dv}{\phi(v)} = dt; \tag{30}$$

whence, when $\phi(v)$ is given, we have v in terms of t.

And if we require the space-integral, since

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = \frac{ds}{dt}\frac{dv}{ds} = v\frac{dv}{ds}, \tag{31}$$

we have
$$\frac{v\,dv}{\phi(v)} = ds; \tag{32}$$

whence we have v in terms of s.

If f is given in terms of two or more of the three quantities t, s, and $\frac{ds}{dt}$, it is only in certain cases that the differential equation admits of integration.

Of all these several forms of f we shall have many examples in the following Chapter.

260.] The truth of the preceding theorems connecting moving force, mass, and expressed velocity, in the case of terrestrial matter is proved by Attwood's machine, for a full description of which I must refer the reader to Attwood's original treatise on rectilineal motion, and to other treatises on experimental mechanics, but of which a concise account is given in Section 3 of the succeeding Chapter. It is shewn by numerous experiments made with it that the expressed velocity-increment in a second of time varies directly as the moving force and inversely as the whole mass moved; and therefore the product of the mass and the velocity-increment varies as the moving force, and may be taken to be a measure of it. The same theorem is also proved by the following experiment: it has been shewn in the preceding Chapter that the earth's attraction on bodies near to its surface is constant; and it will be shewn in Section 3 of the following Chapter that, when bodies move under the action of the constant accelerating force of gravity, the expressed velocities, due to given vertical distances through which the force acts, vary as the square roots of those distances. Suppose two spherical balls m and m' to be suspended from two points in the same horizontal line, and by strings of lengths such that the balls when at rest may just touch, and also have their centres in the same horizontal plane: let these balls be moved in circular arcs; then the velocities acquired by them as they fall from rest to the lowest point vary as the square roots of the versed-sines of the arcs through which they descend; and as the versed-sine varies as the square of the corresponding chord, so the velocities acquired in the descent vary as the chords. If therefore the two balls are raised through arcs, the chords of which are inversely as the masses of the balls, the velocities at the lowest points will also be inversely as the masses; and it is found by experiment that balls which have fallen through arcs, the chords of which are inversely as their masses, and which impinge on each other at the lowest point, are by the collision brought to rest: and therefore the momenta of them must have been equal, and thus being in opposite directions along the same line of action have neutralized each other. This then is an experimental proof that the momentum is equal to the product of the mass and of the velocity. It is also found that, if the arc through which one of the balls moves is greater than that determined above, when the balls come into contact, they are not reduced to rest, but move in the direction of the motion of that which has fallen through the proportionally greater arc.

It is also found by experiment that if two balls of unequal masses are placed in contact, and have a spring so arranged that when the spring is set free it exerts an equal action against both of them, the velocities which are expressed in them are respectively inversely as their masses.

261.] When the matter on which moving force acts rests on a surface, the normal to which is along the line of action of the moving force, the effect is not *velocity* but *pressure:* for the infinitesimal element of velocity, which the moving force would impress in an infinitesimal element of time, is destroyed by the resistance of the surface. But if the surface were removed it would be expressed in the moving matter, and the elements of velocity being added to each other, a finite velocity would be expressed. When therefore a moving force impresses velocity, and the velocity is expressed, the elements of it are added to each other, and the resultant is the whole expressed velocity: but when the elements of velocity are destroyed as soon as they are communicated, the result is pressure. Hence it follows that two pressures are to each other as the product of their masses and the infinitesimal elements of velocity which would be expressed in them in an infinitesimal element of time if they were free. At this point therefore statics becomes a branch of dynamics, and on this principle, which is *the principle of virtual velocities*, the theorems of the latter science are applicable to and become those of the former.

We are hereby supplied with a method, which is in practice most convenient, for determining the mass of terrestrial matter.

Observation shews that at the same place all bodies, whatever are their substances, acquire the same velocity in falling in vacuo in the same time. The earth's attraction therefore is an accelerating force which acts independently of the particular kind and quantity of the matter which moves, and is therefore the same for all matter. Consequently the pressures of bodies under the attraction of the earth vary as their masses: these pressures are the *weights* of the bodies, and therefore the weights at the same place vary as the masses of the bodies; and as the balance affords an easy mode of comparing weights, we can hereby deduce the relative proportions of the masses.

It may probably be thought that this method of determining mass is more simple than that chosen in Art. 255; and practically for the matter of the earth it is: but there are objections to it, so far as the principles of the pure science of motion are concerned: (1) it experimentally assumes the relation between mass, moving force or its measure momentum-increment, and accelerating force or its measure velocity-increment; and this it is adduced to prove: (2) only terrestrial matter can be compared by it, whereas the principles of the science of motion should be laid in breadth sufficient to include matter of all kinds: (3) M. Poisson writes in the Traité de Mécanique, Art. 62, Ed. 2^{de}: "Toutefois, on doit avoir un idée préalable de l'égalité et du rapport des masses, indépendamment de la pesanteur, qui n'est qu'une propriété sécondaire des corps, puisqu'elle deviendrait tout-à-fait insensible, sans que les masses eussent changé, en les transportant à une distance suffisamment grande de la terre." Thus M. Poisson thinks that such a mode of determining mass would not be sufficiently general for even terrestrial matter.

CHAPTER VIII.

THE RECTILINEAR MOTION OF PARTICLES.

SECTION 1.—*Direct impact and collision.*

262.] We proceed now to the application of the principles and equations which have been investigated in the preceding Chapter: and we shall begin with the most simple case, that of the direct impact and collision of two material particles. To fix our thoughts, however, I shall consider these particles to be spherical homogeneous balls, which move so that all the particles describe equal and parallel paths, and the balls have therefore no motion of rotation; the velocities also of the balls will be supposed to be uniform both before and after collision, and the paths along which they move are supposed to be rectilineal; also the line of action of the mutual pressure of the balls during the collision is supposed to pass through their centres; and if this line is that in which the balls are moving the impact is said to be *direct;* but if either of the balls moves in a line not coincident with this line of action the impact is called *oblique.* We shall now investigate the former case: the latter will be considered in Section 1, Chapter X.

Let the masses of the two material particles be m and m'; and, to fix our thoughts, let us suppose them to be moving with uniform velocities in the same direction along the straight line OA, fig. 85, say, from left to right: let v and v' be their respective velocities, and let us suppose v to be greater than v', so that m overtakes and impinges on m': the momenta of the two balls are respectively mv and $m'v'$.

Now no matter is perfectly rigid; all is more or less extensible, compressible, and also elastic. Thus when m impinges on m', a compression of the particles of the two balls at, and about, the point of contact takes place: a change of form of the balls thus takes place, and the molecules of them move one relatively to another: velocity therefore has been impressed on them. The disturbance of the relative positions of the elements of the bodies also brings elastic forces of restitution into action: for the effects of the impact are supposed not to be such that the balls are broken or crushed by them: and the greater the disturbance of the particles is, the greater is this elastic force: now although according to the configuration of the balls which we have imagined, the velocity of m is greater than that of m', yet during the collision momentum is being withdrawn from m and is transferred (1) to m' by the means of all its particles, whereby the velocity of m' is increased; and (2) to the particles which are disturbed in and about the place of contact: the limit of this latter transferred momentum is the elastic force; this transference of momentum continues until m and m' move with the same velocity; which circumstance eventually occurs: for so long as the velocity of m is greater than that of m', the change of the forms of the balls is increased, whereby the elastic force is also increased; and as this increases in a greater proportion than the compressing force, the two balls must ultimately move with the same velocity: at this stage of the process, the compression is, it is to be observed, a maximum.

As soon however as the balls move with the same velocity, there is no mutual pressure between them: there is then no force to counteract the elastic forces which have been brought into action by the compression, and these therefore begin to produce their effects. Now the common velocity of m and m' at the instant, when the compression is a maximum, is from O towards A; and the effect of the elastic forces in the restitution of the figure is to increase the velocity of m' and to diminish that of m in that direction. In other words, during the compression, momentum of the balls is changed into elastic moving force: and in the restitution, this elastic force again produces momentum: and in both processes momentum is abstracted from m and is given to m'.

A question however arises: What relation does the momentum impressed by the elastic force during the restitution of the forms bear to that which was lost by m during the compression? Here, in our ignorance of the constitution of bodies and of their molecular action, we are obliged to have recourse to experiment; and it is found that in two balls of given substances there is always a certain definite ratio between the momentum spent in producing a certain compression, and that acquired during the restitution; the latter quantity being always less than the former: the ratio is called the measure of the restitution of the bodies, and is symbolised by e; the limiting values of e are 0 and 1: the former being its value for substances perfectly inelastic, and perfectly hard, because if a body is perfectly hard there is no compression, and therefore there is no elastic force of restitution: and the latter being the value of e when the bodies are perfectly elastic, and when the momentum recovered during the restitution is equal to that spent in producing the compression. Mr. Hodgkinson has not found in the course of his experiments (see British Association Reports, Vol. III, p. 534). any matter perfectly fulfilling these conditions. Hence the value of e for all known substances is a positive proper fraction. If therefore P represents the momentum impressed during the compression, $e\mathrm{P}$ is that acquired during the restitution.

263.] Let m and m' be the masses of the two balls, which move in the same direction along the straight line OA, see fig. 85, with uniform velocities v and v': and let us suppose v to be greater than v', so that m overtakes and impinges on m': let u be the common velocity of the two balls when the compression is a maximum: let P represent the momentum spent in producing the compression, and $e\mathrm{P}$ that acquired in the restitution of the form of the bodies. Let V and V' be the velocities of m and m', when the collision ceases; and which are their uniform velocities after the collision has taken place. We shall consider the circumstances of the balls as they are (1) at the instant when collision begins, (2) at the instant when the compression is a maximum, (3) when the collision has ceased. Now

mv = the momentum of m at the beginning of the collision,
P = the momentum spent in producing compression,
mu = the momentum of m when the compression is a maximum;

therefore, by reason of (19), Art. 257,

$$mv = mu + \mathrm{P}; \tag{1}$$

$m'v'$ = the momentum of m' at the beginning of the collision,
$m'u'$ = the momentum of m', when the compression is a maximum;

$$\therefore \quad m'v' = m'u - \mathrm{P}: \tag{2}$$

and at the instant when the collision ceases, we have by a similar process,

$$m\mathrm{V} = mu - e\mathrm{P}, \tag{3}$$

$$m'\mathrm{V}' = m'u + e\mathrm{P}: \tag{4}$$

and therefore adding (1) and (2), and (3) and (4),

$$u = \frac{mv + m'v'}{m + m'} = \frac{m\text{V} + m'\text{V}'}{m + m'}; \tag{5}$$

$$\therefore \quad mv + m'v' = m\text{V} + m'\text{V}': \tag{6}$$

therefore the sums of the momenta before and after impact are equal.

From (1) and (5) we have

$$\text{P} = \frac{mm'}{m + m'}(v - v'); \tag{7}$$

therefore the momentum spent in producing the compression varies as the difference between the velocities before impact.

Substituting in (3) and (4) from (5) and (7), we have

$$\text{V} = \frac{mv + m'v'}{m + m'} - \frac{em'}{m + m'}(v - v'); \tag{8}$$

$$\text{V}' = \frac{mv + m'v'}{m + m'} + \frac{em}{m + m'}(v - v'); \tag{9}$$

and thus the velocities of the balls after collision are expressed in terms of their masses, the coefficient of restitution, and their velocities before impact.

The momentum which is impressed on m in a direction opposite to that of its motion, by the elastic force $e\text{P}$ during the restitution of the form of the balls, may be such as either wholly to neutralize the velocity of m and thus to bring it to rest, or to impress on it a velocity in the opposite direction. In the latter case, V will have a negative sign, and we shall have

$$\frac{v}{v'} \text{ greater than } \frac{(1 + e)m'}{em' - m}.$$

If m' before impact moves in a direction opposite to that which we have imagined, and so as to meet m, v' must be affected with a negative sign in all the preceding formulae; in which case if $mv = m'v'$, that is, if the momenta of the impinging balls are equal, $u = 0$, and the balls are at rest at the instant when the compression is a maximum; and after the restitution has taken place, $\text{V} = -ev$, and $\text{V}' = ev'$, and thus the balls move in opposite directions.

Defining vis viva, as in Art. 259, by one half of the product of the mass of the moving particle or ball and the square of the velocity, and noting that this is the equivalent of work, the sum of the vires vivae of the balls before collision is $\frac{mv^2 + m'v'^2}{2}$; and after collision is $\frac{m\text{V}^2 + m'\text{V}'^2}{2}$; so that by (8) and (9) we have

$$\frac{m\mathrm{v}^2+m'\mathrm{v}'^2}{2} = \frac{mv^2+m'v'^2}{2} - \frac{(1-e^2)mm'}{2(m+m')}(v-v')^2; \qquad (10)$$

and therefore in the case of imperfectly elastic balls, when e is less than unity, vis viva, and consequently work, is lost by collision.

Also since the balls after impact move with constant velocities v and v', they in t units of time severally pass over $\mathrm{v}t$ and $\mathrm{v}'t$ units of distance: and therefore the distance between them

$$\begin{aligned} &= (\mathrm{v}'-\mathrm{v})t \\ &= e(v-v')t. \end{aligned}$$

Also $$\mathrm{v}'-\mathrm{v} = e(v-v'),$$

that is, the relative separation after impact is to the relative separation before impact as e is to 1.

264.] Let us consider some special cases of the preceding results.

Ex. 1. Let the elasticity be perfect: $e = 1$: then

$$\mathrm{v} = v - \frac{2m'}{m+m'}(v-v'), \qquad (11)$$

$$\mathrm{v}' = v' + \frac{2m}{m+m'}(v-v'); \qquad (12)$$

and also if $m = m'$, $\mathrm{v} = v'$, $\mathrm{v}' = v$; that is, when a perfectly elastic ball impinges on another equal and perfectly elastic ball, each after impact moves with the velocity of the other before impact; if therefore one is at rest before impact, the impinging ball remains at rest after impact, and the other will move with the velocity of the impinging ball. Hence if there is a row of equal and perfectly elastic balls in a straight line; and if the first ball moves in that line with a velocity v, and impinges on the second, the first will be brought to rest, and the second will move on with the velocity v: similarly will it, after impact on the third ball, be brought to rest, and the third ball will move with the velocity v; and so on through all the balls, until finally the last ball moves with a velocity v and all the others are reduced to rest. Now as this result does not depend on the distances between the balls, it will be true if the balls touch each other; and thus if there is a row of equal and perfectly elastic balls in a straight line, which touch each other, if one of the extreme balls moves with the velocity v, and impinges on the next ball with a velocity v in the direction of the row of balls, the intermediate balls will not be disturbed, and the last will move with the velocity of the impinging ball.

Also if $e = 1$,
$$\frac{m\text{v}^2 + m'\text{v}'^2}{2} = \frac{mv^2 + m'v'^2}{2}; \tag{13}$$
that is, the sum of the vires vivae is the same before and after impact. This is an instance of the general law of dynamics, viz. the conservation of work.

Ex. 2. If the bodies are wholly inelastic, $e = 0$; also if they are perfectly hard, so that no change of form is caused by the impact, then no elastic force is brought into action, and $e = 0$. In these cases
$$\text{v} = \text{v}' = \frac{mv + m'v'}{m + m'}; \tag{14}$$
that is, the balls after impact move together, and of course with the same velocity.

Ex. 3. If m' is infinitely greater than m, and if $v' = 0$, or, which is the same thing, if m impinges on a fixed obstacle, as on a fixed plane, then
$$\text{v} = -ev; \tag{15}$$
that is, the ball rebounds with a velocity which is e times that of impact, and in an opposite direction.

And if the elasticity is perfect, $e = 1$, and
$$\text{v} = -v; \tag{16}$$
that is, the velocity of rebound is equal and opposite to that of impact.

And if $e = 0$, $\text{v} = 0$, and the ball remains in contact with the plane.

265.] The velocity of the centre of gravity or mass-centre is not changed by the alteration which the velocities of the balls undergo by reason of the impact.

Let x and x' be the distances of the centres of m and m' from O, fig. 85, at the time t: so that their velocities along OA at that time are $\frac{dx}{dt}$ and $\frac{dx'}{dt}$; and thus
$$v = \frac{dx}{dt}, \qquad v' = \frac{dx'}{dt}.$$
Let $\bar{x}$ be the distance from O of their mass-centre; then by (110) Art. 125,
$$(m + m')\bar{x} = mx + m'x';$$
$$\therefore \quad (m + m')\frac{d\bar{x}}{dt} = m\frac{dx}{dt} + m'\frac{dx'}{dt}$$
$$= mv + m'v'; \tag{17}$$
and is therefore equal to the sum of the momenta of the balls: but by (6) the sum of the momenta is the same before and after

impact: and therefore the velocity of the centre of gravity is the same before and after impact. The same property is also true of any number of balls directly impinging on each other in a straight row.

266.] Examples illustrative of the preceding equations:

Ex. 1. Determine the velocity of a given ball m which impinges on another equal ball moving with a given velocity, when the impinging ball remains at rest after the collision.

Here $m' = m$, and $\mathrm{v} = 0$: therefore from (8),

$$v = -\frac{1+e}{1-e}v'.$$

Ex. 2. To determine the mass of a ball m_2, which, interposed between m_1 and m_3 is such that the velocity of m_3, which is originally at rest, may after impact from m_1 through the intervention of m_2 be a maximum.

Let v be the velocity of m_1 at first: then

$$\text{the vel. of } m_2 \text{ after impact from } m_1 = \frac{(1+e)m_1 v}{m_1+m_2};$$

$$\therefore \quad \text{the vel. of } m_3 \text{ after impact from } m_2 = \frac{(1+e)^2 m_1 m_2 v}{(m_1+m_2)(m_2+m_3)}$$

$$= f(m_2), \text{ say}:$$

$$\therefore \quad f'(m_2) = \frac{(1+e)^2 m_1 (m_1 m_3 - m_2^2) v}{(m_1+m_2)^2 (m_2+m_3)^2} = 0,$$

if $m_2 = (m_1 m_3)^{\frac{1}{2}}$, and changes sign from + to −: therefore the value of $f(m_2)$ is a maximum, if m_2 is a mean proportional between the two extreme balls.

Ex. 3. n balls $m_1, m_2, m_3, \dots m_n$, perfectly elastic, are placed in a row: find the ratio of their masses, when a momentum $m_1 v$ impressed on the first is after impact equally divided amongst the n balls.

$$\text{vel. of } m_1 \text{ after impact on } m_2 = \frac{m_1-m_2}{m_1+m_2}v; \tag{18}$$

$$\text{vel. of } m_2 \text{ after impact from } m_1 = \frac{2m_1 v}{m_1+m_2};$$

$$\text{vel. of } m_2 \text{ after impact on } m_3 = \frac{m_2-m_3}{m_2+m_3}\,\frac{2m_1 v}{m_1+m_2}; \tag{19}$$

$$\text{vel. of } m_3 \text{ after impact from } m_2 = \frac{2m_2}{m_2+m_3}\,\frac{2m_1 v}{m_1+m_2};$$

$$\text{vel. of } m_3 \text{ after impact on } m_4 = \frac{(m_3-m_4)\,4m_1 m_2 v}{(m_3+m_4)(m_2+m_3)(m_1+m_2)}; \tag{20}$$

and so on: therefore from (18), (19), and (20),

$$\frac{m_1 v}{n} = \frac{m_1 - m_2}{m_1 + m_2} m_1 v = \frac{2 m_1 m_2 (m_2 - m_3) v}{(m_2 + m_3)(m_1 + m_2)}$$

$$= \frac{4 m_1 m_2 m_3 (m_3 - m_4) v}{(m_3 + m_4)(m_2 + m_3)(m_1 + m_2)} = \cdots;$$

$$\therefore \quad \frac{m_1 - m_2}{m_1 + m_2} = \frac{1}{n}; \qquad \frac{m_2 - m_3}{m_2 + m_3} = \frac{1}{n-1}; \qquad \frac{m_3 - m_4}{m_3 + m_4} = \frac{1}{n+2}; \cdots$$

$$\therefore \quad \frac{m_1}{n+1} = \frac{m_2}{n-1}; \qquad \frac{m_2}{n} = \frac{m_3}{n-2}; \qquad \frac{m_3}{n-1} = \frac{m_4}{n-3}; \cdots$$

$$\therefore \quad \frac{m_r}{n-r+2} = \frac{m_{r+1}}{n-r};$$

which gives the ratio of every two successive balls.

267.] The theory of impact may also be applied to the determination of the momentum lost by a body as it moves through a resisting medium.

By the law of inertia, a material particle or body which has a certain momentum continues to move in a rectilinear path, and with a constant velocity, unless it is acted on by some force; that is, unless momentum is abstracted from it or is communicated to it. Now if a particle or body moves in a vacuum, its velocity is not affected by any action of the medium through which it passes, because there are no material particles to be displaced by the body in its passage: but if the particle moves in a resisting medium, such as air or water, or in any other medium gaseous or liquid, whose density is finite, the particles of the medium are to be displaced, to allow the particle to pass through the medium; that is, the particles must move, and must therefore have momentum communicated to them; and this will be abstracted from the moving body; hence it loses momentum, the amount of which it is our object now to determine. And a loss of momentum will arise not only from the displacement of the particles which the passage of the body through the medium requires, but also from the action of the particles on each other, and from their friction against the surface of the moving body, whatever the nature of that friction is: the latter causes of loss of momentum, involving data extraneous to the present subject, we cannot now determine; but of the former cause, and which is the principal one, we can determine, at least approximately, the effects.

Let the moving mass present to the resisting medium a plane face, whose area is ω, and the plane of which is perpendicular to the line of motion of the body: let m be the mass of the moving body, $\rho =$ the density of the resisting medium, and let the plane face ω be at the time t at a distance s from a fixed point in the line of its motion: let ds be the distance through which ω moves in the time dt, and let v be the velocity of the body: so that $ds = v\,dt$. In the time dt, the plane face ω will have passed over a space equal to ds, and will have impinged upon, and communicated momentum to, all the particles of the medium contained within a small cylindrical surface, of which ω is the base and $ds = v\,dt$ is the altitude; then as ρ is the density, and as the particles move with a velocity v so as to allow the body to pass through, a velocity v will have been communicated to the mass $\rho\omega ds$, that is, to $\rho\omega v\,dt$; and as this moves with a velocity v, its momentum is $\rho\omega v^2 dt$; and this has been abstracted from the moving mass; therefore by reason of (22), Art. 258,

$$-m\,dv = \rho\omega v^2 dt;$$

$$\therefore \quad m\frac{dv}{dt} = -\rho\omega v^2: \qquad (21)$$

the resistance of the medium therefore will have caused to the moving body a loss of momentum which varies as the density of the medium, as the plane area of the body on which the medium acts, and as the square of the velocity with which the body moves.

It will be seen hereafter that this result gives a loss of momentum due to the resistance just double of that which is given by the principles of fluid motion as estimated in hydro-mechanics. One source of the discrepancy doubtless arises from the fact that, as the body moves and displaces particles of the medium in front of it, it leaves an empty space behind, into which other particles at once move, and impinging against the body give momentum to it; and thus the loss of momentum, which is given in the preceding expression, is greater than what actually occurs.

From (21) the following results may be derived: since

$$-\frac{dv}{v^2} = \frac{\rho\omega}{m}dt;$$

$$\therefore \quad \frac{1}{v} - \frac{1}{v_0} = \frac{\rho\omega}{m}t, \qquad (22)$$

if v_0 is the value of v when $t = 0$; and this equation gives the velocity at any time t, if the body moves initially with the velocity v_0. Also $t = \infty$ when $v = 0$; so that the body never comes to rest.

Also since $v = \frac{ds}{dt}$, if $v = v_0$, and $s = 0$, when $t = 0$,

$$t = \frac{m}{\rho\omega v_0}\left\{e^{\frac{\rho\omega s}{m}} - 1\right\}; \tag{23}$$

which gives the relation between s and t; and consequently $s = \infty$, when $t = \infty$, and $v = 0$.

SECTION 2.—*Rectilinear motion of particles under the action of an uniformly accelerating force.*

268.] Let m = the mass of the moving particle; and let a point O, fig. 86, in its line of motion be taken as the origin: let P be its position at the time t, let OP $= x$, and let PQ $= dx$ be the space-element described in the time dt: so that if v is the velocity of m at the time t,

$$\frac{dx}{dt} = v; \qquad dx = v\,dt: \tag{24}$$

let f be the impressed velocity-increment due to, and the measure of, the accelerating (or retarding) force: then mf is the impressed momentum-increment of m in an unit of time.

Let dv be the expressed velocity-increment due to the time dt; therefore $m\,dv$ is the expressed momentum-increment due to the same time; and $m\frac{dv}{dt}$ is the expressed momentum-increment due to one unit of time: therefore by reason of (23), Art. 258,

$$f = \frac{dv}{dt} = \frac{d}{dt}\cdot\frac{dx}{dt} = \frac{d^2x}{dt^2}, \tag{25}$$

if t is the equicrescent variable: f also is to be affected with a positive or negative sign according as from (25) the action of the force makes the velocity increase or decrease as the time increases. To fix our thoughts, let f be positive, therefore

$$\frac{d^2x}{dt^2} = f. \tag{26}$$

Now suppose the circumstances of motion to be such that the velocity of $m = u$, when $t = 0$; then, integrating between limits thus assigned, we have

$$d.\frac{dx}{dt} = f\,dt;$$

$$\therefore \quad \frac{dx}{dt} - u = ft; \tag{27}$$

that is, the *increase* of velocity in the time t is ft: u is called the initial velocity.

Again, integrating, and supposing the particle to be at A (OA $= a$), when $t = 0$, we have from (27),

$$dx = u\,dt + ft\,dt;$$

$$\therefore \quad x - a = ut + \frac{ft^2}{2}; \tag{28}$$

$$\therefore \quad x = a + ut + \frac{ft^2}{2}. \tag{29}$$

If m is at the origin, when $t = 0$; $a = 0$, and

$$x = ut + \frac{ft^2}{2}; \tag{30}$$

and also if the particle starts from rest, then $u = 0$, and we have

$$x = \frac{ft^2}{2}; \tag{31}$$

in this last equation x is called the space due to f during the time t: and t is called the time to which x is due under the action of f.

Again, multiplying both sides of (26) by $2\,dx$, we have

$$d.\frac{dx^2}{dt^2} = 2f\,dx;$$

and supposing the velocity of the particle to be u when $x = 0$, so that u and 0 respectively are the inferior limits of the definite integrals of the sides of the equation, we have

$$\frac{dx^2}{dt^2} - u^2 = 2fx; \tag{32}$$

and if the velocity of the particle $= 0$, when $x = 0$, then $u = 0$, and we have

$$\frac{dx^2}{dt^2} = 2fx;$$

$$\therefore \quad \text{the velocity} = (2fx)^{\frac{1}{2}}.$$

Thus, if the particle m stands from rest and moves through the distance x,

$$\text{the vis viva} = \frac{mv^2}{2} = mfx; \tag{33}$$

that is, is equal to the product of the mass, the accelerating force, and the distance.

As (27) gives the relation between the velocity and the time, (33) that between the velocity and the space, and (30) or (31) that between the space and the time, it appears that when a particle moves under the action of a finite accelerating force,

(α) The velocity acquired during a given time varies as the time.

(β) The velocity acquired by the particle during its motion through a certain space varies as the square root of the space.

(γ) The space through which the particle passes varies as the square of the time.

If the force is retarding, f must be affected with a negative sign, and we have

$$\frac{dx}{dt} = \text{velocity} = u - ft, \tag{34}$$

$$\left(\frac{dx}{dt}\right)^2 = u^2 - 2fx, \tag{35}$$

$$x = a + ut - \frac{ft^2}{2}. \tag{36}$$

Also if the initial velocity is in a direction the opposite of that in which the force acts, then u is negative, and the necessary changes must be made in the preceding formulae.

And if the particle is projected with a velocity u from o in a direction contrary to that in which the accelerating force acts, it comes to rest when $\frac{dx}{dt} = 0$; that is, when

$$t = \frac{u}{f}, \quad \text{and} \quad x = \frac{u^2}{2f}. \tag{37}$$

It will be observed that two different modes of integration have been adopted in this Article, the subject of both modes being the equation (26). One mode has been the time-integration, and the other the space-integration. (27) is the result of the former, and (32) of the latter. Thus if we introduce m into both sides of (26), momentum is given by the time-integral and vis viva by the space-integral. In the case of a constant accelerating force, we are able to effect both integrations; hereafter we shall see that the choice is but seldom offered to us; and that the space-integral is the only one that we can effect. The distinction is of great importance, and will come out more prominently than at present in a future section.

If a particle moves from rest, the space described in t units of time is given by (31), and we have

$$x = \frac{ft^2}{2}.$$

Let $x_1, x_2, \dots x_n$ be the spaces described in the first, second, … nth units of time: then we have

$$\left.\begin{array}{lll}
x_1 = \dfrac{f}{2}, & \therefore & x_1 = \dfrac{f}{2} \times 1; \\
x_1 + x_2 = 4\dfrac{f}{2}, & & x_2 = \dfrac{f}{2} \times 3; \\
x_1 + x_2 + x_3 = 9\dfrac{f}{2}, & & x_3 = \dfrac{f}{2} \times 5; \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
x_1 + x_2 + \dots + x_n = n^2\dfrac{f}{2}, & & x_n = \dfrac{f}{2}(2n-1);
\end{array}\right\} \quad (38)$$

that is, the spaces described in the first, second, … nth units of time are as the numbers 1, 3, 5, … $(2n-1)$, and are therefore in an arithmetical progression, the common difference of which is f.

269.] As a full understanding of the results of a constant accelerating force is of great importance for future subjects, let us consider it in its most elementary form, and from first principles.

Let m start from rest at o, fig. 86: and let the time of its motion be resolved into equal infinitesimal elements, each of which we shall represent by τ: and let $x_1, x_2, x_3, \dots x_n$ be the spaces which it describes in the first, second, … nth time-elements; then since f is the velocity which the accelerating force impresses in an unit of time, the velocities of the particle at the end of the first, second, … nth time-elements will be

$$f\tau, \qquad 2f\tau, \qquad 3f\tau, \dots n f\tau.$$

Now imagine each successive space-element to be described in the same time τ, and with an uniform velocity through that space-element: then if θ is a symbol for a positive proper fraction, these successive uniform velocities will be

$$\theta_1 f\tau, \qquad (f+\theta_2 f)\tau, \qquad (2f+\theta_3 f)\tau, \dots \{(n-1)f+\theta_n f\}\tau;$$

and because the space is equal to the product of the time and the velocity,

$$\begin{aligned}
x_1 &= \theta_1 f\tau^2, \\
x_2 &= (1+\theta_2) f\tau^2, \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
x_n &= (n-1+\theta_n) f\tau^2;
\end{aligned}$$

$$\begin{aligned} \therefore \quad \text{the whole space} &= x_1 + x_2 + \ldots + x_n \\ &= \{1+2+\ldots+(n-1)\} f\tau^2 \\ &\qquad + (\theta_1+\theta_2+\ldots+\theta_n) f\tau^2 \quad (39) \\ &= \frac{n(n-1)}{2} f\tau^2 + (\theta_1+\theta_2+\ldots+\theta_n) f\tau^2. \quad (40) \end{aligned}$$

Let the whole space described by the particle $= x$, and let the whole time $= t$: then $t = n\tau$: and since τ is an infinitesimal time-element, n is an infinity of that order of which τ is an infinitesimal, and we have

$$n = \frac{t}{\tau};$$

therefore from (40),

$$x = \frac{t(t-\tau)}{2} f + f\tau^2 \Sigma.\theta;$$

and omitting the infinitesimals, viz. the terms involving τ and τ^2,

$$x = \frac{ft^2}{2};$$

which is the same result as (31).

270.] Some examples are added illustrative of the principles contained in the preceding articles.

Ex. 1. It is required to divide a straight line whose length is a into four parts, such that a particle under the action of a constant accelerating force which acts along the line may, starting from rest, describe each part in an equal time.

Let x_1, x_2, x_3, x_4 be the four parts: then, by equations (38),

$$\begin{aligned} \frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{5} = \frac{x_4}{7} &= \frac{x_1+x_2+x_3+x_4}{1+3+5+7} \\ &= \frac{a}{16}; \end{aligned}$$

$$\therefore \quad x_1 = \frac{a}{16}, \quad x_2 = \frac{3a}{16}, \quad x_3 = \frac{5a}{16}, \quad x_4 = \frac{7a}{16}.$$

Ex. 2. A particle moves in a straight line, under the action of an uniformly accelerating force, and describes spaces P and Q in the pth and qth units of time respectively; determine the velocity of projection, and the magnitude of the accelerating force.

Let $u =$ the velocity of projection, and let f be the accelerating force; then

The space described in one unit of time on account of the velocity of projection is u: and that due to the accelerating force in the nth unit of time $= \frac{f}{2}(2n-1)$;

$$\therefore \quad \text{P} = u + \frac{f}{2}(2p-1), \qquad \text{Q} = u + \frac{f}{2}(2q-1);$$

$$\therefore \quad f = \frac{\text{P}-\text{Q}}{p-q}, \qquad u = \frac{\text{Q}(2p-1)-\text{P}(2q-1)}{2(p-q)}.$$

Ex. 3. A particle is projected with a given velocity u in a line along which an accelerating force acts, and in a direction opposite to that of the force's action: and the time is given between its leaving a given point and its return to it: it is required to determine the velocity of projection and the whole time of motion.

Let u = the velocity with which the particle leaves the origin O: and let the time between the particle's passage through A, at a distance a from O, and its return to it be 2T: let B be the extreme point which the particle reaches: then, by Art. 268,

$$\text{OB} = \frac{u^2}{2f}, \text{ and time from O to B} = \frac{u}{f};$$

$$\therefore \quad \text{the distance AB} = \frac{u^2}{2f} - a;$$

and the time due to this distance = T: therefore by (31),

$$\frac{u^2}{2f} - a = \frac{f\text{T}^2}{2};$$

$$\therefore \quad u = (2af + f^2\text{T}^2)^{\frac{1}{2}};$$

and the whole time of motion $= \frac{2u}{f} = 2\left(\text{T}^2 + \frac{2a}{f}\right)^{\frac{1}{2}}$.

SECTION 3.—*On gravity as an uniformly accelerating force.*

271.] In the Chapter on Attractions it is shewn that the attraction, on an external particle m, of a sphere consisting of homogeneous concentric shells, the density of each one of which may be different, is the same as if the whole sphere were condensed into its centre, and therefore the attraction of such a sphere on an external particle varies as the square of the distance of the particle from the centre of the sphere. Hence if a particle moves in vacuo towards such a sphere, and under the influence of its attraction, the law of force is that of the inverse square of the distance from the centre of the sphere. But when the attracted particle is nearly on the surface of such a sphere, and moves only over distances which are small in comparison of the radius of the sphere, the variation of the attraction is so small that it may be neglected, and the accelerating force may be considered constant. The same result also follows from the investigation of the attraction of a plate of infinite extent on an external particle which is given in Art. 192. If the attracted particle lies within the surface of the sphere, the law of attraction depends on the densities of those concentric shells of which the sphere is composed, and which are within that concentric spherical surface on which the attracted particle is; for the resultant attraction of all the shells external to that one vanishes.

Now these results are approximately applicable to the attraction of the earth on particles and on bodies; only *approximately*, I say: because the mean bounding surface of the earth is not a sphere, but approximately an oblate spheroid, of which the equatorial diameter is 7925 miles, and the polar diameter is 7899 miles; and thus the ratio of these diameters is nearly that of the numbers 299 to 298. Now the effect of this oblateness (1) is an increase in the earth's attraction, and thus in weight and in the accelerating force of gravity, on particles at or near to the earth's surface as we pass from the equator to the poles; and the amount of this increase is in weight about the 590th part of the weight of a body at the equator: (2) is a change of the line of action of the earth's attraction. If the earth were a sphere consisting of homogeneous concentric shells, the line of action on a given particle would be the line joining the position of the particle and the earth's centre: as the case now is, the line of action is, by the principle of fluid-equilibrium, perpendicular to the surface of still water at the place: and is therefore along the normal to the spheroid: all these lines of action therefore touch the evolute-surface of the spheroid, but do not pass through the centre. Laplace has calculated the effect of the oblateness of the earth on the motion of the moon; and observation verifies his results.

Again, as the earth rotates about its polar diameter, the centrifugal force, which diminishes the weight of particles and the earth's accelerating force on particles near to the surface, is greatest at the equator, and is zero at the poles: of this cause of diminution and its measure we shall speak hereafter: I may observe, however, that at the equator the weight of a body is diminished by about the 289th of its true weight, and that the effect of centrifugal force in passing from the equator to the poles varies as the square of the cosine of the latitude.

Also the effect of the earth's accelerating force varies by reason of local causes: it is affected by neighbouring mountains both as to intensity and as to line of action: it is different on an island which is surrounded by a large mass of water, and on a continent: it even varies, as delicate observations with the time-measuring pendulum shew, with the materials of the earth at the place of observation: thus may the pendulum, as M. Poisson observes, and as we shall shew hereafter, become an indicator of geological conditions.

Gravity also manifestly varies with the altitude of a place above the level of the sea: experiments however, by which its value has been determined, are supposed to be made at the level of highwater-mark.

272.] And notwithstanding all these variations of the earth's attraction, for bodies near to the surface the accelerating force due to it is nearly constant at any given place, and increases as we pass from the equator to the pole; and decreases as we remove farther from the centre of the earth. The exact measure of it as an accelerating force, that is, the velocity-increment which it impresses on an unit-particle in one unit of time, for a given place is, of course, to be determined by experiment: and at Greenwich, if one second is the unit of time, at the level of highwater-mark, and in vacuo, the most exact pendulum experiments exhibit a velocity-increment of 386.28 inches, that is, of 32.19 feet. That is, if a particle falls in vacuo towards the earth, the excess of the velocity at the end of any second of time over that at the beginning of the second of time is 32.19 feet.

The velocity-increment is the measure of the accelerating force called *gravity;* and it is independent of the matter, form, and magnitude of bodies. Thus in the common experiments under the exhausted receiver of an air-pump, the heaviest metals and the lightest pith fall from rest through the same distance in the same time, and acquire equal velocities. And also the time of oscillation of a pendulum is independent of the matter of which the pendulum is made: gravity therefore as an accelerating force is independent of the particular kind of matter which it communicates velocity to.

273.] As to the experimental proof that gravity is an uniformly accelerating force: when a heavy particle or body falls freely by itself in vacuo, the velocity of it quickly becomes so great and increases so rapidly that the law of the increase cannot be observed with accuracy; hence arises the need of some contrivance which may diminish the velocity, and not change the law. There are chiefly two contrivances for this purpose: firstly, Attwood's machine; in which two unequal masses, differing slightly from each other in weight, are connected by a very fine, and, as nearly so as may be, flexible and inextensible string: this is suspended over a pulley, see fig. 87, ABC; which, by means of friction-wheels and other appliances whereby friction is diminished, moves as easily as possible. Of course the greater mass descends; and as both the masses move with the same velocities, their expressed momentum-increment is the product of the sum of their masses and their common expressed velocity-increment; and their impressed momentum-increment is that due to the excess of the momentum-increment of the larger mass over that of the smaller; that is, is due to the difference of their weights. And these momentum-increments are equal, except that some small part of the impressed momentum is spent in producing the velocity of the pulley and of the string, which we at present neglect. Now as the difference of the weights of the two masses may be as small as it is convenient, so may the expressed velocity-increment of the masses be diminished as much as we please, and we are thereby enabled to measure the rate of increase of the velocity, and also the whole velocity which is expressed in a given time: and after very careful and numerous observations it is found that,

(1) The velocity of the descending mass varies as the time during which it has been in motion from rest.

(2) The spaces described by the descending mass vary as the squares of the times during which they are described.

(3) The spaces described in successive units of time vary as the odd numbers $1, 3, 5, \dots (2n-1)$.

And as these results are in accordance with those which have been deduced in Art. 268, when the accelerating force is constant; and as the processes by which these results were proved may be inverted; it follows that the moving force by which, in Attwood's machine, moment-increment is impressed, is uniform;

and therefore the earth's attraction, or gravity, is an uniformly accelerating force *.

Secondly, the oscillating pendulum is a contrivance by which great accuracy is attained, in which the velocity-increment is easily measured, and which is actually employed for the purpose. To the lower end of a fine straight rigid rod a body is attached, the mass of which is so large in comparison of that of the rod, that the mass of the latter may approximately be neglected: the upper end of the rod is fixed to a horizontal axis, about which the whole rod and body vibrates freely. Now if the rod is moved out of its position of rest, and turns about this axis, the rod and body will vibrate; let the vibrations be small, and let the motion take place wholly in one plane: then it is observed that the oscillations are isochronous, that is, are performed in equal times. In a future Chapter it will be shewn that such isochronism of bodies moving in small circular arcs can be true only when the accelerating force is constant; and therefore we infer that the force of gravity under the action of which these isochronous oscillations are performed is a constant accelerating force.

274.] In the following examples of the action of gravity, the time-unit is taken to be one second, the space-unit one foot; and the velocity-increment is supposed to be 32.2 feet (rather greater than its correct value 32.19 feet in the latitude of Greenwich) for facility of calculation, and is symbolized by g; and m is the mass of the moving particle.

We will consider the case (1) of a falling body: (2) of a body projected vertically upwards with a certain velocity: and in both cases I would observe that if the time results with a negative sign, it expresses an epoch anterior to that at which we suppose our time to commence.

(1) The motion of a heavy particle m falling towards the earth.

Let a certain point O, (*a*) fig. 88, in the line of the particle's motion be taken as the origin: and let $x = \text{OP}$ be its distance from O at the time t: then, if $dx = \text{PQ}$ is the space described in dt, that is, in dt units of time,

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* From this Article, and from Art. 260, it appears that two principal results are established by Attwood's machine: (1) the matter of the earth is such, that the expressed momentum-increment is equal to the product of the mass and the expressed velocity-increment: (2) gravity is an uniformly accelerating force.

$$\frac{dx}{dt} = \text{the space described in one unit of time,}$$
$$= \text{the velocity at the point P, and at the time } t,$$
$$= v \text{ (say)};$$

and therefore $m\dfrac{dx}{dt} = mv$ is the momentum of the particle at that time. Hence $\dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$ is the expressed velocity-increment in an unit of time, and $m\dfrac{d^2x}{dt^2}$ is the expressed momentum-increment in an unit of time. This last expression is to be equated to mg, which is the earth's impressed momentum-increment on m due to a second of time; so that we have

$$m\frac{d^2x}{dt^2} = mg;$$

$$\therefore \quad \frac{d^2x}{dt^2} = g. \tag{41}$$

Now of this equation let us first take the time-integral; then if u = the velocity of m, when $t = 0$, the definite integral of (41), the superior and inferior limits on both sides corresponding to $t = t$ and to $t = 0$, is

$$\frac{dx}{dt} - u = gt; \tag{42}$$

$$\therefore \quad \frac{dx}{dt} = \text{the velocity of } m = u+gt; \tag{43}$$

that is, the velocity is equal to the sum of the initial velocity, and of that which gravity has impressed in t''; and if the particle is projected upwards from O in a direction contrary to that in which x is measured and g acts, then

$$\frac{dx}{dt} = gt - u. \tag{44}$$

Again integrating (43), and supposing a to be the distance of m from O when $t = 0$, let us take the definite integrals with limits corresponding to $t = t$ and to $t = 0$; and since

$$dx = (u+gt)\,dt; \qquad \therefore \quad x - a = ut + \frac{gt^2}{2};$$

$$x = a + ut + \frac{gt^2}{2}. \tag{45}$$

Next let us take the space-integral of (41), and multiplying both sides by $2\,dx$, we have

$$d.\frac{dx^2}{dt^2} = 2g\,dx;$$

and taking the same limits of integration as before,

$$\therefore \quad \frac{dx^2}{dt^2} - u^2 = 2g(x-a); \tag{46}$$

$$\therefore \quad \frac{dx^2}{dt^2} = (\text{vel.})^2 = u^2 + 2g(x-a). \tag{47}$$

from which the equation of vis viva and of work may be deduced; for from it, if v = the velocity at the time t, we have

$$\frac{m}{2}(v^2-u^2) = mg(x-a), \tag{48}$$

of which the left-hand member expresses vis viva as defined in Art. 259, and the right-hand is the work taken from the mass m as it moves through the vertical distance $x-a$.

If when $t=0$, $x=0$ and the particle is at rest, then $a=0$, and $u=0$, and

$$\frac{dx}{dt} = gt; \qquad x = \frac{gt^2}{2}; \qquad \frac{dx^2}{dt^2} = 2gx. \tag{49}$$

Whence in a particle falling from rest,
(1) the expressed velocity varies as the time;
(2) the expressed velocity varies as the square root of the space;
(3) the space varies as the square of the time.

And generally, (43) gives the velocity in terms of the time: (45) gives the space in terms of the time: and (46) gives the velocity in terms of the space: (45) is of course identical with the equation which would result from the elimination of the velocity by means of (43) and (46).

(2) Suppose the particle m to be projected vertically upwards from O, see (β), fig. 88; OP$=x$, PQ$=dx$; and let OH $= h$, where H is the highest point which m reaches; and let u = the velocity of projection from O: then since g in this case causes both the velocity and the distance to decrease as t increases,

$$m\frac{d^2x}{dt^2} = -mg;$$

$$\therefore \quad \frac{d^2x}{dt^2} = -g: \tag{50}$$

and taking definite integrals with limits the same as heretofore,

$$\frac{dx}{dt} - u = -gt,$$

$$\frac{dx}{dt} = u - gt, \tag{51}$$

$$x = ut - \frac{gt^2}{2}; \tag{52}$$

also from (50), $$\frac{dx^2}{dt^2} - u^2 = -2gx,$$
$$\frac{dx^2}{dt^2} = u^2 - 2gx. \qquad (53)$$

When $\frac{dx}{dt} = 0$, the particle comes to rest; therefore from (51) and (53),

$$\text{the time, when } m \text{ comes to rest, } = \frac{u}{g}; \qquad (54)$$

$$\text{the distance from O to H} = \frac{u^2}{2g}; \qquad (55)$$

after m has come to rest at H, it begins to descend: and from the preceding formula it is manifest that the time of the descent from H to O is equal to that of the ascent from O to H; also that the velocity acquired in the descent is equal to that lost in the ascent. These results are also evident from first principles.

275.] Ex. 1. A particle falls from rest; determine its velocity, and the space which it has described at the end of 6″.

$$\frac{d^2x}{dt^2} = g; \qquad \frac{dx}{dt} = gt; \qquad x = \frac{gt^2}{2};$$

∴ the velocity at the end of 6″ = 6 × 32.2 feet;

the space described during 6″ = $\frac{32.2 \times 36}{2}$ feet.

Ex. 2. A particle is projected vertically upwards with a velocity of 100 feet in one second: find the height to which it ascends, and the time of its ascent.

$$\frac{d^2x}{dt^2} = -g;$$
$$\frac{dx}{dt} - u = -gt;$$
$$\frac{dx^2}{dt^2} - u^2 = -2gx;$$

∴ when $\frac{dx}{dt} = 0$, $\quad t = \frac{u}{g} = \frac{1000}{322}, \qquad x = \frac{u^2}{2g} = \frac{10000}{64.4}.$

Ex. 3. A particle is projected upwards with a velocity u; find the time which intervenes between its leaving, and returning to, a given point in its path.

Let a = the distance of the given point A from O, the point of projection: then if t is the time from O to A,

$$a = ut - \frac{gt^2}{2};$$

$$\therefore \quad t = \frac{u}{g} \pm \frac{(u^2-2ag)^{\frac{1}{2}}}{g}; \tag{56}$$

and the time to the highest point $= \frac{u}{g}$.

In (56) the upper sign refers to the passage of the particle through the given point in its descent, and when it has left the highest point; and the lower sign refers to the passage of m through the given point in its first ascent: therefore

$$\text{the intervening time} = \frac{2(u^2-2ag)^{\frac{1}{2}}}{g}.$$

Ex. 4. With what velocity must a particle be projected downwards, that it may in n'' overtake another particle which has already fallen through a feet.

Let $u =$ the required velocity: therefore the space which the first particle will pass through in n'' is

$$un + \frac{gn^2}{2};$$

and the velocity which the second particle has, when the former starts from rest, is $(2ag)^{\frac{1}{2}}$: therefore at the end of n'', its distance from the origin, is $a+(2ag)^{\frac{1}{2}}n+\frac{gn^2}{2}$; which, being equated to the preceding distance, gives

$$u = \frac{a}{n} + (2ag)^{\frac{1}{2}}.$$

Ex. 5. A particle whose elasticity is e falls through a given vertical distance a and strikes a horizontal plane, whence it rebounds, and falls again; and so on continually: find the whole space which it passes through before it comes to rest.

By (49), the velocity of impact on the plane $= (2ag)^{\frac{1}{2}}$;

$\therefore$ by (15), Art. 264, the velocity of rebound $= e(2ag)^{\frac{1}{2}}$:

$\therefore$ the height to which the particle ascends, by (55), $= e^2 a$; similarly after the second impact, the height $= e^4 a$; and so on: therefore the whole space $= a+2\{e^2a+e^4a+\dots\}$

$$= a + \frac{2e^2a}{1-e^2}$$

$$= a\frac{1+e^2}{1-e^2}.$$

276.] Let us also investigate, and apply to certain examples, the equations of motion of two given masses connected by a fine inextensible and flexible string, which is suspended over a pulley, as in Attwood's machine, see fig. 87: we shall suppose the pulley and the string to be without inertia, so that no part of the impressed momentum is spent in giving velocity to them.

Let m and m' be the masses respectively at P and Q at the time t: of these let us suppose m to be the greater, so that P descends: let $\text{AP} = x$, $\text{BQ} = x'$: then, since the string is inextensible,

$$x + x' = \text{a constant};$$

$$\therefore \quad \frac{dx}{dt} + \frac{dx'}{dt} = 0, \qquad \text{and} \qquad \frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} = 0; \tag{57}$$

whence we conclude that the velocity and the velocity-increments of the two particles are equal and have opposite signs.

Now the whole mass which receives and develops velocity-increment is $m + m'$; and as the whole of this has $\frac{d^2x}{dt^2}$ for its velocity-increment, the expressed momentum-increment

$$= (m + m')\frac{d^2x}{dt^2};$$

and the impressed momentum-increment is the excess of that of m over that of m'; that is, is $mg - m'g$; therefore

$$(m + m')\frac{d^2x}{dt^2} = mg - m'g; \tag{58}$$

$$\frac{d^2x}{dt^2} = \frac{m - m'}{m + m'}g; \tag{59}$$

from which equation all the circumstances of a motion such as we have supposed are to be deduced. The corresponding equation of motion of m' is

$$\frac{d^2x'}{dt^2} = \frac{m' - m}{m' + m}g. \tag{60}$$

As to the circumstances of the initial velocity: suppose m to be projected vertically downwards, so that if it were free it would have the velocity a; and similarly let m' be projected vertically downwards, so that if it were free it would have the velocity a': and let the velocity with which, by virtue of these two separate velocities, m and m' move when connected by the string, be u: then

$$(m + m')u = ma - m'a'; \tag{61}$$

$$\therefore \quad u = \frac{ma - m'a'}{m + m'}; \tag{62}$$

which gives the initial velocity with which m begins to descend, if ma is greater than $m'a'$; and with which m' begins to descend if $m'a'$ is greater than ma.

Let the initial value of x be a: therefore from (59),

$$\frac{dx}{dt} - u = \frac{m-m'}{m+m'} gt; \tag{63}$$

$$\frac{dx^2}{dt^2} - u^2 = 2\frac{m-m'}{m+m'} g(x-a); \tag{64}$$

$$x = a + ut + \frac{m-m'}{m+m'}\frac{gt^2}{2}; \tag{65}$$

u being given by (62).

And if a' is the initial value of x', we have

$$x' = a' - ut - \frac{m-m'}{m+m'}\frac{gt^2}{2}. \tag{66}$$

277.] Examples illustrative of the preceding formulae:

Ex. 1. $m = 16.6$ oz.: $m' = 15.6$ oz., and they start from rest: required the space through which m passes in 5″, and the velocity which it has at the end of the time. From (63) and from (65),

$$\text{the velocity of } m = 5 \text{ feet};$$
$$\text{the space} = 12.5 \text{ feet}.$$

Ex. 2. A mass of 10 lbs. is distributed at the ends of a thin cord passing over a fixed pulley, so that the heavier weight descends through $3g$ feet in 10″: it is required to find the weights at each end of the cord.

Let $w =$ one weight; therefore $10-w =$ the other: then from (65),

$$x = \frac{m-m'}{m+m'}\frac{gt^2}{2}; \qquad \therefore \quad 3g = \frac{2w-10}{10}\,\frac{100g}{2};$$
$$\therefore \quad w = 5.3, \qquad 10-w = 4.7.$$

Ex. 3. A heavy mass m draws another m' by means of a flexible and inextensible string over a pulley: at the starting of the weights, m is thrown downwards through a feet, and m' through a' feet: it is required to determine the distance through which m descends in t''.

Using the notation of equation (62),

$$a = (2ag)^{\frac{1}{2}}, \qquad a' = (2a'g)^{\frac{1}{2}};$$

$$\therefore \quad u = \frac{ma^{\frac{1}{2}} - m'a'^{\frac{1}{2}}}{m+m'}(2g)^{\frac{1}{2}};$$

therefore from (65),

$$x = \frac{ma^{\frac{1}{2}} - m'a'^{\frac{1}{2}}}{m+m'}(2g)^{\frac{1}{2}}\, t + \frac{m-m'}{m+m'}\frac{gt^2}{2}.$$

Ex. 4. It is required to determine the velocity-increment of the centre of gravity of two heavy masses m and m' which are connected by a string passing over a fixed pulley.

Let x and x' be the vertical distances of m and m' at the time t below the horizontal line passing through the centre of the fixed pulley: and let $\bar{x}$ be the vertical distance from the same line of their centre of gravity. Then

$$(m+m')\,\bar{x} = mx+m'x';$$

$$\therefore\quad (m+m')\frac{d^2\bar{x}}{dt^2} = m\frac{d^2x}{dt^2}+m'\frac{d^2x'}{dt^2};$$

and substituting from (59) and (60),

$$(m+m')\frac{d^2\bar{x}}{dt^2} = m\frac{m-m'}{m+m'}g+m'\frac{m'-m}{m'+m}g;$$

$$\therefore\quad \frac{d^2\bar{x}}{dt^2} = \left(\frac{m-m'}{m+m'}\right)^2 g. \tag{67}$$

SECTION 4.—*Rectilinear motion of particles in vacuo under the action of varying accelerating forces.*

278.] The varying accelerating forces whose effects will be considered in this section are supposed to be explicit functions of the distance between the moving particle m and the point wherein the force resides, and whence its influence emanates; and the motion of the particle is supposed to be along this line. Thus the force is only implicitly a function of the time: that is, only so far as the passage of the particle through a certain distance requires time, and the distance may thus become a function of that time; and the equation of motion will be of the form

$$\frac{d^2x}{dt^2} = f(x), \tag{68}$$

and not of the form

$$\frac{d^2x}{dt^2} = f(t). \tag{69}$$

We limit our considerations to the former form chiefly, because it expresses the laws of communication of velocity which present themselves in the salient phaenomena of nature: although in some problems the latter law will also occur.

The point whence the influence of a force emanates is called *the centre of the force;* and according as the force attracts or repels, so is it called an *attractive* or a *repulsive* force.

Let us consider briefly the general case. Let m = the mass of the moving particle, and let the centre of the force be the origin O, fig. 86: let P be the position of m at the time t: let OP $= x$: let the force vary as the nth power of the distance, and let μ, which is called *the absolute* force*, be the value of it when $x = 1$ and $m = 1$; so that the impressed momentum-increment is $\mu m x^n$, which is to be affected with a positive or negative sign according as the force is repulsive or attractive. Now the expressed momentum-increment is $m\frac{d^2x}{dt^2}$: therefore by Art. 258, if the force is repulsive,

$$m\frac{d^2x}{dt^2} = m\mu x^n, \tag{70}$$

because both x and the velocity increase as t increases; and thus both $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = \frac{d^2x}{dt^2}$ are positive; and consequently dividing (70) through by m,

$$\frac{d^2x}{dt^2} = \mu x^n. \tag{71}$$

If the force varies inversely as the nth power of the distance, and is repulsive, (71) becomes

$$\frac{d^2x}{dt^2} = \frac{\mu}{x^n}; \tag{72}$$

and (71) and (72) must have negative signs if the force is attractive, because in that case x decreases as t increases, when the force makes m move towards its centre.

Instead however of deducing from these general values the circumstances of the corresponding rectilinear motion, it will be more convenient to consider the results for particular laws of force: and we shall choose such examples as will either elucidate natural phaenomena or will suggest general methods for solving problems in rectilinear motion.

279.] A particle m moves towards a centre of force which attracts directly as the distance: it is required to determine the circumstances of motion.

Let O, the centre of force, be the origin: and let P, fig. 89, be the position of m at the time t: let OP $= x$ and OA $= a$, where A is the position of the particle when $t = 0$: let $\mu =$ the absolute force: then the equation of motion is

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* In the preceding investigations on attraction, Chap. VI, I have taken the mass of the attracting body to be the absolute force, so that at an unit of distance and on an unit-mass the attraction is equal to the attracting mass: and thus the attraction of one unit-mass on another unit-mass at an unit-distance apart is made the attraction-unit.

$$\frac{d^2x}{dt^2} = -\mu x, \qquad (73)$$

$$\frac{2\,dx\,d^2x}{dt^2} = -2\mu x\,dx;$$

and if the limits of integration are those values which correspond to $t = t$ and to $t = 0$, then if the particle is at rest, when $x = a$ and $t = 0$,

$$\begin{aligned} \frac{dx^2}{dt^2} &= -\mu(x^2 - a^2) \\ &= \mu(a^2 - x^2); \qquad (74) \end{aligned}$$

$$\therefore \quad \frac{-dx}{(a^2 - x^2)^{\frac{1}{2}}} = \mu^{\frac{1}{2}}\,dt,$$

the negative sign of the root being taken, because, according to our configuration, x decreases as t increases: therefore integrating between the limits corresponding to $t = t$ and to $t = 0$,

$$\cos^{-1}\frac{x}{a} = \mu^{\frac{1}{2}} t,$$

$$\therefore \quad x = a\cos\mu^{\frac{1}{2}} t; \qquad (75)$$

$$\therefore \quad \frac{dx}{dt} = -a\mu^{\frac{1}{2}}\sin\mu^{\frac{1}{2}} t. \qquad (76)$$

From (74) it appears that the velocity of the particle is zero when $x = a$, and when $x = -a$; and is a maximum, viz. $a\mu^{\frac{1}{2}}$, when $x = 0$; the particle therefore moves from rest at A; its velocity increases until it reaches O, where it becomes a maximum, and where the force is zero: so that the particle passes through that point, and its velocity decreases, and at A′, at a distance $= -a$, becomes zero: whence the particle under the action of the force returns, and continually oscillates over the distance $2a$, of which O is the middle point. The distance a of A from O is called *the amplitude of the vibration.*

Also from (75) it appears that when $x = 0$, $t = \frac{\pi}{2\mu^{\frac{1}{2}}}$, and when $x = a$, $t = 0$; so that the time of passing from A to O $= \frac{\pi}{2\mu^{\frac{1}{2}}}$; and the time from O to A′ is the same, so that the time of the oscillation from A to A′ is $\frac{\pi}{\mu^{\frac{1}{2}}}$. This result may also be more generally inferred by the following method. The relation between x and t

is given by the periodic function (75), viz. $x = a \cos \mu^{\frac{1}{2}} t$. Now as the greatest value of a cosine is $+1$, and the least value is -1, the greatest and least values of x are $+a$ and $-a$: x also will have passed once through all its values when $\mu^{\frac{1}{2}} t$ is increased by 2π; that is, when t is increased by $\frac{2\pi}{\mu^{\frac{1}{2}}}$; this therefore is the time of a complete double oscillation; and consequently the time of one oscillation, viz. from A to A′, is $\frac{\pi}{\mu^{\frac{1}{2}}}$.

And hence we have the remarkable fact that the time of an oscillation is independent of the distance from the centre of the point from which the particle began to move, and only depends on the absolute force, and is the greater the less that is.

280.] The two following cases in nature, wherein an attraction, the law of which is that of the direct distance, presents itself, deserve mention.

(1) A homogeneous sphere attracts a particle within its bounding surface with a force varying directly as the distance from the centre of the sphere; see Ex. 1, Art. 196. Let us therefore consider the earth to be such a homogeneous sphere, and let us suppose a particle to move under the action of the earth's attraction within the shaft of a mine the direction of which is vertical. Thus if C is the centre of the earth, fig. 90, and P is the position of m at the time t, the force acting on m varies as CP; and thus if the shaft were continued straight through the earth, such as that represented by A′CPA in the figure, and if the particle were free at A, it would move to C, where its velocity would be a maximum, and thence on to A′ on the opposite side, where it would come to rest: and thence it would return through C to A again; and its motion would continue to be oscillatory, and the time of the oscillation would be independent of CA, the earth's radius.

(2) In the undulatory or wave theory of light, all space is supposed to be pervaded in a greater or less degree by the particles of a fluid excessively elastic and jelly-like; in the motion of these particles light is supposed to consist, and when they are at rest, there is darkness. It is also supposed that these particles exercise mutual attractions on each other: that the possible relative displacements of them are very small, and that when displacements occur elastic forces are brought into action,

by virtue of which, in conjunction with their mutual attractions, the motion of them continues: the lines of action, as well as the intensities of such elastic forces, of course vary from one medium to another; and in this variety consists the optical character of the medium. Doubtless the arrangement of the particles of a crystallised substance is different to that of one which is non-crystallised. The sun, the flame of a candle, and the electric spark, are, together with many others, exciting causes of the motion of the particles of ether; and the displacement of each particle is very small. It seems, too, that the force which acts on a particle in its displaced position varies directly as the distance of it from its original position of rest; this force being the resultant of the elastic forces which arise from the disturbance of the medium and of the attracting forces of the particles. Now a ray of light consists in the motion of a series of ethereal molecules which when at rest are in a straight line emanating from the source of motion. The mode of propagation of the motion of the particles it is not my purpose now to inquire into: I shall consider the motion of only a single molecule of a single ray. The displacement of a molecule may be in any direction with reference to the line of propagation of the ray: it might be along that line, or it might, after its first displacement, describe any curve with reference to that line; it is, however, in the theory of light supposed, and not without evidence, that the motion of the molecule takes place in a plane which is perpendicular to the line of propagation of the ray: that is, the displacement of the particle is transversal to the line of propagation. Generally the force acting on the molecule, varying directly as the distance, will have its line of action inclined to the line joining the displaced and the original position of the molecule; and, as we shall shew hereafter, the molecule will move in an ellipse, the centre of which is the original place of rest of the molecule: but in particular constitutions of the ethereal medium, the line of action of the force may be that joining the original and the displaced positions of the molecule: in which case the molecule moves along that line, and is under the action of a force varying directly as the distance from its original position of rest: we have then the case of a particle under the action of a force such as we have supposed that in Art. 279 to be, and the results of that Article are applicable. The particle therefore has an oscillatory motion, and the amplitudes of its vibrations are equal on both sides of its original position; and the time of the oscillation is independent of the amplitude, and depends only on the absolute force at the centre. Now the intensity of light is supposed to depend on the amplitude of the vibration, and the colour of it on the time of vibration, that is, on the value of μ: it follows therefore that, with such incidents of motion as we have imagined, the intensity and the colour may vary independently of each other: the former will depend on the original exciting cause of the motion; the latter on the nature of the medium: and this independence of these properties of light is amply verified by experiment.

If the motion of all the molecules of a ray is in straight lines, and is such as that described above, and if all the lines of motion are parallel to each other, the ray is said to be *plane-polarised;* and as a beam of light consists of an infinite number of rays, if the molecules of all the rays move in lines parallel to each other, the beam is said to be *plane-polarised.* And although there has not been uniformity on the subject, yet the plane, perpendicular to which the motion takes place, may be called *the plane of polarisation.*

281.] If in Art. 279 m is projected from A with a velocity u along the line OA, and towards O, then (74) becomes

$$\frac{dx^2}{dt^2} - u^2 = \mu(a^2 - x^2); \tag{77}$$

$$\therefore \quad \frac{-dx}{\left(a^2 + \frac{u^2}{\mu} - x^2\right)} = \mu^{\frac{1}{2}}\, dt;$$

and taking the definite integrals with limits corresponding to $t = t$ and to $t = 0$, we have

$$\cos^{-1} \frac{x}{\left(a^2 + \frac{u^2}{\mu}\right)^{\frac{1}{2}}} - \cos^{-1} \frac{a}{\left(a^2 + \frac{u^2}{\mu}\right)^{\frac{1}{2}}} = \mu^{\frac{1}{2}} t;$$

$$\therefore \quad x = a \cos \mu^{\frac{1}{2}} t - \frac{u}{\mu^{\frac{1}{2}}} \sin \mu^{\frac{1}{2}} t; \tag{78}$$

$$\text{and} \quad \frac{dx}{dt} = -a\mu^{\frac{1}{2}} \sin \mu^{\frac{1}{2}} t - u \cos \mu^{\frac{1}{2}} t. \tag{79}$$

From (77) it appears that the greatest and least distances of m from O are

$$\left(a^2 + \frac{u^2}{\mu}\right)^{\frac{1}{2}}, \quad \text{and} \quad -\left(a^2 + \frac{u^2}{\mu}\right)^{\frac{1}{2}};$$

and from (78) the time of an oscillation is, as before, $\frac{\pi}{\mu^{\frac{1}{2}}}$.

282.] If the central force varies directly as the distance and is repulsive, the equation of motion is

$$\frac{d^2x}{dt^2} = \mu x.$$

Let us suppose m to be projected from the centre of force with the velocity u; then we have

$$\frac{dx^2}{dt^2} - u^2 = \mu x^2; \tag{80}$$

$$\therefore \quad x = \frac{u}{2\mu^{\frac{1}{2}}}\{e^{\mu^{\frac{1}{2}}t} - e^{-\mu^{\frac{1}{2}}t}\}. \tag{81}$$

Thus as t increases x also increases, and the particle recedes further and further from the centre of force; and the velocity also increases and ultimately $= \infty$, when $x = t = \infty$. Thus in this case we have no oscillatory motion.

283.] From this and the preceding equation we have the following remarkable result, which is of large application and deserves careful consideration; we shall also frequently appeal to it in future parts of our treatise.

The equations of motion (73) and (80) are of the same form; viz. replacing μ by n^2, $\frac{d^2x}{dt^2} = n^2 x$, but in the former n^2 is negative, and in the latter it is positive.

Now in the former case the motion is oscillatory, and the particle never recedes from the centre of force beyond points equally distant from the centre, the position of which is determined by the initial distance of the particle at rest from the centre of force, or by the velocity with which it is projected from the centre of force or from any other given point. Also the time of an oscillation is $\frac{\pi}{n}$; and the complete periodic time is $\frac{2\pi}{n}$, during which the particle has passed through all its possible places, and has undergone all the different circumstances of its motion as to position and velocity, and at the end of which the particle is in precisely the same *phase*, as it is called, as it was at the beginning. This motion is called *harmonic motion*, and the equation

$$\frac{d^2x}{dt^2} = -n^2 x \tag{82}$$

is called *the equation of harmonic motion*. Its most general integral is $x = a\cos(nt+\alpha)$ or $x = a\sin(nt-\alpha)$, where a and α are either arbitrary constants introduced in the course of integration, or are constants determined by the limits of integration, a being the amplitude, and $\frac{a}{n}$ in the latter form the epoch at which the particle is at the centre of force.

In the latter case, where the equation of motion is

$$\frac{d^2x}{dt^2} = n^2 x, \tag{83}$$

as the time increases, the particle recedes further and further from the centre of force, and never returns. If it is originally at rest at a distance a from the centre of force, it never comes nearer to it; and if it is originally at rest at the centre of force, it never moves from that centre.

284.] A particle m moves towards a centre of force which attracts inversely as the square of the distance; it is required to determine the circumstances of motion.

Let the centre of force be the origin; and let P, fig. 91, be the position of m at the time t; let A be the position of m at rest, when $t = 0$, so that the particle is moving towards O: let $\text{OP} = x$, $\text{OA} = a$; let $\mu =$ the absolute force: and let the limits of the definite integrals correspond to $t = t$ and to $t = 0$. Then the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2};$$

$$\frac{2\,dx\,d^2x}{dt^2} = -\frac{2\mu\,dx}{x^2};$$

$$\frac{dx^2}{dt^2} = \frac{2\mu}{x} - \frac{2\mu}{a}$$

$$= \frac{2\mu(ax-x^2)}{ax^2}; \tag{84}$$

$$\therefore \quad \frac{-x\,dx}{(ax-x^2)^{\frac{1}{2}}} = \left(\frac{2\mu}{a}\right)^{\frac{1}{2}} dt,$$

the negative sign being taken, because x decreases as the time increases, according to the arrangement of our figure. Therefore integrating again, and taking the limits corresponding to $t = t$ and to $t = 0$, we have

$$(ax-x^2)^{\frac{1}{2}} - \frac{a}{2}\,\text{versin}^{-1}\frac{2x}{a} + \frac{\pi a}{2} = \left(\frac{2\mu}{a}\right)^{\frac{1}{2}} t;$$

$$\therefore \quad t = \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \left\{ (ax-x^2)^{\frac{1}{2}} + a\cos^{-1}\left(\frac{x}{a}\right)^{\frac{1}{2}} \right\}. \tag{85}$$

From (84) it appears that the velocity $= 0$, when $x = a$; and $= \infty$, when $x = 0$: thus the velocity increases as the particle approaches the centre of force, and ultimately, when it arrives at the centre, becomes infinite; and from (85) it appears that the time of passing from A to O is $\frac{\pi a^{\frac{3}{2}}}{(8\mu)^{\frac{1}{2}}}$.

If m moves from an infinite distance towards O, then $a = \infty$, and the velocity at a distance x from O $= \left(\frac{2\mu}{x}\right)^{\frac{1}{2}}$.

If m is projected from A with a velocity u, then we have

$$\frac{dx^2}{dt^2} - u^2 = \frac{2\mu}{x} - \frac{2\mu}{a},$$

and the process of integration is the same as the preceding.

This problem is that of a particle moving in vacuo from a given place above the surface of the earth towards the earth's centre, the distance through which it moves being so great that the variation of the earth's attraction due to the distance must be taken account of. In this case if R is the radius of the earth, and g is the earth's impressed velocity-increment at the surface, and is such as we have taken g to be in the preceding section, and x is the distance from the centre of the earth of the moving particle at the time t, then the equation of motion is

$$\frac{d^2x}{dt^2} = -g\frac{\text{R}^2}{x^2}; \tag{86}$$

and if the particle is projected upwards from the surface of the earth with the velocity u, we have

$$\frac{dx^2}{dt^2} - u^2 = 2g\text{R}^2\left\{\frac{1}{x} - \frac{1}{\text{R}}\right\}:$$

and the particle comes to rest, when

$$x = \frac{2g\text{R}^2}{2g\text{R} - u^2}. \tag{87}$$

If however the particle falls towards the earth, and also passes from above to below its surface, as, for instance, down a mine, the law of force changes at the surface: and having varied inversely as the square of the distance, then varies (approximately) directly as the distance.

285.] Again, let the force vary inversely as the square root of the distance and be attractive; and suppose the particle to be at rest at a distance a from the centre of force; it is required to determine the circumstances of motion.

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^{\frac{1}{2}}};$$

$$\frac{dx^2}{dt^2} = 4\mu(a^{\frac{1}{2}}-x^{\frac{1}{2}});$$

$$\frac{-dx}{(a^{\frac{1}{2}}-x^{\frac{1}{2}})^{\frac{1}{2}}} = 2\mu^{\frac{1}{2}}\,dt,$$

$$\frac{4}{3}(x^{\frac{1}{2}}+2a^{\frac{1}{2}})\,(a^{\frac{1}{2}}-x^{\frac{1}{2}})^{\frac{1}{2}} = 2\mu^{\frac{1}{2}}\,t;$$

$$\therefore\quad t = \frac{2}{3\mu^{\frac{1}{2}}}(x^{\frac{1}{2}}+2a^{\frac{1}{2}})\,(a^{\frac{1}{2}}-x^{\frac{1}{2}})^{\frac{1}{2}};$$

and thus the velocity at any point of the path and the time occupied in arriving at that point are known: and when the particle arrives at the centre, $x = 0$, and the velocity $= 2\mu^{\frac{1}{2}}a^{\frac{1}{4}}$, and the time $= \dfrac{4a^{\frac{3}{4}}}{3\mu^{\frac{1}{2}}}$.

286.] Let us briefly consider the case in which the force varies inversely as the nth power of the distance, and investigate the laws under which the time of moving over a finite distance can be found.

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^n};$$

$$\therefore\quad \frac{dx^2}{dt^2} = \frac{2\mu}{n-1}\left\{\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right\}, \tag{88}$$

if the particle is at rest at a distance a from the centre of force. This equation, which is that of vis viva, assigns the velocity in terms of the distance. To find the time; from (88), if T is the time from $x = a$ to $x = x$,

$$\text{T} = -\left(\frac{n-1}{2\mu}\right)^{\frac{1}{2}} a^{\frac{n-1}{2}} \int_a^x x^{\frac{n-1}{2}} (a^{n-1}-x^{n-1})^{-\frac{1}{2}}\,dx. \tag{89}$$

As the element-function under the sign of integration is of the form (86), Art. 43 (Integral Calculus), the expression is integrable by rationalization, (1) when $n = \dfrac{2m+1}{2m-1}$, (2) when $n = \dfrac{m+1}{m}$, m in both cases being an integer. The series of values of n in the two cases are

$$\left.\begin{array}{ll}(1) & \cdots\frac{5}{7},\ \frac{3}{5},\ \frac{1}{3},\ -1,\ \frac{3}{1},\ \frac{5}{3},\cdots; \\ (2) & \cdots\frac{3}{4},\ \frac{2}{3},\ \frac{1}{2},\ 0,\ 2,\ \frac{3}{2},\ \frac{4}{3},\cdots.\end{array}\right\} \quad (90)$$

287.] If the force varies inversely as the distance, and the particle moves to the centre from a given finite distance, the time will be expressed by means of the gamma-function. Thus

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x};$$

$$\frac{dx^2}{dt^2} = 2\mu\log\frac{a}{x},$$

if the particle is at rest, when $x = a$. If T is the time of passage from $x = a$ to $x = 0$,

$$\mathrm{T} = -\frac{1}{(2\mu)^{\frac{1}{2}}}\int_a^0 \frac{dx}{\left(\log\frac{a}{x}\right)^{\frac{1}{2}}}. \tag{91}$$

Let $\left(\log\frac{a}{x}\right)^{\frac{1}{2}} = y$; then

$$\begin{aligned}\mathrm{T} &= a\left(\frac{2}{\mu}\right)^{\frac{1}{2}}\int_0^\infty e^{-y^2}dy \\ &= a\left(\frac{\pi}{2\mu}\right)^{\frac{1}{2}}; \end{aligned} \tag{92}$$

that is, the time varies directly as the distance.

288.] Let us now take the case of a particle moving along the straight line joining two centres of force of equal absolute intensity which vary directly as the distance.

Let A and A′, fig. 92, be the centres of the forces, at a distance $2a$ apart: let O, the middle point of AA′, be the origin: let μ be the absolute force of each centre: let B be the position of m at rest, P its position at the time t: $\mathrm{OA} = \mathrm{OA}' = a$: $\mathrm{OB} = b$: $\mathrm{OP} = x$. Then the equation of motion is

$$m\frac{d^2x}{dt^2} = m\mu\,\mathrm{AP} - m\mu\,\mathrm{A'P};$$

$$\begin{aligned}\frac{d^2x}{dt^2} &= \mu(a-x) - \mu(a+x) \\ &= -2\mu x;\end{aligned}$$

$$\therefore\quad \frac{dx^2}{dt^2} = 2\mu(b^2 - x^2),$$

$$x = b\cos(2\mu)^{\frac{1}{2}}t. \tag{93}$$

Thus it appears that the velocity of the particle is zero when $x = \pm b$: the particle therefore moves from rest at B, and comes to rest again at a point B′ on the opposite side of O, and at a distance from it equal to that of B: also the velocity is a maximum at O: and the particle returns from B′ to O, and again to B, and thus oscillates continually: and from (93) it appears that the time of an oscillation is $\frac{\pi}{(2\mu)^{\frac{1}{2}}}$.

289.] A particle m is placed at rest at a certain point in the line joining the centres of two forces, which vary inversely as the square of the distance: it is required to determine the circumstances of motion of m.

Let A and A′ be the centres of force, fig. 94, of which let the absolute forces be μ and μ': let the point O, which is the neutral point of attraction between them, be the origin; $\text{OA} = a$, $\text{OA}' = a'$; let $\text{AA}' = c$: then

$$\frac{\mu}{a^2} = \frac{\mu'}{a'^2};$$

$$\therefore \quad \frac{\mu^{\frac{1}{2}}}{a} = \frac{\mu'^{\frac{1}{2}}}{a'} = \frac{\mu^{\frac{1}{2}} + \mu'^{\frac{1}{2}}}{c};$$

whereby a and a' are known. Let B and P be respectively the places of m when $t = 0$, and when $t = t$: let $\text{OB} = b$, $\text{OP} = x$: then the equation of motion is

$$\frac{d^2x}{dt^2} = \frac{\mu}{(a-x)^2} - \frac{\mu'}{(a'+x)^2}; \tag{94}$$

therefore multiplying by $2\,dx$ and integrating, and taking the limits corresponding to $t = t$ and to $t = 0$,

$$\frac{dx^2}{dt^2} = \frac{2\mu}{a-x} + \frac{2\mu'}{a'+x} - \frac{2\mu}{a-b} - \frac{2\mu'}{a'+b};$$

which equation involves an elliptic transcendent, and does not generally admit of further integration.

Suppose however the circumstances to be such that the particle is projected from B with a velocity u, and comes to rest at O: then from (94) by integration we have generally

$$\frac{dx^2}{dt^2} - u^2 = \frac{2\mu}{a-x} + \frac{2\mu'}{a'+x} - \frac{2\mu}{a-b} - \frac{2\mu'}{a'+b};$$

and since $\frac{dx}{dt} = 0$, when $x = 0$,

$$u^2 = \frac{2\mu b}{a(a-b)} - \frac{2\mu' b}{a'(a'+b)}. \tag{95}$$

If the velocity of projection is less than that thus determined, m will not reach O, but will come to rest at some point short of it, and will, as the force at A is greater than that at A′, return to A: similarly if the velocity of projection is greater, the particle will pass beyond O, and will eventually fall into A′.

Now if A and A′ are the centres of two spheres, each of which is composed of concentric homogeneous shells, every particle of which attracts with a force varying directly as the mass, and inversely as the square of the distance, then each sphere will attract an external particle with a force which varies directly as its mass, and inversely as the square of the distance of the particle from its centre; see Ex. 4, Art. 193. Now suppose A′ and A to be the centres of the earth and the moon, which are assumed to be spheres, and to be at rest: and suppose P to be the position at the time t of a particle in the line joining their centres, and acted on by their attractions; then we have the circumstances of the preceding problem: and since the mass of the earth is about 75 times that of the moon, as determined by tidal observation and by the phaenomenon of nutation, we have

$$\mu' = 75\mu;$$

and also since the mean distance of the moon's centre from that of the earth is about 60 (actually 59.9643) of the earth's equatorial radii, or about 237000 miles, we have $a+a' = 237000$ miles: therefore $a' = 212466$ miles, $a = 24534$ miles; such are the distances from the centres of the earth and the moon of the neutral point of attraction of the two bodies.

Suppose now a particle to be projected from the surface of the moon towards the earth, and with such a velocity as just to arrive at the neutral point, and to remain at rest there. Then since the mean radius of the moon is 1080 miles,

$$b = (24534-1080) \text{ miles} = 23454 \text{ miles}: \text{and therefore}$$

$$a-b = 1080 \text{ miles}, \qquad a'+b = 235920 \text{ miles}.$$

Also if g is gravity at the earth's surface, $g = \dfrac{32.2}{1760 \times 3}$ miles; and if r is the earth's mean radius, $r = 4000$ miles,

$$g = \frac{\mu'}{r^2}; \qquad \therefore \quad \mu' = (4000)^2 \frac{32.2}{1760 \times 3};$$

therefore substituting in (95), and reducing, we have ultimately,

$$u = 7852 \text{ feet in } 1'';$$

and therefore if the moon were not moving, and if there is no atmosphere, so that the projectile does not meet with a resisting medium, a particle thrown from its surface with a velocity greater than 7582 feet in 1″ towards the earth, will pass beyond the neutral point of attraction, and will move towards the earth.

290.] Suppose two particles m and m', which attract each other directly as their masses, and as the distance between them, to be placed at rest at two given points, and then to be left to their mutual action: it is required to determine their positions at a given time, and the other circumstances of their motion.

Let a and a' be the distances of m and m' from a certain point O, fig. 93, in the line joining them, when they are at rest, and when $t = 0$: and let x and x' be the distances of them from the same point O, when $t = t$: let $\text{OA} = a$, $\text{OA}' = a'$: $\text{OP} = x$, $\text{OP}' = x'$: then the equations of motion of m and m' respectively are

$$\frac{d^2x}{dt^2} = m'(x'-x); \qquad \frac{d^2x'}{dt^2} = m(x-x'); \tag{96}$$

$$\therefore \quad m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = 0;$$

$$m\frac{dx}{dt} + m'\frac{dx'}{dt} = 0, \tag{97}$$

the initial values of $\frac{dx}{dt}$ and of $\frac{dx'}{dt}$ being simultaneously zero; and taking the definite integral, with limits corresponding to $t = t$ and to $t = 0$, we have

$$m(x-a) + m'(x'-a') = 0. \tag{98}$$

If $\bar{x}$ refers to the centre of gravity of m and m', then

$$\begin{aligned}(m+m')\bar{x} &= mx + m'x' \\ &= ma + m'a';\end{aligned} \tag{99}$$

and therefore the centre of gravity remains at rest. Again from (96),

$$\frac{d^2x'}{dt^2} - \frac{d^2x}{dt^2} = -(m'+m)(x'-x);$$

let $x' - x = z$; and let $m' + m = \mu$;

$$\therefore \quad \frac{d^2z}{dt^2} = -\mu z; \quad \text{and} \quad \frac{dz^2}{dt^2} = \mu\{(a'-a)^2 - z^2\},$$

because when $t = 0$, $\frac{dz}{dt} = \frac{dx'}{dt} - \frac{dx}{dt} = 0$: therefore

$$\frac{-dz}{\{(a'-a)^2 - z^2\}^{\frac{1}{2}}} = \mu^{\frac{1}{2}}\,dt;$$

$$\therefore \quad \cos^{-1}\frac{z}{a'-a} = \mu^{\frac{1}{2}}\,t,$$

because when $t = 0$, $z = a' - a$. Therefore substituting

$$x' - x = (a'-a)\cos(m'+m)^{\frac{1}{2}}t;$$

$$\therefore \quad x' = \bar{x} + \frac{m(a'-a)}{m+m'} \cos (m'+m)^{\frac{1}{2}} t,$$

$$x = \bar{x} - \frac{m'(a'-a)}{m+m'} \cos (m'+m)^{\frac{1}{2}} t.$$

Thus the motion of the particles is oscillatory, the periodic time being $\frac{\pi}{(m+m')^{\frac{1}{2}}}$; this result may be inferred from the preceding equation in terms of z, which is the equation of harmonic motion.

291.] A centre of force, which varies as the distance, moves with an uniform velocity along a straight line, and attracts a particle, which is projected with a given velocity from a given point in the line of motion of the centre of the force and along that line: it is required to determine the circumstances of motion of the particle.

Let O, fig. 95, the position of the centre of force when $t = 0$, be taken as the origin; let a = the constant velocity with which the centre of force moves along OA, and let C be its position at the time t, so that $\text{OC} = at$: let A and P be respectively the positions of m when $t = 0$, and when $t = t$: $\text{OA} = a$, $\text{OP} = x$: and let m be projected from A along the line, and in the direction AP, with a velocity u. The equation of motion is

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\mu\,\text{CP} \\ &= -\mu(x-at); \end{aligned} \qquad (100)$$

$$\therefore \quad \frac{d^2x}{dt^2} + \mu x = \mu a t;$$

whence by integration we have

$$x = at + \text{C}_1 \sin \mu^{\frac{1}{2}} t + \text{C}_2 \cos \mu^{\frac{1}{2}} t;$$

where C_1 and C_2 are arbitrary constants introduced in integration, and which are determined by the following conditions: when $t = 0$, $x = a$, therefore $\text{C}_2 = a$; also when $t = 0$, $\frac{dx}{dt} = u$; therefore $\text{C}_1 = \frac{u-a}{\mu^{\frac{1}{2}}}$: and thus

$$x = at + \frac{u-a}{\mu^{\frac{1}{2}}} \sin \mu^{\frac{1}{2}} t + a \cos \mu^{\frac{1}{2}} t; \qquad (101)$$

whence it appears that the *mean* value of x varies directly as the time: that the particle is sometimes before and sometimes behind the centre of force; and that while it has on the whole a progressive motion, it oscillates from one side to the other of the centre of force; and that the period of an oscillation is $\frac{\pi}{\mu^{\frac{1}{2}}}$.

292.] Thus far we have referred the place and the velocity of a moving particle to a fixed origin and to a particle at rest respectively. It is however frequently convenient to refer to an origin moving either uniformly or with a varying velocity. In the former case the motion is called *absolute*, and in the latter it is said to be *relative*. The problem of the preceding Article affords so good and so simple an instance of rectilineal relative motion that it is worth while to consider it from that point of view, although we shall somewhat anticipate the complete discussion which will be made in the following Chapter.

Let the moving centre of force be the point relative to which the motion of the particle is to be estimated; and let the distance of the particle from it at the time t be z; then employing the notation of the preceding Article, $z = x - at$. Consequently

$$\frac{dz}{dt} = \frac{dx}{dt} - a; \qquad \frac{d^2z}{dt^2} = \frac{d^2x}{dt^2}:$$

and the equation of motion is

$$\frac{d^2z}{dt^2} + \mu z = 0; \tag{102}$$

which is the equation of harmonic motion; and thus the motion is oscillatory about the centre of force, the particle being sometimes in advance and at other times in the rear of the moving centre, and the periodic time being $\frac{\pi}{\mu^{\frac{1}{2}}}$. If the particle is projected from the moving centre with a relative velocity β, that is, with an absolute velocity $\beta + a$, the integral of (102) is

$$z = \frac{\beta}{\mu^{\frac{1}{2}}} \sin \mu^{\frac{1}{2}} t.$$

SECTION 5.—*Rectilinear motion of particles in resisting media.*

293.] A particle is projected from a given point with a given velocity, in a medium of which the density is constant, and of which the resistance varies as the square of the velocity, and where no other force acts on the particle: it is required to determine the circumstances of motion.

Let the point from which the particle is projected be taken as the origin: and the line in which it is projected as the axis of x: let u be the velocity of projection, and let x be the distance of the particle from the origin at the time t: let the particle be of the mass m, and be of such a form as to present a plane surface ω to the medium in the direction in which it moves: then by (21), Art. 267, the equation of motion is

$$m\frac{d^2x}{dt^2} = -\rho\omega\left(\frac{dx}{dt}\right)^2;$$

and as ρ is constant, let $\rho\omega = mk$;

$$\therefore \quad \frac{d^2x}{dt^2} = -k\left(\frac{dx}{dt}\right)^2; \tag{103}$$

k is commonly called *the coefficient of resistance.* Now putting (103) in the following form,

$$\frac{d.\frac{dx}{dt}}{\frac{dx}{dt}} = -k\,dx,$$

and taking the definite integral at limits corresponding to $t = t$ and to $t = 0$; we have

$$\log\frac{dx}{dt} - \log u = -kx;$$

$$\therefore \quad \frac{dx}{dt} = ue^{-kx}; \tag{104}$$

$$e^{kx}dx = u\,dt,$$

$$e^{kx} - 1 = kut. \tag{105}$$

(104) gives the relation between the velocity and the distance through which the particle has passed: and (105) gives the relation between the distance and the time. From (104) it appears that $\frac{dx}{dt} = 0$, or that the particle comes to rest, when $x = \infty$: in which case also $t = \infty$, as appears by (105).

294.] A heavy particle m acted on by gravity (a constant accelerating force) moves in the air, which is supposed to be a resisting medium, whose density is uniform, and the resistance of which (according to Art. 267) varies as the square of the velocity: it is required to determine the circumstances of motion.

Firstly, let us suppose the particle to descend towards the earth and to start from rest: then if ρ is the constant density of the air, and if ω is the area of the face, which m presents to the medium, transversal to the direction of its line of motion,

$$m\frac{d^2x}{dt^2} = mg - \rho\omega\left(\frac{dx}{dt}\right)^2, \tag{106}$$

affecting the resistance with a negative sign, because it tends to diminish the velocity:

$$\therefore \quad \frac{d^2x}{dt^2} = g - \frac{\rho\omega}{m}\left(\frac{dx}{dt}\right)^2.$$

Let $\frac{\rho\omega}{m} = k$, the coefficient of resistance: so that we have

$$\frac{d^2x}{dt^2} = g - k\left(\frac{dx}{dt}\right)^2; \tag{107}$$

$$\therefore \quad \frac{d.\frac{dx}{dt}}{\frac{g}{k} - \left(\frac{dx}{dt}\right)^2} = k\,dt; \tag{108}$$

whence integrating, and taking the definite integrals corresponding to $t = t$ and to $t = 0$, we have

$$\frac{k^{\frac{1}{2}}}{2g^{\frac{1}{2}}}\log\frac{g^{\frac{1}{2}} + k^{\frac{1}{2}}\frac{dx}{dt}}{g^{\frac{1}{2}} - k^{\frac{1}{2}}\frac{dx}{dt}} = kt;$$

$$\therefore \quad \frac{dx}{dt} = \left(\frac{g}{k}\right)^{\frac{1}{2}}\frac{e^{2(kg)^{\frac{1}{2}}t} - 1}{e^{2(kg)^{\frac{1}{2}}t} + 1}. \tag{109}$$

Also again from (107),

$$\frac{d.\left(\frac{dx}{dt}\right)^2}{\frac{g}{k} - \left(\frac{dx}{dt}\right)^2} = 2k\,dx;$$

therefore integrating, and taking the limits as before,

$$\log\frac{g - k\left(\frac{dx}{dt}\right)^2}{g} = -2kx;$$

$$\frac{dx^2}{dt^2} = \frac{g}{k}\{1 - e^{-2kx}\}. \tag{110}$$

(109) gives the velocity in terms of the time, and (110) in terms of the distance through which m has passed. Also from (109),

$$k\,dx = (kg)^{\frac{1}{2}}\frac{e^{(kg)^{\frac{1}{2}}t} - e^{-(kg)^{\frac{1}{2}}t}}{e^{(kg)^{\frac{1}{2}}t} + e^{-(kg)^{\frac{1}{2}}t}}\,dt;$$

therefore integrating, and taking the same limits as before,

$$kx = \log\{e^{(kg)^{\frac{1}{2}}t}+e^{-(kg)^{\frac{1}{2}}t}\}-\log 2;$$

$$\therefore \quad 2e^{kx} = e^{(kg)^{\frac{1}{2}}t}+e^{-(kg)^{\frac{1}{2}}t}; \qquad (111)$$

which gives the relation between the distance and the time to which it is due. This equation might have been found by eliminating the velocity between (109) and (110).

When $t = \infty$, $x = \infty$; that is, an infinite time is required for an infinite space: but when $x = \infty$, and $t = \infty$, the velocity $= \left(\frac{g}{k}\right)^{\frac{1}{2}}$, that is, becomes uniform; in which case, as appears from (107), $\frac{d^2x}{dt^2} = 0$; and although this state is never attained to, yet it is that to which the circumstances of motion approach; also this limiting velocity is greater, the less k is; but k varies directly as the density of the medium, directly as the surface which the particle presents, and inversely as the mass of the particle: therefore the terminal velocity is greater, the greater the mass of the particle is, and the less the density of the medium is, and the less the area of the face is which the particle presents to it in its motion. These results are in accordance with experience. From the form of (107) it appears that the equation is satisfied if

$$\frac{dx^2}{dt^2} = \frac{g}{k}, \qquad (112)$$

because in that case $\frac{d^2x}{dt^2} = 0$: this therefore is a solution of the equation, and is a singular one, because it does not arise by giving any particular values to the arbitrary constants, to which the limits of the integrals are equivalent, and which are therefore dependent on the initial circumstances of motion. It appears therefore that the general integral represents the circumstances until the velocity attains its constant value; and that then the singular solution represents the motion. Other and similar peculiar properties of singular solutions will be exhibited hereafter.

295.] Secondly, let us suppose m to be projected upwards, that is, in a direction contrary to that of the action of gravity, with a given velocity u: it is required to determine the circumstances of motion.

Let us moreover suppose m to be of such a form, that it presents to the medium an equal area transversal to the line of motion, whether it falls, or whether it moves upwards: then if x is measured upwards, gravity and the resistance of the medium both tend to diminish the velocity as t increases: so that the equation of motion is

$$m\frac{d^2x}{dt^2} = -mg - \rho\omega\left(\frac{dx}{dt}\right)^2;$$

and if $\rho\omega = mk$, we have

$$\frac{d^2x}{dt^2} = -g - k\left(\frac{dx}{dt}\right)^2; \qquad (113)$$

$$\therefore \quad \frac{d.\frac{dx}{dt}}{\frac{g}{k} + \left(\frac{dx}{dt}\right)^2} = -k\,dt;$$

therefore integrating, and taking the limits which correspond to $t = t$ and to $t = 0$,

$$\left(\frac{k}{g}\right)^{\frac{1}{2}} \left\{ \tan^{-1}\left(\frac{k}{g}\right)^{\frac{1}{2}} \frac{dx}{dt} - \tan^{-1}\left(\frac{k}{g}\right)^{\frac{1}{2}} u \right\} = -kt; \qquad (114)$$

$$\therefore \quad \frac{dx}{dt} = \left(\frac{g}{k}\right)^{\frac{1}{2}} \frac{uk^{\frac{1}{2}} - g^{\frac{1}{2}} \tan (kg)^{\frac{1}{2}}\, t}{g^{\frac{1}{2}} + uk^{\frac{1}{2}} \tan (kg)^{\frac{1}{2}}\, t}, \qquad (115)$$

which gives the velocity in terms of the time.

Again, from (113), if we multiply both sides by $2dx$,

$$\frac{d.\left(\frac{dx}{dt}\right)^2}{\frac{g}{k} + \left(\frac{dx}{dt}\right)^2} = -2k\,dx;$$

therefore integrating, and taking limits the same as before,

$$\log \frac{g + k\left(\frac{dx}{dt}\right)^2}{g + ku^2} = -2kx;$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = u^2 e^{-2kx} - \frac{g}{k}(1 - e^{-2kx}), \qquad (116)$$

which gives the velocity in terms of the distance.

Also, from (115),

$$dx = \left(\frac{g}{k}\right)^{\frac{1}{2}} \frac{uk^{\frac{1}{2}} \cos (kg)^{\frac{1}{2}}\, t - g^{\frac{1}{2}} \sin (kg)^{\frac{1}{2}}\, t}{uk^{\frac{1}{2}} \sin (kg)^{\frac{1}{2}}\, t + g^{\frac{1}{2}} \cos (kg)^{\frac{1}{2}}\, t}\, dt;$$

and therefore integrating, and taking the limits the same as before,

$$x = \frac{1}{k} \log \frac{uk^{\frac{1}{2}} \sin (kg)^{\frac{1}{2}}\, t + g^{\frac{1}{2}} \cos (kg)^{\frac{1}{2}}\, t}{g^{\frac{1}{2}}}; \qquad (117)$$

which gives the space described by the particle in terms of the time to which it is due.

From (115) and (116), when $\frac{dx}{dt} = 0$, that is, when m has reached the highest point,

$$t = \frac{1}{(kg)^{\frac{1}{2}}} \tan^{-1} u \left(\frac{k}{g}\right)^{\frac{1}{2}}, \tag{118}$$

$$x = \frac{1}{2k} \log\left(1 + \frac{k}{g} u^2\right), \tag{119}$$

which give the distance of the highest point, and the time of reaching it. After which the particle begins to fall, and the investigations of the preceding Article are applicable.

Since k is the same in this and the preceding Article, that is, since m presents an equal area ω in the ascent and the descent, by (110) the velocity acquired by m in descending to the point whence it was projected with u is

$$\frac{u g^{\frac{1}{2}}}{(g + ku^2)^{\frac{1}{2}}}, \tag{120}$$

which is less than u: hence the velocity acquired in the descent is less than that lost in the ascent, the reason being that momentum is withdrawn from m both in the ascent and in the descent, and is transferred to the molecules of the medium.

Again, substituting (119) in (111), the time occupied in the descent is

$$\frac{1}{2(kg)^{\frac{1}{2}}} \log \frac{(g + ku^2)^{\frac{1}{2}} + uk^{\frac{1}{2}}}{(g + ku^2)^{\frac{1}{2}} - uk^{\frac{1}{2}}}, \tag{121}$$

which is different to that required for the ascent, as given in (118).

296.] Let us also consider the motion of a particle under the action of a constant force in the line of its motion, and moving in a medium, the resistance of which varies as the velocity; and let us suppose the particle to be projected with a velocity u, when $t = 0$ and $x = 0$. The equation of motion is, in terms of velocity-increment,

$$\frac{d^2x}{dt^2} = f - k\frac{dx}{dt}; \tag{122}$$

wherein f expresses the constant force, and k is the coefficient of resistance. Therefore integrating, and taking the limits which correspond to $t = t$ and to $t = 0$,

$$\frac{dx}{dt} - u = ft - kx;$$

$$\therefore \quad \frac{dx}{dt} + kx = u + ft; \tag{123}$$

$$\therefore \quad x = \left(\frac{d}{dt} + k\right)^{-1}(u + ft)$$

$$= e^{-kt}\int_0^t e^{kt}(u + ft)\,dt$$

$$= \frac{ft}{k} + \frac{f - uk}{k^2}(e^{-kt} - 1). \tag{124}$$

Thus from (123) we have the velocity in terms of x and t; and in (124) the relation is given between x and t: hence also

$$\frac{dx}{dt} = \frac{f}{k} - \frac{f - uk}{k} e^{-kt}. \tag{125}$$

And if $t = \infty$, $x = \infty$, and $\frac{dx}{dt} = \frac{f}{k}$; that is, the velocity has this finite limiting value, which it attains only when $t = \infty$. This result also follows from the equation of motion: $\frac{dx}{dt} - \frac{f}{k} = 0$ is a singular solution of it: and thus the particular integrals (123) and (124) express the circumstances of the motion, so long as the time is finite; but when $t = \infty$, the singular solution expresses them.

297.] Lastly, let us consider the case of a particle moving in a resisting medium, where the density of the medium varies; and let us suppose the resistance to vary as the square of the velocity, and the density to vary inversely as the square of the distance from a given point; and the particle also to move under the action of an attracting force which varies inversely as the cube of the distance from the same point.

Let a and x be the distances of m from the given point when $t = 0$ and when $t = t$. Let $u =$ the velocity of m when $t = 0$, and let μ be the absolute force of the central force: then the equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{m\mu}{x^3} + \frac{k'}{x^2}\,\omega\left(\frac{dx}{dt}\right)^2.$$

Let $k'\omega = mk$: so that we have

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^3} + \frac{k}{x^2}\left(\frac{dx}{dt}\right)^2. \tag{126}$$

Multiplying by $2\,dx$, we have

$$d\,.\,\frac{dx^2}{dt^2} = -\frac{2\mu}{x^3}\,dx + \frac{2k}{x^2}\left(\frac{dx}{dt}\right)^2 dx;$$

$$\therefore\quad d\,.\,\frac{dx^2}{dt^2} - \frac{2k}{x^2}\left(\frac{dx}{dt}\right)^2 dx = -\frac{2\mu}{x^3}\,dx;$$

a linear differential equation, of which the integrating factor is $e^{\frac{2k}{x}}$; therefore integrating, and taking the limits which correspond to $t=t$ and to $t=0$, we have

$$e^{\frac{2k}{x}}\left(\frac{dx}{dt}\right)^2 - e^{\frac{2k}{a}}u^2 = -2\mu\int_a^x e^{\frac{2k}{x}}\,\frac{dx}{x^3}$$

$$= \frac{\mu}{2k^2}\left\{\frac{2k-x}{x}\,e^{\frac{2k}{x}} - \frac{2k-a}{a}\,e^{\frac{2k}{a}}\right\},$$

which gives the velocity in terms of the distance; but does not admit of further integration.

CHAPTER IX.

THE THEORY OF CURVILINEAR MOTION.

SECTION 1.—*The Kinematics of a particle moving in a curvilinear path.*

298.] The motion, whose incidents we have thus far considered, has been that of a particle describing a rectilineal path; but there is a much more general case, viz. that in which the path is curvilinear; and I propose to consider the kinematics of such motion with the object of applying the results dynamically. Here as elsewhere some system of reference is needed, to which the path of the particle may be referred, and whereby its position at any time may be determined; the systems usually taken are the Cartesian, whether of plane geometry or of geometry in space; the two corresponding systems of polar coordinates; and sometimes peculiar facilities for the solution of a problem are offered by the intrinsic equation of a curve. We shall hereafter have examples of all these.

The conception and the definition of velocity and of velocity-increment acceleration which are given in Arts. 246, 247 are evidently just as applicable to a particle describing a curvilinear path as to one moving along a straight line; and consequently what has been said on these subjects need not be repeated. In reference however to a curvilinear path, if s is the length of an arc measured along the curve from any fixed point in it, and dt is the time during which an infinitesimal arc-element ds is described, then $\frac{ds}{dt}$, and $\frac{d^2s}{dt^2}$ are respectively the velocity and the velocity-increment of a particle moving along the curve.

299.] Let us first suppose the path of the particle to be a plane curve, and refer its place to a system of rectangular axes in that plane, and let us take (x, y) to be its place at the time t, so that x and y are functions of t; and consequently if t is eliminated by means of these two equations, the resulting equation in terms of x and y is that to the path described. This path is technically called *the trajectory of the particle.*

Let (x, y) be the place of the particle at the time t, and $(x+dx, y+dy)$ at the time $(t+dt)$, so that ds, which is equal to $(dx^2+dy^2)^{\frac{1}{2}}$, is the path described in the time dt; and dx and dy are the increments of x and y in that time; and consequently, according to the definition of velocity, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the velocities of the particle relatively to the axes of x and y respectively; these velocities being estimated positively or negatively according as the coordinates are increased or diminished as the time increases.

Also since
$$ds^2 = dx^2+dy^2; \tag{1}$$
$$\therefore \quad \frac{ds^2}{dt^2} = \frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}; \tag{2}$$
and accordingly the square of the expressed velocity is equal to the sum of the squares of the expressed velocities relative to the coordinate axes of x and y.

Also if τ is the angle at which the tangent to the path at (x, y) is inclined to the axis of x,
$$dx = ds\cos\tau; \qquad dy = ds\sin\tau;$$
$$\therefore \quad \frac{dx}{dt} = \frac{ds}{dt}\cos\tau, \qquad \frac{dy}{dt} = \frac{ds}{dt}\sin\tau, \tag{3}$$
that is, the velocities relative to, or along, the coordinate axes are severally the product of the expressed velocity and the cosine of the angle contained between the two lines of estimation. Consequently as the lines of the axes are arbitrary, this law holds universally; and the cosine is the projective factor of velocities, as it is also of lines, areas, and statical pressures.

This kinematical theorem is of the greatest importance in the treatment of complicated problems of Mechanics. It is called *the parallelogram of velocities*, and yields results of composition and resolution of velocities which enable us to analyse and solve questions otherwise beyond our powers. Thus if ds is the diagonal of a rectangle whose sides are dx and dy, all these three lines meeting in one point, the velocity along ds which is expressed by $\frac{ds}{dt}$ may be resolved into two velocities $\frac{dx}{dt}$ and $\frac{dy}{dt}$ which are effective along the sides dx and dy respectively; and as one velocity may be resolved into two, so may also two or more be compounded into a single one. We shall have illustrations of this theorem from a dynamical point of view in the following section.

300.] In the most general case of the motion of a particle, the velocity of it will vary so that $\frac{ds}{dt}$ will not be constant; and the resolved velocities along the coordinate axes will also vary; and thus $\frac{dx}{dt}$ as also $\frac{dy}{dt}$ will not be constant. Hence during equal and successive dt's, the ds's, dx's, dy's will not all be equal, and we shall have increments of them expressed by d^2s, d^2x, d^2y. And consequently, there will be velocity-increments or accelerations along the curve, and along the axes of x and y respectively; these will be represented by $\frac{d^2s}{dt^2}$, $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$ respectively if t is equicrescent; and if t is not equicrescent severally by

$$\frac{d^2s\,dt-d^2t\,ds}{dt^3}, \quad \frac{d^2x\,dt-d^2t\,dx}{dt^3}, \quad \frac{d^2y\,dt-d^2t\,dy}{dt^3}. \qquad (4)$$

Before I apply these expressions to the solution of particular problems I would observe that as they express velocity-increments, they are subject to the same laws of composition and resolution as velocities; that is, to the law given in the preceding Article; and consequently the velocity or velocity-increment of the particle along any line is the sum of the resolved parts of the axial velocities or velocity-increments along that line. The following are examples in which the preceding expressions are applied to cases in which the laws of velocity and of acceleration are given:

301.] Ex. 1. A particle moves so that the axial-components of its velocity vary as the corresponding coordinates; it is required to find the equation of its path.

$$\frac{dx}{dt} = kx; \qquad \frac{dy}{dt} = ky;$$

$$\therefore \quad \frac{dx}{x} = \frac{dy}{y} = k\,dt;$$

$$\therefore \quad \log\frac{x}{a} = \log\frac{y}{b} = kt,$$

if (a, b) is the initial place of the particle;

$$\therefore \quad x = a e^{kt}, \qquad y = b e^{kt};$$

$$\frac{x}{a} = \frac{y}{b},$$

and this last is the equation to the path.

In this case the axial velocity-increments are

$$\frac{d^2x}{dt^2} = k^2x, \qquad \frac{d^2y}{dt^2} = k^2y.$$

Ex. 2. If $\frac{dx}{dt} = ky$, $\frac{dy}{dt} = kx$, the path is an equilateral hyperbola, and the axial accelerations are

$$\frac{d^2x}{dt^2} = k^2x, \qquad \frac{d^2y}{dt^2} = k^2y.$$

Ex. 3. A wheel rolls along the straight line at a constant velocity; compare the velocity of a given point in the wheel with that of the centre of the wheel.

Let the line along which the wheel rolls be the axis of x, and let u be the velocity of its centre: then a point in the circumference of the wheel describes a cycloid, of which, the origin being conveniently taken, the equation is

$$x = a \operatorname{versin}^{-1}\frac{y}{a} - (2ay-y^2)^{\frac{1}{2}}:$$

$$\therefore \quad \frac{dx}{y^{\frac{1}{2}}} = \frac{dy}{(2a-y)^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}}.$$

$$\text{Now} \quad u = \frac{d}{dt}\cdot a \operatorname{versin}^{-1}\frac{y}{a} = \frac{a}{(2ay-y^2)^{\frac{1}{2}}}\frac{dy}{dt};$$

$$\therefore \quad \frac{ds}{dt} = \left(\frac{2y}{a}\right)^{\frac{1}{2}} u;$$

and this gives the velocity of the point in the circumference of the wheel. Thus the highest point of the wheel moves with a velocity twice as great as that of the point at which $y = \frac{a}{2}$.

This is a problem in which a curve is given, and one axial-component of the velocity is given. From these data the other axial-component and the velocity can of course be found.

302.] Let us now take some cases in which two out of the three quantities, viz. the path and the two axial accelerations, being given, the third is required.

Ex. 1. A particle describes an ellipse with a constant velocity $= a$ parallel to the axis of x: find the velocity and velocity-increment parallel to the axis of y, and the time of describing the ellipse.

Let the equation to the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

and let (x, y) be the place of m at the time t: so that $\frac{dx}{dt} = a$;

$$\therefore \quad \frac{dy}{dt} = -\frac{b^2 x}{a^2 y}\frac{dx}{dt}$$
$$= -\frac{\alpha b^2}{a^2}\frac{x}{y},$$

which gives the velocity parallel to the axis of y.

$$\frac{d^2 y}{dt^2} = -\frac{b^2 \alpha}{a^2}\,\frac{y\dfrac{dx}{dt} - x\dfrac{dy}{dt}}{y^2}$$
$$= -\frac{b^4 \alpha^2}{a^2 y^3};$$

thus the acceleration parallel to the axis of y varies inversely as the cube of the ordinate of the ellipse, and acts towards the axis of x, as is shewn by the negative sign.

Since $\frac{dx}{dt} = \alpha$, $x = \alpha t$, if we assume the position of the particle to be at the extremity of the minor axis when $t = 0$. Hence the time of passing from the extremity of the minor axis to that of the major axis is $\frac{a}{\alpha}$, and the time of describing the whole ellipse is $\frac{4a}{\alpha}$.

If the orbit is a circle $b = a$, and the acceleration parallel to the axis perpendicular to that along which the velocity is constant is $-\frac{\alpha^2 a^2}{y^3}$.

If the velocity parallel to the y-axis is constant and is equal to β, then

$$\frac{dx}{dt} = -\frac{a^2 \beta}{b^2}\frac{y}{x};$$
$$\frac{d^2 x}{dt^2} = -\frac{a^4 \beta^2}{b^2 x^3};$$

and the periodic time $= \frac{4b}{\beta}$.

Ex. 2. A particle describing a parabola moves with a constant velocity α in a direction perpendicular to the axis. Find the velocity and the velocity-increment parallel to the axis.

Let the equation to the parabola be $y^2 = 4ax$; then $\frac{dy}{dt} = \alpha$; $\frac{dx}{dt} = \frac{\alpha y}{2a}$, and $\frac{d^2 x}{dt^2} = \frac{\alpha^2}{2a}$, and is constant; and as it has a positive sign, it shews that the particle moves away from the tangent to the curve at the vertex with a constant acceleration.

Hence as the earth acts on particles near to its surface with a constant acceleration in vertical lines, if a particle m is projected with a velocity α in a horizontal line, and is attracted towards the earth in a vertical line, m will move in a parabolic path.

By a similar process, if Y represents the acceleration parallel to the axis of y, when the velocity parallel to the axis of x is constant; and if X represents the acceleration parallel to the axis of x, when the velocity parallel to the axis of y is constant; and if α and β are the constant velocities parallel to the axes of x and y respectively in each case; then, if a particle m describes

A Hyperbola, $xy = k^2$, $\quad \mathrm{X} = \dfrac{2\beta^2}{k^4} x^3$,

$$\mathrm{Y} = \frac{2\alpha^2}{k^4} y^3.$$

A Parabola, $y^2 = 4ax$, $\quad \mathrm{Y} = -\dfrac{4a^2\alpha^2}{y^3}$.

A Hyperbola, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $\quad \mathrm{X} = \dfrac{a^4\beta^2}{b^2 x^3}$,

$$\mathrm{Y} = -\frac{b^4\alpha^2}{a^2 y^3}.$$

The Logarithmic Curve, $y = a^x$, $\quad \mathrm{X} = -\dfrac{\beta^2}{\log a \times a^{2x}}$,

$$\mathrm{Y} = \alpha^2 (\log a)^2 y.$$

The Cycloid, starting point being origin, $\mathrm{X} = \dfrac{\beta^2 ay}{(2ay - y^2)^{\frac{3}{2}}}$,

$$\mathrm{Y} = -\frac{\alpha^2 a}{y^2}.$$

The Catenary, $y = \dfrac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}$, $\quad \mathrm{X} = -\dfrac{\beta^2 ay}{(y^2 - a^2)^{\frac{3}{2}}}$,

$$\mathrm{Y} = \frac{\alpha^2}{a^2} y.$$

Ex. 3. To determine the laws of acceleration parallel to the axes of x and y, so that a particle m may describe the parabola with a constant velocity.

Let the equation to the parabola be

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}};$$

$$\therefore \quad \frac{dx}{x^{\frac{1}{2}}} = \frac{-dy}{y^{\frac{1}{2}}} = \frac{ds}{(x+y)^{\frac{1}{2}}}.$$

And let the constant velocity along the curve $= c$; so that

$$ds = c\,dt;$$

$$\therefore \quad \frac{dx^2}{dt^2} = \frac{c^2 x}{x+y}, \qquad \frac{dy^2}{dt^2} = \frac{c^2 y}{x+y};$$

and differentiating,

$$\frac{d^2x}{dt^2} = \frac{c^2 (ay)^{\frac{1}{2}}}{2\,(x+y)^2}, \qquad \frac{d^2y}{dt^2} = \frac{c^2 (ax)^{\frac{1}{2}}}{2\,(x+y)^2}.$$

As very many examples of a similar kind will arise hereafter when we treat the subject from a dynamical point of view, it is unnecessary to insert others in this place.

303.] And we will now consider the motion of a particle in a plane curve from another point of view and in relation to another system of reference. The method of resolving velocities and velocity-increments along two rectangular axes chosen arbitrarily is artificial, and has not been deduced from considerations of the actual motion of the particle; but it is convenient, and adapts itself to the Cartesian system of algebraical geometry, and to the ordinary equations of curves. Let us however consider whether the actual motion of m does not lead us to another and more natural method.

$\frac{d^2s}{dt^2}$ is the velocity-increment in the line of motion of m at any time t; and therefore if there is velocity-increment only in this line, the path will be rectilinear: generally however the particle moves in a curvilinear path, and there is therefore a deflexion from the rectilinear path, and a corresponding velocity-increment: the question is, What is the mathematical representative of this velocity-increment? In fig. 98, let PQ $(= ds)$ be the element of the curvilinear path described in the time dt; let t be equicrescent; then if the particle is not deflected from its rectilinear path, it will in the next dt describe QR; but suppose QT $(= ds + d^2s)$ to be the element of the curve succeeding PQ, and to be the path taken by the particle in the second dt; then at the point Q and along the line QS the particle has received a velocity with which it moves over QS, in the time dt, so that at the end of dt, m is at the point T; our object is to determine the value of the acceleration which acts along QS.

P, Q, and T being three consecutive points in the curve, the angle RQT is the angle of contingence; see Art. 284, Vol. I. Let ρ be the radius of curvature of the path at P; that is, ρ is the radius of the circle passing through P, Q, and T: and therefore from the geometry,

$$\text{QS} = \frac{\text{QT}^2}{2\rho} = \frac{(ds+d^2s)^2}{2\rho} = \frac{ds^2}{2\rho}. \tag{5}$$

Now whatever is the law of acceleration with which the particle m describes QS in the time dt, we may consider it to be constant for that infinitesimal element of time; and therefore if f is the velocity-increment, by Ex. 2, Art. 249,

$$\text{QS} = \frac{f}{2}dt^2; \tag{6}$$

therefore from (5) and (6),

$$\begin{aligned} f &= \frac{1}{\rho}\frac{ds^2}{dt^2} \\ &= \frac{v^2}{\rho}, \end{aligned} \tag{7}$$

if v is the velocity of m at P: and the line of action of it is along the radius of curvature, that is, along the normal.

Hence at any point P of the trajectory, if the velocity-increment is resolved along the tangent to the curve at P and along the normal, the velocity-increments along these two lines are respectively $\frac{d^2s}{dt^2}$ and $\frac{v^2}{\rho}$.

304.] These results may be deduced from the expressions for the axial accelerations; for as accelerations are velocities they may be compounded and resolved along any line according to the laws of compositions and resolution of velocities. Hence if T and N are the velocity-increments along the tangent to the curve of which the direction cosines are $\frac{dx}{ds}$ and $\frac{dy}{ds}$, and along the normal respectively, we have

$$\text{T} = \frac{d^2x}{dt^2}\frac{dx}{ds} + \frac{d^2y}{dt^2}\frac{dy}{ds}, \tag{8}$$

$$\text{N} = \frac{d^2y}{dt^2}\frac{dx}{ds} - \frac{d^2x}{dt^2}\frac{dy}{ds}. \tag{9}$$

Since however $ds^2 = dx^2+dy^2$, $ds\,d^2s = dx\,d^2x+dy\,d^2y$: and if ρ is the radius of curvature at P, by (5), Art. 282, Differential Calculus,

$$dx\,d^2y - dy\,d^2x = \frac{ds^3}{\rho};$$

$$\therefore \quad \text{T} = \frac{d^2s}{dt^2}, \qquad \text{N} = \frac{1}{\rho}\frac{ds^2}{dt^2} = \frac{v^2}{\rho}, \tag{10}$$

if v is the velocity of the particle at the point (x, y).

These results may also be found by the following short process. Let ψ be the angle between the normal to the path at (x,y) and the x-axis. Then

$$\frac{dx}{dt} = \frac{ds}{dt}\sin\psi;$$

$$\therefore\quad \frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}\sin\psi + \cos\psi\frac{ds}{dt}\frac{d\psi}{dt}.$$

But when $\psi = 90°$, the path is parallel to the x-axis and $\frac{d^2x}{dt^2}$ is the tangential acceleration; and when $\psi = 0$, the path is perpendicular to the x-axis, and $\frac{d^2x}{dt^2}$ is the normal acceleration. Also $d\psi = \frac{ds}{\rho}$; consequently employing the same notation as heretofore,

$$\text{T} = \frac{d^2s}{dt^2}, \qquad \text{N} = \frac{v^2}{\rho}.$$

It will have been observed that $\frac{d^2s}{dt^2}$ is not the resultant of $\frac{d^2x}{dt^2}$ and of $\frac{d^2y}{dt^2}$, because it is not the resultant acceleration, there being also the normal component. To shew this still further let R be the resultant acceleration, then

$$\begin{aligned}\text{R}^2 &= \left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 \\ &= \frac{(d^2x)^2 + (d^2y)^2 - (d^2s)^2}{dt^4} + \left(\frac{d^2s}{dt^2}\right)^2 \\ &= \left(\frac{v^2}{\rho}\right)^2 + \left(\frac{d^2s}{dt^2}\right)^2, \qquad (11)\end{aligned}$$

since by (15), Art. 285, Vol. I. (Differential Calculus),

$$\frac{ds^4}{\rho^2} = (d^2x)^2 + (d^2y)^2 - (d^2s)^2; \qquad (12)$$

and the form of (11) shews that the resultant may be resolved into two components whose action-lines are at right angles to each other; and of which one, viz. $\frac{d^2s}{dt^2}$, is evidently the tangential component, and consequently the other, viz. $\frac{v^2}{\rho}$, is the normal component.

305.] This mode of resolution is convenient when the tangential velocity-increment is constant or is given as a function of s; and also generally when a condition is given in terms of the quantities of which these components of the velocity-increments are functions. Thus if a particle describes a curve with a constant velocity, the velocity-increment along the curve vanishes, and the normal acceleration at any point varies inversely as the radius of curvature of the curve at that point. Consequently if a particle describes a circle with uniform velocity, the normal acceleration varies as the square of the velocity and inversely as the radius of the circle.

As we shall have several illustrations of this mode of resolution of velocity-increments in the dynamical portion of the work, it is unnecessary to insert others in this place.

306.] The direction of the resultant acceleration may always be found by means of the following construction due to Sir W. R. Hamilton:

From a given point O let radii-vectores OP, OQ, ... be drawn, representing by their length the intensity of the velocity and by their direction the line of motion of a moving particle at each successive point of its continuous path. In such a path there will be no abrupt changes of velocity, and no abrupt deviations of lines of motion; because directions of the tangents of the curvilinear path of the particle vary continuously; and the locus of the extremities of all the radii-vectores thus drawn will form a continuous curve. Now suppose OP to represent the velocity and the line of motion of the particle at the time t, and OQ, which is infinitesimally near to it, to be a similar representative at the time $t+dt$; then the line PQ represents the resultant of the velocity-increment, since OQ is the resultant of OP and PQ. But as dt is infinitesimal, PQ is an arc-element of the curve-locus of P, Q, ...; and consequently the arc-element of this curve is the line-representative both in magnitude and direction of the acceleration of the moving particle. Sir W. R. Hamilton has named this curve *the hodograph*.

The preceding properties of this curve may also thus be found. Let (ξ, η) be the point on it which corresponds to (x, y) on the path of the particle; then by the definition of the hodograph,

$$\frac{\xi}{\frac{dx}{dt}} = \frac{\eta}{\frac{dy}{dt}} = \frac{\rho}{\frac{ds}{dt}} = k, \tag{13}$$

if ρ is the radius-vector of the hodograph, and k is a constant. Consequently

$$\frac{d\xi}{\frac{d^2x}{dt^2}} = \frac{d\eta}{\frac{d^2y}{dt^2}} = \frac{d\rho}{\frac{d^2s}{dt^2}} = k. \tag{14}$$

And thus the tangent to the hodograph at the point (ξ, η) is parallel to the action-line of the resultant acceleration at (x, y);

and the differential of the radius-vector of the hodograph is proportional to the tangential acceleration.

307.] Before we consider velocities and velocity-increments in reference to the system of polar coordinates, it is necessary to inquire into a mode of estimating the rate or velocity with which a radius-vector revolves in one plane about a fixed point and generates angular quantity at that point.

Let θ be the angle between a fixed straight line passing through the fixed point, and the position of the revolving radius-vector at the time t, and let us suppose the line to revolve uniformly, that is, to pass through, or to generate, equal angles in equal times; let ω be the angle generated in an unit of time; then if the generating line coincided with the fixed originating line when $t=0$, and θ is the angle generated in t units of time,

$$\theta = \omega t: \tag{15}$$

$$\therefore \quad \omega = \frac{\theta}{t}. \tag{16}$$

ω is called *angular velocity*: for as the linear velocity of a particle moving uniformly is the linear space described by it in an unit of time, so the angle generated in an unit of time by an uniformly revolving straight line is called *the angular velocity of the line*, and *the velocity of rotation*. The line passing through the fixed point and perpendicular to the plane of the rotating line is called *the axis of rotation*. If however the radius-vector does not revolve uniformly, and consequently does not generate equal angles in equal times, then the angular velocity is a function of the time. Let the time be resolved into infinitesimal elements, and let us suppose the angular velocity at the time t to be ω, and at the time $t+dt$ to be $\omega+d\omega$, and $d\theta$ be the angle generated in the time dt. Then since ω is the angular velocity at the time t and $\omega+d\omega$ at the time $t+dt$, the mean uniform angular velocity with which $d\theta$ is generated is $\omega+\phi\, d\omega$, where ϕ is a proper fraction, and is positive or negative according as the velocity is increasing or decreasing; consequently by reason of (15),

$$d\theta = (\omega+\phi\, d\omega)\, dt = \omega\, dt, \tag{17}$$

omitting the infinitesimal of the second order. Thus $d\theta$ is the angle generated in dt units of time by the radius-vector revolving with the angular velocity ω at the beginning of dt; and consequently dividing both sides by dt,

$$\omega = \frac{d\theta}{dt}; \tag{18}$$

and ω or $\frac{d\theta}{dt}$ is the angle generated in an unit of time, and is the angular velocity of the revolving radius-vector.

Thus in both cases, of uniform and of continuously varying angular velocity, angular velocity is the angle described by the radius-vector in an unit of time; and is the ratio of the angle described in a given time to the time in which it is described; in the case of varying velocity this ratio is the ratio of two infinitesimals.

The unit angular velocity is that with which the radius vector rotates through an unit angle in an unit of time; and if the angular velocity is ω, ω is a number designating the number of unit-angles through which the radius rotates in an unit of time.

308.] Hence is derived the principle on which angular velocities are measured; if two radii rotate with angular velocities such that each generates equal angles in equal times, the angular velocities being uniform during that time, these angular velocities are said to be equal. And this mode of determining equal angular velocities being adopted, it is evident that one angular velocity may be double, or treble, or n times another. If the equal angles are described by both radii in the same direction, the angular velocities are equal and in the same direction; but if the equal angles are described in opposite directions, the angular velocities are equal and opposite. Angular velocities may therefore be affected with signs. Thus if ω represents the angular velocity with which a line rotates in a given direction, $-\omega$ will represent the equal angular velocity of a line rotating in the opposite direction. As angular velocities have rotation-axes, intensities, and directions, it is evidently desirable to have some geometrical representative of them, as of linear velocities. This is supplied by a straight line on a principle similar to that by which the line-representatives of couples have been determined in Art. 46. Along the rotation-axis let a length be taken containing the same number of linear-units as ω contains angle-units; then this line by its position and its length represents the axis of rotation and the intensity of the angular velocity. Let a point on this rotation-axis be taken as a fixed pole; as the body may rotate about this axis in either of two directions, so may the line-representative of the angular velocity be measured in either of two opposite directions, and therefore we must choose a principle by which direction of rotation may be determined. Let it be this; if, as we look along the axis from the pole, the body rotates from left to right, like the hands of a watch when we face it, let that rotation be called *positive*, and let its line-representative be measured from the pole in the direction in which we look; but if the body rotates from right to left, that is, in the direction opposite to that of the motion of the hands of a watch, let that rotation be negative, and let the line-representative be measured from the pole in a direction opposite to that along which we look. Hereafter, however, it will be convenient to affect angular velocities with signs in reference to certain systems of coordinate axes in the manner explained in Art. 69.

309.] If the angular velocity varies, this variation may take place either uniformly or at a variable rate. In this case we have angular velocity-increments, or, as they are often called, *angular accelerations*. These changes also may take place when the angular velocity either increases or decreases: we will at present at least assume the angular velocity to increase, as such an assumption will give solidity to our ideas.

Firstly, let us suppose this increase to take place at an uniform pace; and let ϕ be the angular velocity added in an unit of time; so that if the angular velocity is zero at the beginning of t, and ω is the angular velocity at the end of t,

$$\omega = \phi t: \qquad (19)$$

and if ω_0 is the angular velocity at the commencement of t, and ω the angular velocity at the end of t,

$$\omega - \omega_0 = \phi t; \qquad (20)$$

so that the increment of angular velocity varies as t, and also as ϕ.

If the angular velocity decreases uniformly, and ϕ is the angular velocity taken from it in an unit of time, then if ω and ω_0 are respectively the angular velocities at t and when $t = 0$,

$$\omega = \omega_0 - \phi t; \qquad (21)$$

and the generating line will come to rest, when $t = \frac{\omega_0}{\phi}$.

Secondly, let us suppose the increase of angular velocity to take place at a varying rate, so that the increments of angular velocity are not equal in equal times; and accordingly the increase of angular velocity is a function either explicit or implicit of the time.

Let the time be resolved into equal elements; that is, let t be equicrescent; let us suppose the radius which generates the angles to be revolving at the time t with an angular velocity ω, and at the time $t+dt$ with an angular velocity $\omega+d\omega$; then if Φ is the angular acceleration at the time t, and $\Phi+d\Phi$ at the time $t+dt$, $\Phi+\phi\, d\Phi$, where ϕ is a proper fraction, is the mean angular velocity-increment during the time dt; and consequently, by reason of (19),

$$\begin{aligned} d\omega &= (\Phi+\phi\, d\Phi)\, dt \\ &= \Phi\, dt, \end{aligned} \tag{22}$$

if we omit the infinitesimal of the second order; that is, $d\omega$ units of angular velocity are added in the time dt; and dividing through by dt,

$$\begin{aligned} \Phi &= \frac{d\omega}{dt} = \frac{d}{dt}\frac{d\theta}{dt} \\ &= \frac{d^2\theta}{dt^2}, \end{aligned} \tag{23}$$

if t is equicrescent. And thus whether the increase of angular velocity is uniform or is variable, in each case the angular velocity-increment is the increase of angular velocity in an unit of time.

Thus we have the unit of angular velocity-increment or of angular acceleration, when the increase of angular velocity is an unit in an unit of time.

310.] The following examples are illustrations of the preceding mode of estimating velocity and velocity-increments:

Ex. 1. If a particle is placed on the revolving line which generates angle at a distance r from the origin, and the line revolves with an uniform angular velocity ω, the relation between the linear velocity of the particle and the angular velocity may thus be found.

Let $d\theta$ be the angle through which the radius revolves in dt, and let ds be the path described by the particle, so that $ds = r\, d\theta$; then

$$\frac{ds}{dt} = r\frac{d\theta}{dt} = \omega r; \tag{24}$$

so that the linear velocity varies as the angular velocity and as the length of the radius conjointly.

Hence if a particle revolves in a circular orbit with an uniform velocity ω, the normal component of the velocity-increment is equal to $\omega^2 r$.

Ex. 2. If the angular acceleration is constant, and equal to Φ, say; then from (23),

$$\frac{d^2\theta}{dt^2} = \Phi,$$

$$\frac{d\theta}{dt} = \omega_0 + \Phi t,$$

$$\theta = \theta_0 + \omega_0 t + \Phi \frac{t^2}{2},$$

where θ_0 and ω_0 are the initial values of θ and of ω.

Hence, if a line revolves from rest with a constant angular acceleration, we have

$$\theta = \frac{\Phi t^2}{2};$$

and the angle described by it varies as the square of the time.

Ex. 3. If the angular velocity-increment varies as the angle generated from a given fixed line, then the equation which expresses the motion is of the form

$$\frac{d^2\theta}{dt^2} = k^2\theta;$$

and is a harmonic equation or not according as k^2 is negative or positive. If k^2 is negative, and a is the initial value of θ,

$$\frac{d\theta^2}{dt^2} = k^2(a^2 - \theta^2);$$

$$\therefore \quad \theta = a\cos kt.$$

Ex. 4. If a particle revolves in a circle with uniform velocity, its angular velocity about any point in the circumference is also uniform; and is one-half of what it is about the centre.

Ex. 5. If a particle revolves uniformly in a circle, and its place is continually projected on a given diameter, the linear acceleration along that diameter varies directly as the distance of the projected place from the centre.

Let ω be the constant angular velocity, θ the angle between the fixed diameter and the radius drawn from the centre to its place at the time t, x the distance of this projected place from the centre, so that $x = a\cos\theta$, a being the radius of the circle;

$$\therefore \quad \frac{dx}{dt} = -a\sin\theta\frac{d\theta}{dt} = -a\omega\sin\theta,$$

$$\frac{d^2x}{dt^2} = -a\omega\cos\theta\frac{d\theta}{dt} = -\omega^2 x;$$

which proves the theorem.

Let this suffice at present for the general explanation of angular velocity and angular velocity-increment; we shall hereafter return to the subject when we treat of the motion of rigid bodies; we have now to consider these expressions in another relation.

311.] As the curvilinear paths of particles are frequently expressed conveniently in terms of polar coordinates, it is necessary to investigate the mathematical values of the components of velocity and of velocity-increment along and perpendicular to the radius-vector of the particle at any point of its path. The former are termed *the radial or the paracentric velocity and velocity-increment,* the latter *the transversal velocity and velocity-increment* respectively. The required values are thus found:

Let (r, θ) be the place of the moving particle at the time t, (x, y) being its place referred to a system of rectangular axes having the same origin, and the x-axis coincident with the prime radius. Then

$$x = r\cos\theta; \tag{25}$$

$$\frac{dx}{dt} = \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}; \tag{26}$$

$$\frac{d^2x}{dt^2} = \cos\theta\frac{d^2r}{dt^2} - 2\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}. \tag{27}$$

In all these expressions if $\theta = 0$, $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ are respectively the radial velocity and radial velocity-increment; and if $\theta = -90°$, we have the transversal velocity and velocity-increment. Hence

$$\text{the radial velocity} = \frac{dr}{dt}; \tag{28}$$

$$\text{the transversal velocity} = r\frac{d\theta}{dt}; \tag{29}$$

$$\text{the radial velocity-increment} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2; \tag{30}$$

$$\begin{aligned}\text{the transversal velocity-increment} &= 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} \\ &= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).\end{aligned} \tag{31}$$

The values given in (28) and (29) are evidently from first principles the radial and transversal components of the velocity of the particle. The expressions for the radial and transversal velocity-increments may also be deduced from similar principles. All these expressions may also be deduced by differentiation from $y = r\sin\theta$.

312.] Two particular forms of (30) and (31) deserve notice. If the acceleration is only radial, so that the transversal acceleration is zero, then

$$r^2 \frac{d\theta}{dt} = \text{a constant} = h, \text{ say};$$

$$\therefore \quad \frac{d\theta}{dt} = \frac{h}{r^2}; \qquad \frac{dr}{dt} = \frac{h}{r^2}\frac{dr}{d\theta};$$

$$\therefore \quad \frac{d^2r}{dt^2} = \frac{h^2}{r^4}\frac{d^2r}{d\theta^2} - \frac{2h^2}{r^5}\Big(\frac{dr}{d\theta}\Big)^2;$$

and the radial acceleration

$$= \frac{h^2}{r^4}\frac{d^2r}{d\theta^2} - \frac{2h^2}{r^5}\Big(\frac{dr}{d\theta}\Big)^2 - \frac{h^2}{r^3}, \tag{32}$$

and thus is expressed independently of t.

This expression however may be put into a more convenient form. Let $r = \frac{1}{u}$; then

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta},$$

$$\frac{d^2r}{d\theta^2} = -\frac{1}{u^2}\frac{d^2u}{d\theta^2} + \frac{2}{u^3}\Big(\frac{du}{d\theta}\Big)^2:$$

substituting which in (32) we have

$$\text{the radial acceleration} = -h^2 u^2 \Big\{\frac{d^2u}{d\theta^2} + u\Big\}. \tag{33}$$

From these formulae the law of radial acceleration may be deduced when the curve is given; and the curve may be deduced when the law of radial acceleration is given. But as very many examples of these processes will be given in a subsequent chapter, it is unnecessary to insert them in this place.

If the angular velocity is constant, so that $\frac{d\theta}{dt}$ = a constant, $= \omega$ (say), then

$$\text{the radial acceleration} \quad = \frac{d^2r}{dt^2} - \omega^2 r, \tag{34}$$

$$\text{the transversal acceleration} = 2\omega\frac{dr}{dt}; \tag{35}$$

and these are independent of θ.

In illustration of these formulae let us take the following example:

A particle describes a path with a constant angular velocity, and without radial acceleration; find the equation to the path, and the transversal acceleration.

Let Q be the required transversal acceleration so that the required equations are

$$\frac{d^2r}{dt^2} - \omega^2 r = 0,$$

$$Q = 2\omega\frac{dr}{dt}.$$

Consequently if $r = a$ when $\frac{dr}{dt} = 0$, $\theta = 0$, and $t = 0$, we have by integration

$$\frac{dr^2}{dt^2} - \omega^2(r^2 - a^2) = 0;$$

$$\therefore \quad \frac{dr}{(r^2-a^2)^{\frac{1}{2}}} = \omega\, dt; \qquad \log\frac{r+(r^2-a^2)^{\frac{1}{2}}}{a} = \omega t;$$

$$\therefore \quad r = \frac{a}{2}\{e^{\omega t} + e^{-\omega t}\}.$$

Also, as $\frac{d\theta}{dt} = \omega$, therefore $\theta = \omega t$, because $\theta = 0$ when $t = 0$;

$$r = \frac{a}{2}\{e^{\theta} + e^{-\theta}\};$$

and this is the equation to the path described by m; also

$$\frac{dr}{dt} = \frac{a\omega}{2}\{e^{\omega t} - e^{-\omega t}\};$$

$$\begin{aligned} \therefore \quad Q &= a\omega^2\{e^{\omega t} - e^{-\omega t}\} \\ &= a\omega^2\{e^{\theta} - e^{-\theta}\} \\ &= 2\omega^2(r^2 - a^2)^{\frac{1}{2}}; \end{aligned}$$

which is the transversal acceleration.

313.] It remains for us still to investigate the kinematics of a particle describing a curvilinear path in space; and we will at first refer its motion to a system of rectangular axes, and suppose (x, y, z) to be its place at the time t. If three relations can be found between x, y, z and t, the position of the particle at the time t will be completely determined; and if t is eliminated, two equations in terms of x, y, z will result, which will represent two surfaces, the line of intersection of which will be the trajectory of the particle.

Now if (x, y, z) is the place of the particle at the time t, and $(x+dx, \; y+dy, \; z+dz)$ at the time $t+dt$, and if

$$dx^2 + dy^2 + dz^2 = ds^2, \tag{36}$$

then ds is the space described in dt, and the velocity of the particle in its path is $\frac{ds}{dt}$, and the components of this along the three axes are respectively $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$; these results following from the law of resolution of velocities which has just now been investigated.

In the most general case the velocity of the moving particle will be variable along its path, that is, $\frac{ds}{dt}$ will be variable; moreover the velocity-increment along the path, viz. $\frac{d^2s}{dt^2}$, if t is equicrescent, will be also variable; and consequently, generally, dx, dy, dz will vary, and d^2x, d^2y, d^2z will have values; and thus the velocity-increments or accelerations along the coordinate axes will be $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$.

If t is not an equicrescent variable, these quantities severally are

$$\frac{d^2s\,dt-d^2t\,ds}{dt^3}; \tag{37}$$

$$\frac{d^2x\,dt-d^2t\,dx}{dt^3}, \quad \frac{d^2y\,dt-d^2t\,dy}{dt^3}, \quad \frac{d^2z\,dt-d^2t\,dz}{dt^3}. \tag{38}$$

314.] This process of axial resolution of velocity and of velocity-increment is, as we have before observed, artificial; whereas the motion itself suggests tangential and normal resolution; the tangential acceleration being the velocity-increment along the tangent at the point (x, y, z), and the normal acceleration being that with which the particle is deflected from its rectilineal tangential path.

The tangential velocity-increment is evidently $\frac{d^2s}{dt^2}$; the normal velocity-increment is thus found.

Let P, Q, T, fig. 98, be three consecutive points in the curve, when PQ $= ds$ and is the element of the path described in dt; let t be equicrescent, so that QT, which is equal to $ds+d^2s$, is the path described in the next dt. Consequently the particle describes QT with two effective accelerations; one, viz. along QR, of which the mathematical expression is $\frac{d^2s}{dt^2}$, and another along QS, the mathematical expression for which is to be determined. As P, Q, T are three consecutive points in the curve, the plane PQT is the osculating plane of the curve at P, and the angle RQT is the angle of contingence; and QS is the action-line of the normal acceleration, and is the distance through which the particle is displaced. Since however, whatever is the law of acceleration, the acceleration may in the beginning of the motion be taken to be constant, the relation between it the distance and the time is given in Ex. 2, Art. 249; and we have

$$\text{the normal acceleration} = \frac{2.\text{QS}}{dt^2}.$$

But if ρ is the radius of absolute curvature at P; that is, is the radius of the circle passing through P, Q, T, $\text{QS} = \frac{\text{QT}^2}{2\rho} = \frac{ds^2}{2\rho}$;

$$\therefore \text{ the normal acceleration } = \frac{1}{\rho}\frac{ds^2}{dt^2} = \frac{v^2}{\rho}, \tag{39}$$

if v is the velocity at the point P.

Hence if the velocity-increment at any point of the trajectory is resolved along and perpendicular to the tangent, these components are severally expressed by $\frac{d^2s}{dt^2}$ and $\frac{v^2}{\rho}$, the action-line of the latter being in the osculating plane. Thus these quantities do not express torsion; torsion is due to their variations only.

315.] These values may also be deduced from the expressions for the axial accelerations: for resolving these latter along the tangent to the path,

$$\text{the tangential component } = \frac{d^2x}{dt^2}\frac{dx}{ds}+\frac{d^2y}{dt^2}\frac{dy}{ds}+\frac{d^2z}{dt^2}\frac{dz}{ds}$$

$$= \frac{d^2s}{dt^2}, \tag{40}$$

because from (36), $ds\,d^2s = dx\,d^2x + dy\,d^2y + dz\,d^2z$;

and the normal acceleration, by (22) Art. 377, Differential Calculus,

$$= \frac{\rho}{ds}\left\{\frac{d^2x}{dt^2}d.\frac{dx}{ds} + \frac{d^2y}{dt^2}d.\frac{dy}{ds} + \frac{d^2z}{dt^2}d.\frac{dz}{ds}\right\}$$

$$= \frac{\rho}{ds}\,\frac{(d^2x)^2+(d^2y)^2+(d^2z)^2-(d^2s)^2}{ds\,dt^2}$$

$$= \frac{v^2}{\rho}. \tag{41}$$

316.] If the path of the particle is referred to the system of polar coordinates in space, which is explained in Art. 165, Integral Calculus, $\frac{dr}{dt}$, $r\frac{d\theta}{dt}$, $r\sin\theta\frac{d\phi}{dt}$ are the components of the resultant velocity along the radius-vector, perpendicular to r, in the plane containing r and the pole and perpendicular to this latter plane respectively; the line of the last being also tangential to the parallel of latitude which passes through the place of the particle at the time t; and thus these velocities are along lines perpendicular to each other. These values are evident from the explanation given in Art. 165, Integral Calculus. Consequently

$$\frac{ds^2}{dt^2} = \frac{dr^2}{dt^2} + r^2\frac{d\theta^2}{dt^2} + r^2(\sin\theta)^2\frac{d\phi^2}{dt^2}. \tag{42}$$

The components of the velocity-increment along these three lines may be deduced from the axial-components by the ordinary process of resolution. The work however is long and cumbrous, and it is unnecessary to insert it: because hereafter, in the Chapter on Theoretical Dynamics in Vol. IV, a general method will be given by which these results may be immediately obtained from (42). If the reader wishes to exercise himself in the longer process, and P, Q, R are the velocity-increments respectively along the radius-vector, in the plane of r and the pole perpendicular to the radius-vector, and perpendicular to this plane, he will find the following results; viz.

$$\text{P} = \frac{d^2 r}{dt^2} - r\frac{d\theta^2}{dt^2} - r(\sin\theta)^2\frac{d\phi^2}{dt^2}; \tag{43}$$

$$\text{Q} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - r\sin\theta\cos\theta\frac{d\phi^2}{dt^2}; \tag{44}$$

$$\text{R} = \frac{1}{r\sin\theta}\frac{d}{dt}\left\{r^2(\sin\theta)^2\frac{d\phi}{dt}\right\}. \tag{45}$$

If the motion of the particle is confined to one plane, viz. that of (r, θ), $\frac{d\phi}{dt} = 0$, and we have the results already given in (30) and (31).

317.] In the preceding cases we have considered the motion of the particle in reference to a fixed origin, and in the cases of rectangular coordinate axes in reference to fixed axes; and the velocity and other incidents of motion are thus determined absolutely. In many cases however, and especially in astronomical inquiries, it is convenient to refer the motion of the moving particle to the place of another moving particle, taking this latter to be the origin of coordinates, the system of coordinates originating at it also moving with it. The motion of the moving particle is said to be *relative* to this latter particle, and the case is called *relative motion.* I propose at present to take only some simple cases of it.

Let us in the first place suppose the moving axes to be always in the same plane, and to have the same directions. Let the place of the moving particle at the time t be (x, y) in reference to a fixed system, and (ξ, η) in reference to the moving system of axes; (x_0, y_0) being the place of the origin at the same time in reference to the fixed system: then

$$\xi = x - x_0, \qquad \eta = y - y_0; \tag{46}$$

$$\therefore \quad \frac{d\xi}{dt} = \frac{dx}{dt} - \frac{dx_0}{dt}; \qquad \frac{d\eta}{dt} = \frac{dy}{dt} - \frac{dy_0}{dt}; \tag{47}$$

$$\frac{d^2\xi}{dt^2} = \frac{d^2x}{dt^2} - \frac{d^2x_0}{dt^2}; \qquad \frac{d^2\eta}{dt^2} = \frac{d^2y}{dt^2} - \frac{d^2y_0}{dt^2}; \tag{48}$$

and these equations assign the relative velocities and velocity-increments in terms of the absolute velocities and velocity-increments of the moving particle and of the origin.

These expressions may also be found by the following process: In the case of velocity as given by (47), let us suppose a velocity, of which the axial-components are $\frac{dx_0}{dt}$ and $\frac{dy_0}{dt}$, to be communicated to both the moving particle and to the moving origin in directions towards the origin; then it is evident that the *relative* positions and velocities of the particle and origin are not hereby changed; but the effect is to bring to rest the moving origin, and to leave the moving particle with a velocity relatively to it, of which the axial-components are $\frac{dx}{dt} - \frac{dx_0}{dt}$, $\frac{dy}{dt} - \frac{dy_0}{dt}$, which are the same as those given in (47). The system may be subjected to a similar process with reference to velocity-increments, and we shall procure the results given in (48).

Similar results are also true for the relative motion of a particle in space referred to three rectangular axes.

318.] Let us now take a more general case in which the origin describes a curve in the plane of (x, y), and the moving axes rotate in that plane through an angle θ in the time t: let us suppose the fixed and the moving axes to have coincided at the time $t = 0$. Then we have

$$\begin{aligned} x &= x_0 + \xi \cos\theta - \eta \sin\theta, \\ y &= y_0 + \xi \sin\theta + \eta \cos\theta; \end{aligned} \tag{49}$$

therefore

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{dx_0}{dt} + \frac{d\xi}{dt}\cos\theta - \frac{d\eta}{dt}\sin\theta - (\xi\sin\theta + \eta\cos\theta)\frac{d\theta}{dt}, \\ \frac{dy}{dt} &= \frac{dy_0}{dt} + \frac{d\xi}{dt}\sin\theta + \frac{d\eta}{dt}\cos\theta + (\xi\cos\theta - \eta\sin\theta)\frac{d\theta}{dt}. \end{aligned}\right\} \tag{50}$$

Let v_ξ and v_η be the components of the absolute velocity parallel to the axes of ξ and η: then

$$\begin{aligned} v_\xi &= \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta \\ &= \frac{dx_0}{dt}\cos\theta + \frac{dy_0}{dt}\sin\theta - \eta\frac{d\theta}{dt} + \frac{d\xi}{dt}; \end{aligned} \tag{51}$$

$$
\begin{aligned}
v_\eta &= -\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta \\
&= -\frac{dx_0}{dt}\sin\theta + \frac{dy_0}{dt}\cos\theta + \xi\frac{d\theta}{dt} + \frac{d\eta}{dt}; \qquad (52)
\end{aligned}
$$

in which values of v_ξ and v_η all the terms except the last of each are due to the motion of the coordinate system of reference, and the last expresses the axial-component of the velocity of m relatively to the moving axes.

Let us next take the t-differentials of (50): then

$$
\begin{aligned}
\frac{d^2x}{dt^2} = \frac{d^2x_0}{dt^2} - 2\left(\frac{d\xi}{dt}\sin\theta + \frac{d\eta}{dt}\cos\theta\right)\frac{d\theta}{dt} &- (\xi\cos\theta - \eta\sin\theta)\left(\frac{d\theta}{dt}\right)^2 \\
&- (\xi\sin\theta + \eta\cos\theta)\frac{d^2\theta}{dt^2} + \frac{d^2\xi}{dt^2}\cos\theta - \frac{d^2\eta}{dt^2}\sin\theta;
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2y}{dt^2} = \frac{d^2y_0}{dt^2} + 2\left(\frac{d\xi}{dt}\cos\theta - \frac{d\eta}{dt}\sin\theta\right)\frac{d\theta}{dt} &- (\xi\sin\theta + \eta\cos\theta)\left(\frac{d\theta}{dt}\right)^2 \\
&+ (\xi\cos\theta - \eta\sin\theta)\frac{d^2\theta}{dt^2} + \frac{d^2\xi}{dt^2}\sin\theta + \frac{d^2\eta}{dt^2}\cos\theta.
\end{aligned}
$$

Let v'_ξ and v'_η be the axial-components of the absolute velocity-increment parallel to the axes of ξ and η: then

$$
\begin{aligned}
v'_\xi &= \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta \\
&= \frac{d^2x_0}{dt^2}\cos\theta + \frac{d^2y_0}{dt^2}\sin\theta - 2\frac{d\eta}{dt}\frac{d\theta}{dt} - \xi\left(\frac{d\theta}{dt}\right)^2 - \eta\frac{d^2\theta}{dt^2} + \frac{d^2\xi}{dt^2} \\
&= \frac{d^2x_0}{dt^2}\cos\theta + \frac{d^2y_0}{dt^2}\sin\theta - \xi\left(\frac{d\theta}{dt}\right)^2 - \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right) + \frac{d^2\xi}{dt^2}; \qquad (53)
\end{aligned}
$$

$$
\begin{aligned}
v'_\eta &= -\frac{d^2x}{dt^2}\sin\theta + \frac{d^2y}{dt^2}\cos\theta \\
&= -\frac{d^2x_0}{dt^2}\sin\theta + \frac{d^2y_0}{dt^2}\cos\theta + 2\frac{d\xi}{dt}\frac{d\theta}{dt} - \eta\left(\frac{d\theta}{dt}\right)^2 + \xi\frac{d^2\theta}{dt^2} + \frac{d^2\eta}{dt^2} \\
&= -\frac{d^2x_0}{dt^2}\sin\theta + \frac{d^2y_0}{dt^2}\cos\theta - \eta\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right) + \frac{d^2\eta}{dt^2}; \qquad (54)
\end{aligned}
$$

of which expressions for v'_ξ and v'_η all the terms except the last in each are due to the motion of the moving coordinate system of reference; and the last expresses the axial-component of the expressed relative velocity-increment.

319. On examining equations (53) and (54), it appears that the first two terms in the right-hand member of each express those parts of the velocity-increment which arise from the motion of translation of the moving origin, and that the last three

in each arise from the angular motion of the coordinate system of reference and from the relative motion of *m* in reference to that moving system. Now these last are in accordance with the results of radial and transversal resolution which have been discussed in Art. 311. The axial-components of the velocity-increment of *m* at P are the sums of the components of the velocity-increments of equal particles at L and N, which are the projections of P on the axes of ξ and η; for as $\frac{d\theta}{dt}$ is the angular velocity of these axes about the moving origin,

the radial component of L $= \frac{d^2\xi}{dt^2} - \xi\left(\frac{d\theta}{dt}\right)^2$,

the transversal component of L $= \frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right)$;

the radial component of N $= \frac{d^2\eta}{dt^2} - \eta\left(\frac{d\theta}{dt}\right)^2$,

the transversal component of N $= \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right)$;

and consequently of the velocity-increment,

the ξ-component $= \frac{d^2\xi}{dt^2} - \xi\left(\frac{d\theta}{dt}\right)^2 - \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right)$;

the η-component $= \frac{d^2\eta}{dt^2} - \eta\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right)$.

Thus the axial-components along the moving axes of ξ and η respectively of the velocity-increment at the time t are

$$\frac{d^2\xi}{dt^2} + \frac{d^2x_0}{dt^2}\cos\theta + \frac{d^2y_0}{dt^2}\sin\theta - \xi\left(\frac{d\theta}{dt}\right)^2 - \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right),$$

$$\frac{d^2\eta}{dt^2} - \frac{d^2x_0}{dt^2}\sin\theta + \frac{d^2y_0}{dt^2}\cos\theta - \eta\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right).$$

If the origin of the moving axes does not move, and the axes revolve with an uniform angular velocity ω, then (53) and (54) become

$$\frac{d^2\xi}{dt^2} - \omega^2\xi - 2\omega\frac{d\eta}{dt}, \quad \text{and} \quad \frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega\frac{d\xi}{dt}.$$

These equations however refer to a very special case of the general motion.

Kinematics of other and more complicated cases of relative motion will be discussed hereafter: the preceding is sufficient for our present purpose.

SECTION 2.—*The dynamics of a particle moving in a curvilinear path.*

320.] The incidents of motion of a particle moving in a curvilinear path having been considered in the preceding section, we have to investigate the equations connecting these results with the forces of which they are the effects.

When two or more forces act simultaneously on a material particle in motion, the effects are different, and require separate consideration, according as their lines of action are in the line of motion of the particle or make angles with that line; in the former case the effect is an acceleration or retardation of the particle in the line along which it is moving: and the total effect of many such forces is the sum of their several effects; in the latter case, the effect of a force acting along a line which is inclined at a given angle to the line of motion of a particle is partly to produce a deflexion from the rectilinear path in which by the law of inertia the particle would move, and partly to produce an acceleration or a retardation along that line. Such forces therefore will generally cause a particle to describe a curvilinear path: for it is to be observed that although a particle may have a certain velocity, yet that velocity is not an impediment to the independent action of another force on the particle: the material particle, whether in motion or at rest, has the same property of inertia. And another force will produce its own effect on it, and precisely in the same manner as if the particle was not moving. Thus the expressed velocity of the particle will be the resultant of these two several velocities, and its line of motion will depend on the lines of action and the intensities of the two component velocities, and according to a law which has already been investigated. The law of inertia however becomes extended, and we have the following proposition:

When two or more forces impress velocity on a particle, the change in velocity and line of motion of the particle due to each is the same as if the others did not act.

This proposition arises partly from the inertia of matter, and partly from the fact that an adequate and intelligible conception of force requires that it acts on matter and causes it to move along a definite line, and impresses a definite velocity; and consequently by the laws of composition and resolution of velocities, which have been investigated in the preceding section, the resultant velocity will be represented by the diagonal of the parallelogram, of which the two adjacent sides meeting at the position of the particle are the representatives of the separately impressed velocities. This proposition is commonly called *the Second Law of Motion.*

321.] This theorem may be worked out by the following process:

Let O, fig. 7, be the place of the particle *m* at rest at the beginning of the time: let two impulsive forces P and Q act on it, of which the lines of action are OP and OQ; and let us suppose the force P to impress a velocity on *m* so that it would describe the space OP uniformly in *t* units of time: similarly let the force Q impress on *m* a velocity such that it would describe uniformly the space OQ in *t* units of time. Let the figure be constructed as in the diagram; where OR is the diagonal of the parallelogram of which OP and OQ are two containing and adjacent sides; where Q″OP″ is perpendicular to OR, and OQ″= OP″= QQ′= PP′, and where these four lines are all parallel to each other. Now the velocity of which OP is the line-representative may be resolved into two velocities, one of which will be represented by OP′ and the other by OP″; similarly may the velocity of which OQ is the line-representative be resolved into two, of which OQ′ and OQ″ are the line-representatives. Then OQ″ and OP″, being equal and in opposite directions, destroy each other; and OP′ and OQ′ acting along the same line and in the same direction must be added, and of their resultant the line-representative is the sum of OP′ and OQ′, that is, is OR; OR therefore is the line-representative of the velocity which the particle has, and therefore of the resultant of the two component velocities of which OP and OQ are the line-representatives.

Thus if on a particle *m* two impulsive forces act, the lines of action of which are inclined at an angle γ, and if these acting singly would impress on *m* velocities u and v along their lines of action, then, if w is the velocity which one force acting would impress on *m* and produce the same effect as the other two acting in combination,

$$w^2 = u^2 + 2uv\cos\gamma + v^2; \tag{55}$$

and if α and β are the angles between the lines of action of v and w, and of w and u respectively, then

$$\frac{u}{\sin\alpha} = \frac{v}{\sin\beta} = \frac{w}{\sin\gamma}. \tag{56}$$

Similarly if three forces, whose lines of action are mutually inclined at angles α, β, γ, act on a material particle, and are such that each acting singly would impress on it velocities u, v, w along their lines of action, then the one force which would impress on m the same velocity as these three acting simultaneously is that which would impress a velocity σ, where

$$\sigma^2 = u^2 + v^2 + w^2 + 2vw\cos\alpha + 2wu\cos\beta + 2uv\cos\gamma; \quad (57)$$

and its line of action would be parallel to the line whose equations are

$$\frac{x}{u} = \frac{y}{v} = \frac{z}{w}. \quad (58)$$

If $\alpha = \beta = \gamma = 90°$ these results are identical with these determined kinematically in the preceding section.

322.] This result may be illustrated by the following experiment: Let ABC, fig. 96, be the horizontal deck of a boat which is moving with a constant velocity in the direction indicated by the arrow, so that in the time t the point A moves into the position A′, and all the other points on the deck describe straight lines equal and parallel to AA′; and suppose at A a particle m to be placed, and from a force to receive a velocity in the direction AQ, so that if the boat is at rest, in the time t it describes the line AQ: now if the boat is moving, this latter velocity will be combined with that of the boat, and the result is the effect of the two; but neither of them alters the effect of the other; and thus at the end of the time t the particle is found at the point R, having described the diagonal AR, and which is therefore the line-representative of its velocity, because it is described uniformly and in the time t.

Experiments and observations such as the following shew the law to be true in the matter of the earth.

A small heavy particle let fall from the top of a mast of a ship sailing uniformly, falls at the foot of the mast, although the force under the action of which it falls acts vertically and is uniformly accelerating. Thus the particle retains the horizontal velocity which it had at the top of the mast, and this is combined with the vertical impressed velocity.

If a carriage moves evenly along a railway, and if an impulse is given to a ball in it, the effect of the impulse is the same, whatever is the direction in which it is given: the motion of the carriage does not alter the effect of the impulse, and the path and absolute velocity of the ball are of course compounded of the two velocities.

The earth revolves on its polar axis from west to east, and therefore all points on its surface move with a velocity due to this rotation. If therefore the law is not true, a body struck in a direction north or south, would deviate to the west, and this is not found to be the case. And this fact admits of most accurate examination: for suppose a pendulum to be suspended and to oscillate, the time and the extent of oscillation would be different for the different directions of the plane of oscillation: no difference however is observed, whatever is the azimuth of the plane: and the smallest difference, as Laplace has shewn in the Mécanique Céleste, cannot fail of being appreciable.

Again: of a lofty and vertical tower the top is of course farther from the centre of the earth than the bottom, and therefore as the earth rotates from west to east, the horizontal velocity of the top is greater than that of the bottom. Let a heavy ball fall from the top: it will have the horizontal velocity of the top, and this is greater than that of the bottom: if therefore the ball falls on the west side of the tower, it will strike the tower before it reaches the earth: but if it falls on the east side of the tower, it will strike the earth at a small distance from the tower towards the east. These results have been actually observed; and from them we infer the law of which they are the effects.

Also the phaenomena of the aberration of light, and the accordance with observation of the results arrived at, yield another proof of the truth of the law of composition of velocities which we have here investigated. Suppose, see fig. 97, S to be the place of a star, and E to be the place of the earth in its path at the same time: now light travels with a velocity of 186,000 miles in one second of time, and the earth moves in its elliptic path through 19 miles in a second: and let us suppose that in the time during which the light of the star has travelled from S to E, the earth has moved from E to E′, where EE′ is to SE as 19 to 186,000; then the effect to us is the same as if the earth had been at rest, and light had a velocity represented by EE′ from E′ to E along EE′, in combination with its velocity along SE, so that the ray of light would come in the direction S′E, where S′E is the diagonal of the parallelogram of which SE and EE′ are two adjacent containing sides: the star therefore appears to us to be before its real place in the direction of the line of motion of the earth. See also Herschel's "Outlines of Astronomy," Arts. 328–335. Ed. I. 1849.

And, omitting other experiments and observations, I may remark that the most conclusive evidence of the truth of the law of composition of velocities is the *a posteriori* proof arising out of the results of physical astronomy. The expressed velocities and velocity-increments of the planets are resolved and estimated according to it, and their places calculated at particular times; when these are compared with the observed places, no discrepancies are discovered; and thus one of the severest tests of the truth of such a law is applied and is satisfied.

323.] Thus much being all that is necessary to be said as to the effects of force on matter, when two or more forces act simultaneously on a particle along different lines of action and cause it to move in a definite curvilinear path, it remains for us to investigate equations by which that path may be determined when the forces are given.

Let m be the mass of the moving particle; then $m\frac{ds}{dt}$ is its expressed momentum in the line of its motion; $m\frac{dx}{dt}$, $m\frac{dy}{dt}$, $m\frac{dz}{dt}$ are its expressed momenta along the axes of x, y, z respectively if the path is referred to coordinate axes in space; and $m\frac{dx}{dt}$, $m\frac{dy}{dt}$ are the axial-components of the resultant momentum if the motion takes place in the plane of (x, y): momenta are resolved and compounded according to the law of geometrical projection.

Hence also $m\frac{d^2s}{dt^2}$ is the expressed momentum-increment of m in an unit of time along the line of its motion; and $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$, $m\frac{d^2z}{dt^2}$ are the several expressed momentum-increments of m in an unit of time along the coordinate axes of x, y, z.

Also the expressed momentum-increment in the line of motion of a particle at a given time is the sum of the resolved parts along that line of the several expressed momentum-increments along the coordinate axes.

Hence also impressed momenta and momentum-increments and their causes, accelerating forces and moving forces respectively, are resolved and compounded according to the law of geometrical projection.

And as statical pressures, see Art. 261, are virtual dynamical momenta, it follows that statical pressures are resolved and compounded according to the same law: hence we have a proof of the parallelogram of statical forces.

324.] If the motion of the particle m takes place wholly in one plane, the equations which determine that motion are thus found.

Let the motion be referred to a system of rectangular axes; and let X', Y' be the axial-components of the impressed momentum-increment on m at the point (x, y) at the time t; then equating the impressed and the expressed momentum-increments by reason of the law explained in Art. 258, we have

$$m\frac{d^2x}{dt^2} = \text{X}', \qquad m\frac{d^2y}{dt^2} = \text{Y}'; \tag{59}$$

and if X', Y' are proportional to the mass of m,* so that

$$\text{X}' = m\text{X}, \qquad \text{Y}' = m\text{Y},$$

the equations of motion are

$$\frac{d^2x}{dt^2} = \text{X}, \qquad \frac{d^2y}{dt^2} = \text{Y}, \tag{60}$$

in which case X, Y are the impressed velocity-increments which are the effects of the accelerating forces.

If the resultant velocity-increment is resolved tangentially and normally, and T and N are the corresponding components of the impressed velocity-increment, then by (10)

$$\frac{d^2s}{dt^2} = \text{T}, \qquad \frac{v^2}{\rho} = \text{N}. \tag{61}$$

If the motion is referred to a system of polar coordinates, and P and Q are the radial and the transversal components of the impressed velocity-increment, then by (30) and (31)

$$\left.\begin{aligned} \frac{d^2r}{dt^2} - r\frac{d\theta^2}{dt^2} &= \text{P}, \\ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) &= \text{Q}. \end{aligned}\right\} \tag{62}$$

325.] The three following Chapters will contain so many illustrations of these equations, that it is unnecessary to shew their application immediately; but the attention of the reader must be at once called to the manner in which they are consistent with the principle of the conservation of work.

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* The impressed velocity-increments are not always proportional to m: instances of the non-proportionality have already occurred in the equations of motion in Articles 293 and 294.

Let the two equations (60) be multiplied by $m\,dx$ and $m\,dy$ respectively, where dx and dy are the actual axial displacements of m in the time dt: so that according to the definition of work given in Art. 259, $m\,\text{X}\,dx$ and $m\,\text{Y}\,dy$ are the elements of the work done by the accelerating forces X and Y in the time dt, and the integrals of these expressions are the works done by those forces through the assigned limits, whatever they are. Let the equations after multiplication by $m\,dx$ and $m\,dy$ respectively be added; then we have

$$m\frac{dx\,d^2x+dy\,d^2y}{dt^2} = m(\text{X}\,dx+\text{Y}\,dy);$$

$$\therefore \quad \frac{m}{2}(v^2-v_0^2) = \int_{x_0,\,y_0}^{x,\,y} m(\text{X}\,dx+\text{Y}\,dy); \tag{63}$$

the left-hand member of which is the vis viva of m acquired in its motion from (x_0, y_0) to (x, y) under the action of the given forces; and the right-hand member is the work done by the forces through the spaces assigned by the limits. This equation is called *the equation of vis viva and of work*. Also from (61) and from (62) we have

$$\frac{m}{2}(v^2-v_0^2) = \int_{s_0}^{s} m\,\text{T}\,ds; \tag{64}$$

$$\frac{m}{2}(v^2-v_0^2) = \int_{r_0,\,\theta_0}^{r,\,\theta} m(\text{P}\,dr+\text{Q}\,r\,d\theta); \tag{65}$$

each of these being the equation of work in its own system of reference. The circumstances under which these equations are possible and are interpretable will be explained in a future Chapter.

326.] As the normal component of the impressed momentum-increment does not appear in the equation of vis viva, viz. (64), the force, which by its action impresses it, does no work; the reason being that the path of m is always at right angles to the action-line of this force. Since however m would, if left to itself or if left to the action of a force which acted along the tangent, by the law of inertia, continue to move along that tangent, so $\frac{mv^2}{\rho}$ is the effect of the force which deflects m from its otherwise rectilinear path, and draws it towards the centre of absolute curvature. This force is called *the centripetal force;*

and the expressed normal momentum-increment is called *the centrifugal force.*[*] Hence

$$\text{the centrifugal force of } m = \frac{mv^2}{\rho}. \tag{66}$$

Centripetal and centrifugal forces are therefore the same quantity under different aspects. Centripetal force is the force which impresses the momentum, and is spoken of in reference to that force: centrifugal force is spoken of with reference to the particle m, and is that force of which $m\frac{v^2}{\rho}$ is the expressed momentum.

327.] As an exact knowledge of the nature of centrifugal force is necessary for a complete comprehension of the theory of curvilinear motion, let us examine it in two or three applications.

Let m move in a circle with a constant velocity v; let a = the radius of the circle, and T = the time of a complete revolution: then $2\pi a = \text{vT}$;

$$\therefore \quad \text{the centrifugal force of } m = m\frac{4\pi^2 a}{\text{T}^2}; \tag{67}$$

thus the centrifugal force in a circle varies directly as the radius of the circle, and inversely as the square of the periodic time.

Again, if m moves in the circle with a constant angular velocity ω, then by (24), Art. 310, $\text{v} = a\omega$;

$$\therefore \quad \text{the centrifugal force of } m = m\omega^2 a; \tag{68}$$

and therefore varies directly as the radius of the circle.

Thus if a particle of mass m is fastened by a string of length a to a point in a horizontal plane, and describes a circle in the plane about the given point as centre, the centrifugal force produces a tension of the string, and if ω is the constant angular velocity, the tension $= m\omega^2 a$.

A railway-carriage of given weight and dimensions moves at a given velocity along a curved railway of which the radius of curvature is known. It is required to find the height to which the outer rail should be raised above the inner one so as to counteract the effects of centrifugal force.

Let $2a$ be the horizontal breadth of the railway, and h the height of the centre of gravity of the carriage above the rails when they are horizontal; let z be the elevation of the outer rail, and θ the angle of inclination to the horizon of the transverse section of the road when the outer rail is raised; so that $z = 2a\tan\theta$; let $v =$ the given velocity of the carriage, ρ the radius of the curve; then as the outer rail is to be raised so that at the given maximum velocity there shall be no pressure on the inner rail, we may take about the outer rail moments of the centrifugal force and of the weight of the carriage and may equate them: whereby if m is the mass of the carriage, we have

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[*] This term is inaccurate; $\frac{v^2}{\rho}$ is the expressed effect of an accelerating force.

$$\frac{mv^2}{\rho}(h\cos\theta - a\sin\theta) = mg(a\cos\theta + h\sin\theta);$$

$$\therefore \quad z = 2a\tan\theta = 2a\frac{v^2 h - \rho a g}{v^2 a + \rho h g}.$$

328.] When a solid body rotates about an axis, all its particles describe in equal times circles, the planes of which are perpendicular to the axis of rotation, the centres of which are in this axis, and the radii of which are the perpendiculars from each point on the axis: therefore from (68), as ω is the same for all the points, the centrifugal forces vary as these perpendiculars. Now as the earth revolves about its polar axis, the centrifugal forces of particles on its surface vary as the perpendicular distances from the particle on the polar axis; and therefore the centrifugal force of each particle varies as the radius of the parallel of latitude which the particle describes: and the line of action, being the radius of the circular path in which the particle moves, is perpendicular to the polar axis. As the radius of the circle of the parallel of latitude decreases from the equator to the pole, so does the centrifugal force which varies as this radius by reason of (68); the centrifugal force therefore is greatest at the equator and least at the poles, where it vanishes.

Let us consider the earth to be a perfect sphere, and to be revolving with an angular velocity such that the time of the revolution is 24 hours; and let us consider a place on it of which the latitude is λ; then, if r is the radius of the earth, the radius of the circle of the parallel of latitude is $r\cos\lambda$; and therefore if ω is the angular velocity, the centrifugal force of m in this line, which is perpendicular to the polar axis, is $m\omega^2 r\cos\lambda$; and if this is resolved into two parts, one of which is horizontal, and the other is vertical, at the place, the vertical component of the centrifugal force is $m\omega^2 r(\cos\lambda)^2$; by this quantity therefore will the attraction of the earth on a particle m on its surface at the latitude λ be diminished: so that if mg is the weight of m

when diminished by the centrifugal force, and $m\mathrm{G}$ were its weight if the earth were at rest, then

$$mg = m\mathrm{G} - m\omega^2 r(\cos\lambda)^2; \tag{69}$$

$$\therefore\quad g = \mathrm{G} - \omega^2 r(\cos\lambda)^2; \tag{70}$$

therefore $\omega^2 r(\cos\lambda)^2$ is the diminution of the earth's gravity due to the centrifugal force.

The diminution of gravity is the greatest when $\lambda = 0$, that is, at the equator; in which case

$$\begin{aligned} g &= \mathrm{G}\left\{1 - \frac{\omega^2 r}{\mathrm{G}}\right\} \\ &= \mathrm{G}\left\{1 - \frac{\omega^2 r}{g}\right\}, \end{aligned} \tag{71}$$

since the difference between g and G is very small. Let T = the time of rotation of the earth about its axis; therefore $\mathrm{T} = 24 \times 60 \times 60$ seconds. Also $2\pi = \omega\mathrm{T}$; and $g = 32.2$ feet; $\pi = 3.14159$; therefore $r = 4000 \times 1760 \times 3$ feet,

$$\begin{aligned} \frac{\omega^2 r}{g} &= \frac{4\pi^2 r}{g\mathrm{T}^2} \\ &= \frac{1}{289} \text{ nearly}; \end{aligned}$$

$$\therefore\quad g = \mathrm{G}\left\{1 - \frac{1}{289}\right\}; \tag{72}$$

that is, the diminution of gravity at the earth's equator due to the centrifugal force is the 289th part of that which the earth's attraction at the equator would be if the earth did not rotate.

Thus also the weight of a body m at the equator is diminished by its 289th part, and the diminution of its weight at the latitude λ is

$$\frac{mg(\cos\lambda)^2}{289}.$$

The preceding calculation is made on the hypothesis that the earth is a perfect sphere, whereas it is an oblate spheroid: and the attraction of the earth on particles at its surface decreases as we pass from the poles to the equator according to the law investigated in Art. 212, and given in (91) of that Article. The present inquiry gives the law of diminution of gravity on account of centrifugal force; the combination of these two effects produces the result given in Art. 124, viz.

$$g = \mathrm{G}\{1 + .005133(\sin\lambda)^2\}; \tag{73}$$

and the whole diminution is nearly a 200th part of the original gravity. Hence also the weight of a body at the poles is one 200th more that its weight at the equator.

Since 289 is the square of 17, it follows that if the earth completed a revolution about its polar axis in the 17th part of a day, the centrifugal force at the equator would be equal to, and would neutralize, the earth's attraction; in which case matter at the equator would have no weight.

329.] If the path of the moving particle m is a curvilinear path in space and is referred to three fixed rectangular axes, then if $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ are the axial-components of the impressed velocity-increment, the equations of motion are

$$\frac{d^2x}{dt^2} = \mathrm{X}, \qquad \frac{d^2y}{dt^2} = \mathrm{Y}, \qquad \frac{d^2z}{dt^2} = \mathrm{Z}; \tag{74}$$

and these are applicable to the solution of every problem involving such a motion and such forces.

Thus, if the laws of the impressed velocity-increments are given, and if the problem is the deduction from them of the equations of the trajectory, (74) must be integrated, whereby three relations will be given between x, y, z and t; whence t may be eliminated, and two equations will result in terms of x, y, z, which will represent two surfaces, the line of intersection of which will be the trajectory. In the course of integration, twelve limiting values will be required, viz. the six components of the velocities corresponding to $t = t$ and to $t = 0$; and the six coordinates of m corresponding to the same values of t: of these, six, viz. those corresponding to $t = t$, will be left in the general equations in their general form: the other six, which correspond to $t = 0$, will enter into the final equations as arbitrary constants, because the beginning of the time t is arbitrary.

330.] If the particle m is not acted on by any forces, so that $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = 0$, then

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = 0, \qquad \frac{d^2z}{dt^2} = 0. \tag{75}$$

Let (a, b, c) be the place of the particle when $t = 0$, and (x, y, z), when $t = t$: also let α, β, γ be the components of its velocity when $t = 0$; then integrating (75) between the limits corresponding to these values of t,

$$\frac{dx}{dt} - \alpha = 0; \qquad \frac{dy}{dt} - \beta = 0; \qquad \frac{dz}{dt} - \gamma = 0;$$

$$x - a - \alpha t = 0; \qquad y - b - \beta t = 0; \qquad z - c - \gamma t = 0;$$

$$\therefore \quad \frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma} = t;$$

which are the equations to a straight line, whose direction-cosines are proportional to the components of the velocity when $t = 0$, and which passes through the point (a, b, c). If there is no initial velocity, $\alpha = \beta = \gamma = 0$; in which case $x = a$, $y = b$, $z = c$; that is, the particle remains at rest and in its original position. The result is of course in accordance with the law of inertia.

331.] If the place and motion of the particle are referred to a system of polar coordinates in space, and P, Q, R are the impressed velocity-increments along the radius-vector, along a line in the plane of r and the pole and perpendicular to r, and along a line perpendicular to this plane, then the equations of motion are

$$\left.\begin{aligned} \frac{d^2 r}{dt^2} - r\frac{d\theta^2}{dt^2} - r(\sin\theta)^2\frac{d\phi^2}{dt^2} &= \mathrm{P}, \\ \frac{1}{r}\frac{d}{dt}\Big(r^2\frac{d\theta}{dt}\Big) - r\sin\theta\cos\theta\frac{d\phi^2}{dt^2} &= \mathrm{Q}, \\ \frac{1}{r\sin\theta}\frac{d}{dt}\Big\{r^2(\sin\theta)^2\frac{d\phi}{dt}\Big\} &= \mathrm{R}. \end{aligned}\right\} \qquad (76)$$

332.] The equations of relative motion, of which the kinematics have been investigated in Arts. 317–319, are formed on the same principle of inertia. Thus as to equations (48), let X, Y be the axial-components of the impressed velocity-increment acting on m, and let X_0, Y_0 be the axial-components of the impressed velocity-increment acting on a particle of the mass m at the origin or the axial-components of the acceleration of the origin; then the equations of motion are

$$\frac{d^2\xi}{dt^2} = \mathrm{X} - \mathrm{X}_0, \qquad \frac{d^2\eta}{dt^2} = \mathrm{Y} - \mathrm{Y}_0. \qquad (77)$$

If the path described by m is referred to three axes in space, a third equation in terms of ζ has to be added.

If the motion of m is referred to a system of moving axes as also to a moving origin, then from (53) and (54) the equations of motion are

$$\frac{d^2\xi}{dt^2} - \xi\frac{d\theta^2}{dt^2} - \frac{1}{\eta}\frac{d}{dt}\Big(\eta^2\frac{d\theta}{dt}\Big) = (\mathrm{X} - \mathrm{X}_0)\cos\theta + (\mathrm{Y} - \mathrm{Y}_0)\sin\theta; \quad (78)$$

$$\frac{d^2\eta}{dt^2} - \eta\frac{d\theta^2}{dt^2} + \frac{1}{\xi}\frac{d}{dt}\Big(\xi^2\frac{d\theta}{dt}\Big) = (\mathrm{Y} - \mathrm{Y}_0)\cos\theta - (\mathrm{X} - \mathrm{X}_0)\sin\theta. \quad (79)$$

CHAPTER X.

APPLICATION OF THE EQUATIONS OF THE PRECEDING CHAPTER TO PARTICLES MOVING FREELY IN SPACE.

SECTION 1.—*Oblique impact and collision of particles and of smooth spherical balls.*

333.] The laws of resolution of velocities and momenta, and the principles of the theory of curvilinear motion have been investigated in the preceding Chapter; our object now is further to elucidate these, and to apply them to the special case of particles moving freely in space, reserving to subsequent Chapters the cases of particles moving on curves or on surfaces or in tubes; cases, that is, of constrained motion.

The most simple case of resolution of momenta is that of a material particle or of a smooth homogeneous spherical ball, which is supposed to have motion of translation only, which moves in a rectilinear path with a constant velocity, and impinges at an oblique angle on a given plane. As the velocity of the ball is constant before impact, so will the velocity be also constant after impact: and thus the inquiry is limited to the circumstances of change during the collision, and to the determination of the line of motion and of the velocity of the ball after impact. The elasticity of the ball is supposed to be the same as that which was explained in Art. 262; that is, the momentum acquired during the restitution of the figure of the ball is supposed to bear a constant ratio to that lost during the compression of the figure: the limiting values of the ratio being 1 and 0, according as the ball and plane are perfectly elastic or are perfectly inelastic.

One observation however is to be made on the circumstances of oblique impact, which was not applicable in that of direct impact. In oblique impact we assume that the mutual action of the balls during collision is along the line joining their centres at the instant when compression is a maximum, and along that line only; that is, in other words, we assume the balls to be perfectly smooth. For suppose a ball of mass m, see fig. 102, to move uniformly along the line aA and to impinge on the ball of mass m', which is moving at an uniform velocity along the line bB: and suppose the line OAB to be that which passes through their centres A, B at the instant at which compression is a maximum: we assume the action of the two balls on each other for the time during which the collision takes place to be wholly along this line; along this line has momentum been lost during the compression, along this line will momentum be acquired during the restitution: the momenta in a line perpendicular to AB have not been altered by the collision: thus by virtue of the statements made in Art. 262, if P represents the momentum along the line AB lost during the compression, eP represents that acquired during the restitution along the same line. Thus although for the time during which the balls are in collision, they, by reason of their velocity which is perpendicular to AB, slide on each other, and thereby the momentum in that line would be changed if the balls are not perfectly smooth, yet for the sake of simplicity we assume the roughness of the balls to be so far diminished, that it is not necessary to take account of the change of momentum along the line perpendicular to AB.

Hence if a smooth ball impinges obliquely on a smooth plane, the line of reaction of the plane will be perpendicular to its surface, and the momentum of the impinging ball will be affected along that line only, and not along the plane.

334.] The results of the direct impact of a ball on a plane are given in equations (15), (16), Art. 264, so that if v is the velocity of impact, ev is the velocity of rebound.

But suppose a perfectly smooth and spherical ball, whose mass is m, and whose elasticity is e, to move with an uniform velocity v, and to impinge at P on a smooth plane in the line aA, making an angle α with the normal to the plane at the point P, fig. 103, so that $a\text{AN} = \alpha$; let u be the velocity of m when the compression is a maximum: at which instant the motion of the ball is wholly along the plane; and suppose V to be the velocity of rebound, and $\beta = \text{NA}b$ to be the angle which the line of motion of the ball after rebound at P makes with the normal: let P be the momentum in the line AN which is lost by the ball during the compression, and let eP be that recovered during the restitution, the line of action of both these momenta being the normal AN. Let us resolve the momenta along and perpendicular to the plane: then at the instant when the compression is a maximum, we have (1) along the plane,

$mv \sin \alpha =$ the impressed momentum of m,

$mu =$ the expressed momentum of m;

(2) perpendicular to the plane,

$mv \cos \alpha =$ the impressed momentum of m,

$\text{P} =$ momentum lost by m during compression;

$$\therefore \quad mv \sin \alpha = mu; \tag{1}$$

$$mv \cos \alpha = \text{P}: \tag{2}$$

by a similar process when restitution ends, we have (1) along the plane,

$mu =$ the impressed momentum of m,

$m\text{V} \sin \beta =$ the expressed momentum of m;

(2) perpendicular to the plane,

$e\text{P} =$ the impressed momentum of m,

$m\text{V} \cos \beta =$ the expressed momentum of m;

$$\therefore \quad mu = m\text{V} \sin \beta, \tag{3}$$

$$e\text{P} = m\text{V} \cos \beta; \tag{4}$$

$$\therefore \quad v \sin \alpha = \text{V} \sin \beta, \qquad e \tan \beta = \tan \alpha;$$

$$\therefore \quad \tan \beta = \frac{\tan \alpha}{e}; \tag{5}$$

$$\text{V} = \frac{\sin \alpha}{\sin \beta} v; \tag{6}$$

whereby β and V are known in terms of given quantities; α and β are called respectively the angles of *incidence* and *reflexion.* Generally as e is less than unity, β is greater than α. If the ball is perfectly elastic, $e = 1$: in which case $\alpha = \beta$, that is, the angles of incidence and reflexion are equal; and $\text{V} = v$, that is, the velocities of incidence and reflexion are equal to each other.

If the ball is perfectly inelastic, $e = 0$; in which case, $\beta = 90°$, and $\text{V} = v \sin \alpha$; the ball, that is, after impact moves along the plane with the component in that line of its velocity of impact.

These results are equally true, when the ball impinges on a curved surface, the plane of impact being in this case the tangent plane to the surface at the point of impact.

335.] Examples in illustration of the preceding principles:

Ex. 1. To find the line along which a ball of given elasticity e must be projected from a given point A, so that after reflexion at a given plane it may strike another given ball at B.

Let OD, fig. 104, be the given plane, A and B the given positions of the balls. From A and B draw AC and BD at right angles to the plane OD; produce AC to A′, making CA′ = e.CA. Join BA′ cutting CD in P, and join AP; APB is the required path. At P draw the line PN normal to the plane: then APN is the angle of incidence, and NPB is the angle of reflexion: and since

$$\tan \text{APN} = \cot \text{APC} = \frac{\text{CP}}{\text{AC}};$$

and
$$\tan \text{BPN} = \cot \text{BPD} = \cot \text{A}'\text{PC} = \frac{\text{CP}}{\text{A}'\text{C}} = \frac{\text{CP}}{e.\text{AC}};$$

$$\therefore \quad \tan \text{BPN} = \frac{1}{e} \tan \text{APN};$$

and therefore the path APB satisfies the condition (5), and is such that a ball projected from A along AP will strike another ball at B.

Ex. 2. To determine the path which a ball of elasticity e must take with reference to two given inclined planes, so that when projected from a given point A it may after reflexion successively at the two planes strike another ball at B, A and B being in the plane which is perpendicular to the line of intersection of the two planes of reflexion.

Let the plane of the paper, fig. 105, be that in which the two points A and B are, and thus the planes of reflexion are perpendicular to the plane of the paper.

From A draw ADA′ perpendicular to the plane DO, and make A′D = e.AD: also from B draw BCB′ perpendicular to OC, and such that BC = e.B′C: draw A′B′ cutting DO in P and OC in Q, join AP and BQ; APQB shall be the path required.

It is evident by the construction in the former example that the lines AP and PQ satisfy at P the condition required in (5): and also that PQ and QB satisfy the same condition at Q: therefore APQB is the required path.

Also by a similar process may the path be determined, by which a ball of given elasticity may after projection from a given point and reflexion at given planes strike a ball placed at another given point.

Ex. 3. A ball of given elasticity e is projected from a given point in the circumference of a circle, and after two reflexions from the circle returns to the same point: determine the angle at which it was projected.

Let A, fig. 106, be the point in the circle whence the ball is projected, and let AP, PQ, QA be the paths which it successively describes: let $\text{CAP} = \theta = \text{CPA}$, $\text{CPQ} = \phi = \text{CQP}$, $\text{CQA} = \psi = \text{CAQ}$. Then by (5),

$$\tan\phi = \frac{1}{e}\tan\theta, \qquad \tan\psi = \frac{1}{e}\tan\phi = \frac{1}{e^2}\tan\theta;$$

$$\text{also} \qquad \theta + \phi + \psi = 90°;$$

$$\therefore \quad \tan\theta = \cot(\phi+\psi) = \frac{1-\tan\phi\tan\psi}{\tan\phi+\tan\psi}$$

$$= \frac{e^3-(\tan\theta)^2}{(e^2+e)\tan\theta};$$

$$\therefore \quad \tan\theta = \left(\frac{e^3}{1+e+e^2}\right)^{\frac{1}{2}}.$$

336.] Two smooth spheres of given elasticity and of masses m and m', moving in given lines, and with given velocities, and with their centres in the same plane, impinge on each other; it is required to determine their velocities and lines of motion after impact.

Let us suppose the two spheres to be moving in the directions indicated by the arrows, fig. 102, and along the lines aA and bB, and let OABx be the line passing through their centres at the instant when the compression is a maximum: and let m, whose centre is A, impinge on m', whose centre is B. Now by Art. 333, the elastic action of the two spheres takes place along the line OAB only: let $e =$ the elasticity, and $\text{P} =$ the momentum lost during compression, so that $e\text{P}$ is that acquired during the restitution of the figure of the balls. The momenta will be resolved along, and at right angles to, the line OAB.

Let v, v' be the velocities of m, m' before collision begins,

u, u' - - - - - - - when compression is a maximum,

V, V' - - - - - - - when collision ends;

α, α' be the angles between OAB, and the lines of motion of m, m' when collision begins,

θ, θ' - - - - - - - - - when compression is a maximum,

β, β' - - - - - - - - - when collision ends.

Then at the instant when the compression is a maximum, the equations of resolved momenta are,

$$\text{for } m \begin{cases} mv\cos\alpha = mu\cos\theta + \mathrm{P}, & (7) \\ mv\sin\alpha = mu\sin\theta, & (8) \end{cases}$$

$$\text{for } m' \begin{cases} m'v'\cos\alpha' = m'u'\cos\theta' - \mathrm{P}, & (9) \\ m'v'\sin\alpha' = m'u'\sin\theta'; & (10) \end{cases}$$

and at the instant when collision ends,

$$\text{for } m \begin{cases} mu\cos\theta = m\mathrm{V}\cos\beta + e\mathrm{P}, & (11) \\ mu\sin\theta = m\mathrm{V}\sin\beta, & (12) \end{cases}$$

$$\text{for } m' \begin{cases} m'u'\cos\theta' = m'\mathrm{V}'\cos\beta' - e\mathrm{P}, & (13) \\ m'u'\sin\theta' = m'\mathrm{V}'\sin\beta'. & (14) \end{cases}$$

But when the compression is a maximum, both balls are moving with the same velocity along OAB: therefore

$$u\cos\theta = u'\cos\theta'. \qquad (15)$$

From these nine equations, V, V', β, β' are to be determined.

From (7), (9) and (15),

$$u\cos\theta = u'\cos\theta' = \frac{mv\cos\alpha + m'v'\cos\alpha'}{m+m'}; \qquad (16)$$

from which and from (7) and (11), we have

$$\mathrm{V}\cos\beta = \frac{mv\cos\alpha + m'v'\cos\alpha'}{m+m'} - \frac{em'}{m+m'}(v\cos\alpha - v'\cos\alpha'). \qquad (17)$$

Similarly,

$$\mathrm{V}'\cos\beta' = \frac{mv\cos\alpha + m'v'\cos\alpha'}{m+m'} + \frac{em}{m+m'}(v\cos\alpha - v'\cos\alpha'). \qquad (18)$$

Also from (8) and (12), and from (10) and (14),

$$\mathrm{V}\sin\beta = v\sin\alpha, \qquad (19)$$

$$\mathrm{V}'\sin\beta' = v'\sin\alpha'; \qquad (20)$$

so that V, V', β, β' are completely determined.

Also from the preceding we have

$$\left.\begin{aligned} m\mathrm{V}\cos\beta + m'\mathrm{V}'\cos\beta' &= mv\cos\alpha + m'v'\cos\alpha', \\ m\mathrm{V}\sin\beta + m'\mathrm{V}'\sin\beta' &= mv\sin\alpha + m'v'\sin\alpha'; \end{aligned}\right\} \qquad (21)$$

that is, the sum of the momenta both along, and at right angles to, OAB is the same before and after impact.

337.] Let the balls be perfectly elastic, that is, let $e=1$: then

$$\mathrm{V}\cos\beta = \frac{mv\cos\alpha + m'v'\cos\alpha'}{m+m'} - \frac{m'(v\cos\alpha - v'\cos\alpha')}{m+m'}, \qquad (22)$$

$$\mathrm{V}'\cos\beta' = \frac{mv\cos\alpha + m'v'\cos\alpha'}{m+m'} + \frac{m(v\cos\alpha - v'\cos\alpha')}{m+m'}, \qquad (23)$$

$$\mathrm{V}\sin\beta = v\sin\alpha, \qquad (24)$$

$$\mathrm{V}'\sin\beta' = v'\sin\alpha'; \qquad (25)$$

$$\therefore \quad m\mathrm{V}^2 + m'\mathrm{V}'^2 = mv^2 + m'v'^2; \qquad (26)$$

that is, the sum of the vires vivæ is the same before and after impact, and no work is lost by the collision.

Again, let the balls be perfectly inelastic, that is, let $e=0$: then

$$\mathrm{v}\cos\beta = \mathrm{v}'\cos\beta' = \frac{mv\cos a + m'v'\cos a'}{m+m'}, \tag{27}$$

$$\mathrm{v}\sin\beta = v\sin a, \tag{28}$$

$$\mathrm{v}'\sin\beta' = v'\sin a'; \tag{29}$$

that is, the balls after impact have the same velocity along the line OAB, but unequal velocities at right angles to that line; also

$$\frac{\tan\beta}{\tan\beta'} = \frac{v\sin a}{v'\sin a'}. \tag{30}$$

338.] The velocity and the line of motion of the centre of gravity of two smooth balls which impinge on each other is the same before and after impact.

Let e be the elasticity of the balls: and let us take a line parallel to that which joins the centres of the balls at the instant when the compression is a maximum to be the axis of x; and a line perpendicular to it to be that of y: let (x, y), (x', y') be the places of the centres of the balls at the time t, either before or after impact: and let $(\bar{x}, \bar{y})$ be the place of the centre of gravity at the same time: then

$$\left.\begin{aligned}(m+m')\,\bar{x} &= mx+m'x',\\ (m+m')\,\bar{y} &= my+m'y';\end{aligned}\right\} \tag{31}$$

$$\therefore\quad \left.\begin{aligned}(m+m')\frac{d\bar{x}}{dt} &= m\frac{dx}{dt}+m'\frac{dx'}{dt},\\ (m+m')\frac{d\bar{y}}{dt} &= m\frac{dy}{dt}+m'\frac{dy'}{dt};\end{aligned}\right\} \tag{32}$$

but $\frac{dx}{dt}$ and $\frac{dx'}{dt}$ are before impact respectively $v\cos a$ and $v'\cos a'$; and are after impact respectively $\mathrm{v}\cos\beta$ and $\mathrm{v}'\cos\beta'$: and $\frac{dy}{dt}$ and $\frac{dy'}{dt}$ are before impact respectively $v\sin a$ and $v'\sin a'$, and are after impact respectively $\mathrm{v}\sin\beta$ and $\mathrm{v}'\sin\beta'$: therefore by virtue of equations (21), the right-hand members of (32) are the same before and after impact; therefore also the left-hand members are: and therefore $\frac{d\bar{x}}{dt}$ and $\frac{d\bar{y}}{dt}$ are the same before and after impact: and therefore the velocity and the line of motion of the centre of gravity of m and m' are the same before and after impact.

339.] Examples illustrative of the preceding equations:

Ex. 1. A smooth and homogeneous sphere of mass m and elasticity e moves with a velocity v and impinges directly on another of the mass m', which is moving with a velocity v' in a line of motion at right angles to that of m; it is required to find the velocities and the lines of motion of both balls after the collision.

In this case $\alpha = 0$, $\alpha' = 90°$; therefore from (17), &c.,

$$\mathrm{v}\cos\beta = v\,\frac{m-em'}{m+m'},$$

$$\mathrm{v}\sin\beta = 0,$$

$$\mathrm{v}'\cos\beta' = (1+e)\,\frac{mv}{m+m'},$$

$$\mathrm{v}'\sin\beta' = v';$$

$$\therefore\quad \beta = 0, \qquad \tan\beta' = \frac{v'}{v}\,\frac{m+m'}{(1+e)m},$$

$$\mathrm{v} = v\,\frac{m-em'}{m+m'}, \qquad \mathrm{v}'^2 = v'^2 + \left\{\frac{(1+e)mv}{m+m'}\right\}^2.$$

Hence the ball m will continue to move in the same line of motion, but with a velocity diminished in the ratio of $m-em'$ to $m+m'$ to its former velocity; the velocity of m' will be increased, and its line of motion will make an angle β' with that of m's motion.

Ex. 2. Two balls m and $2m$, whose elasticity is $\frac{2}{3}$, move with velocities $2v$ and v, and impinge, so that the line of motion of each one makes an angle of 60° with the line joining their centres at the instant when compression is a maximum; it is required to determine their motion after collision.

Here $m' = 2m$, $v' = \frac{v}{2}$, $e = \frac{2}{3}$, $\alpha = \alpha' = 60°$; therefore from (17), &c.,

$$\mathrm{v}\cos\beta = \frac{2v}{9}, \qquad \mathrm{v}'\cos\beta' = \frac{7v}{18};$$

$$\mathrm{v}\sin\beta = \frac{(3)^{\frac{1}{2}}v}{2}, \qquad \mathrm{v}'\sin\beta' = \frac{3^{\frac{1}{2}}v}{4};$$

whence v, v', β, and β' are easily determined.

340.] In Article 267 we estimated the loss of momentum of a body in its passage through a resisting medium, when the body presented to the medium a plane surface of area ω, which is perpendicular to the line of motion of the body. The investigation can now be extended to the case in which the surface on which the medium acts is inclined at any angle to the line of motion. The velocity of the body will be resolved into two components, of which one is perpendicular to the plane of the surface, and the other is along the surface: the former alone causes the resistance by reason of momentum being transferred to the elements of the resisting medium; the latter produces only a friction along the surface, the calculation of which does not belong to the present part of our work.

Let there then be a plane surface, of which the area is ω, perpendicular to the plane of the paper, and of which the section by the plane of the paper is the line OP, fig. 107; and suppose it to be moving in the line MO, and its normal to be inclined to MO at the angle i; let v be the velocity of the body along the line of motion; then $v \cos i$ is the velocity of OP in the line of its normal; and therefore by a process similar to that of Art. 267 it follows, that the momentum which is impressed by ω on the particles of the resisting medium during the time dt, and which has therefore been withdrawn from the moving body, is

$$\rho \omega v^2 (\cos i)^2 \, dt;$$

but the line of action of this resistance is in the normal to OP; therefore its component in the line of motion of the moving body is

$$\rho \omega v^2 (\cos i)^3 \, dt; \qquad (33)$$

and therefore if m is the mass of the moving body, and dv its loss of velocity in the line of its motion during the time dt owing to the resistance of the medium,

$$-m\,dv = \rho \omega v^2 (\cos i)^3 \, dt;$$

$$\therefore \quad m \frac{dv}{dt} = -\rho \omega v^2 (\cos i)^3. \qquad (34)$$

Hence it appears that the resistance of a plane rudder passing through the water varies as the cube of the sine of the angle at which it is inclined to the keel of the vessel.

A few examples illustrative of (33) are subjoined.

Ex. 1. An isosceles triangular wedge, of which the vertical angle is $2a$, the depth is b, and the altitude is a, moves in a resisting medium, firstly with its edge forward, secondly with its top forward: compare the resistances in the two cases.

Let R_1 and R_2 be the resistances in the first and second cases respectively, then

$$R_1 = 2\rho v^2 (\sin\alpha)^3 ab \sec\alpha, \qquad R_2 = 2\rho v^2 ab \tan\alpha;$$

$$\frac{R_1}{R_2} = (\sin\alpha)^2.$$

Ex. 2. A semicircular lamina of given thickness τ moves in a fluid, firstly with its convex edge forwards, secondly with its base forwards: compare the resistances in the two cases.

Let the resistances be R_1 and R_2: let $a =$ the radius of the semicircle: then, fig. 108, if $POA = \theta$, $QOP = d\theta$, $PQ = a\,d\theta$,

$$R_1 = 2a\tau\rho v^2 \int_0^{\frac{\pi}{2}} (\cos\theta)^3\, d\theta$$

$$= \frac{4a\tau\rho v^2}{3};$$

$$R_2 = 2a\tau\rho v^2;$$

$$\therefore \quad \frac{R_1}{R_2} = \frac{2}{3}.$$

341.] By means of (33) also can be determined the resistance which a solid of revolution meets with in its passage through a resisting medium, such as water or air.

Let OPB, fig. 109, be the generating curve of the bounding surface of the solid, and let its equation be $y = f(x)$; $OM = x$, $MP = y$; and let PG be the normal to the curve at the point P, so that

$$\cos PGO = \frac{dy}{ds}.$$

Let an element of the curve at $P = ds$; so that of a surface-element at P, ds is the section by the plane of the paper: also let the surface-element subtend at an angle $d\theta$ at M; and thus, if $\omega =$ the surface-element,

$$\omega = ds\, y\, d\theta;$$

and therefore the loss of momentum corresponding to ω in the line AO and in the time dt is

$$\rho v^2 y \left(\frac{dy}{ds}\right)^3 ds\, d\theta\, dt;$$

and as the loss of momentum corresponding to every equal element of the ring generated by the revolution of ds about AO is the same, therefore the loss of momentum due to the ring is

$$2\pi\rho v^2 y \left(\frac{dy}{ds}\right)^3 ds\, dt;$$

and the loss of momentum of the whole surface in the time dt is

$$2\pi\rho v^2\, dt \int y \left(\frac{dy}{ds}\right)^2 ds; \tag{35}$$

and the loss of momentum to the moving body in an unit of time, or the resistance of the medium, as it is called, is

$$2\pi\rho v^2 \int y \left(\frac{dy}{ds}\right)^2 ds, \tag{36}$$

the limits of the integral being quantities assigned by the conditions of the problem.

Ex. 1. Let the surface COB be a hemisphere; it is required to compare the resistance of the hemisphere with the resistance of the base.

Let a be the radius; then if the line of motion is the axis of x, and O is the origin,

$$y^2 = 2ax - x^2;$$

$$\therefore \quad \frac{dy}{a-x} = \frac{dx}{y} = \frac{ds}{a};$$

$$\therefore \quad \text{the resistance} = 2\pi\rho v^2 \int_0^a \frac{(a-x)^2}{a^2}\, dx$$

$$= \frac{\pi\rho v^2 a^2}{2};$$

and the resistance of the hemisphere moving with the base forwards $= \pi\rho v^2 a^2$.

Therefore the resistance of a hemisphere moving with its convex surface forwards is one-half of its resistance when it moves with its base forwards.

Ex. 2. A right cone passes through a resisting medium, firstly with its vertex forwards, secondly with its base forwards: it is required to compare the resistances in the two cases.

Let the resistances be R_1 and R_2: let $a =$ the altitude of the cone, $b =$ the radius of its base; so that in the first case

$$\frac{y}{b} = \frac{x}{a}; \qquad \therefore \quad \frac{dy}{b} = \frac{dx}{a} = \frac{ds}{(a^2+b^2)^{\frac{1}{2}}};$$

$$\therefore \quad R_1 = \frac{2\pi\rho v^2 b^4}{a^2(a^2+b^2)} \int_0^a x\, dx$$

$$= \frac{\pi\rho v^2 b^4}{a^2+b^2};$$

$$\text{also} \qquad R_2 = \pi\rho v^2 b^2;$$

$$\therefore \quad \frac{R_1}{R_2} = \frac{b^2}{a^2+b^2}.$$

In these investigations no account has been taken of the action of the particles of the fluid on each other, nor of the friction of the particles against the surface of the moving body: also as the body moves forward it leaves a space behind it, which the particles of the resisting medium rush into and occupy; doubtless some momentum is imparted by these to the moving body: it is not therefore to be expected that the preceding results will be entirely accordant with experiment; and they are not; and in fact it appears that the law of the resistance is not, caeteris paribus, that of the square of the velocity. It is however worth while even to approximate to a solution of a problem of some difficulty, and therefore I have inserted the preceding theory of resistance, springing as it does out of that of impact and collision. We shall discuss it hereafter from a hydrodynamical point of view, and it will appear that on that theory the coefficient of resistance is only one-half of what it is on the present aspect of the case. There is also one other problem in the subject which deserves insertion, requiring as it does the Calculus of Variations, and of which the solution was first given by Sir Isaac Newton.

342.] To determine the form of a surface of revolution COB, so that the resistance of a fluid, through which it moves in the line of its axis, may be the least.

Let u represent the resistance: then

$$u = 2\pi\rho v^2 \int_0^1 y\frac{dy^3}{ds^2};$$

and taking the variation, and equating it to zero, we have

$$0 = \int_0^1 \left\{\frac{dy^3}{ds^2}\delta y + y\frac{3\,dy^2\,ds\,\delta.dy - 2\,dy^3\,\delta.ds}{ds^3}\right\};$$

but since $$ds^2 = dx^2 + dy^2,$$

$$\therefore \quad \delta.ds = \frac{dx}{ds}\delta.dx + \frac{dy}{ds}\delta.dy;$$

$$\therefore \quad 0 = \int_0^1 \left\{\frac{dy^3}{ds^2}\delta y + \frac{3y\,dy^2}{ds^2}d.\delta y - \frac{2y\,dy^4}{ds^4}d.\delta y - \frac{2y\,dy^3\,dx}{ds^4}\delta.\delta x\right\};$$

$$\therefore \quad 0 = \left[\left(3y\frac{dy^2}{ds^2} - 2y\frac{dy^4}{ds^4}\right)\delta y - \frac{2y\,dy^3\,dx}{ds^4}\delta x\right]_0^1$$
$$+\int_0^1 \left\{\left(\frac{dy^3}{ds^2} - d.\frac{3y\,dy^2}{ds^2} + d.\frac{2y\,dy^4}{ds^4}\right)\delta y + d.\frac{2y\,dy^3\,dx}{ds^4}\delta x\right\}. \quad (37)$$

To determine the function which represents the curve, we have, by the Calculus of Variations, the two following equations, viz.

$$d.\frac{2y\,dy^3\,dx}{ds^4} = 0,$$

$$\frac{dy^3}{ds^2} - d.\frac{3y\,dy^2}{ds^2} + d.\frac{2y\,dy^4}{ds^4} = 0; \tag{38}$$

either of which gives the equation to the curve; from the former,

$$\frac{2y\,dy^3\,dx}{ds^4} = \text{a constant} = c \text{ (say)};$$

$$\therefore \quad c(dy^2+dx^2)^2 = 2y\,dy^3\,dx,$$

$$\text{or,} \qquad c\left(1+\frac{dy^2}{dx^2}\right)^2 = 2y\frac{dy^3}{dx^3}; \tag{39}$$

and replacing y in (38) by its value from (39), we have

$$2\frac{dy^3}{ds^2} - cd.\frac{dy}{dx} - 3cd.\frac{dx}{dy} = 0,$$

$$2\,dx - cd.\frac{dy}{dx}\left(\frac{dx}{dy}+\frac{dx^3}{dy^3}\right) - 3c\frac{dx}{dy}\left(1+\frac{dx^2}{dy^2}\right)d.\frac{dx}{dy} = 0,$$

$$\therefore \quad 2x - c\log\frac{dy}{dx} + \frac{c}{2}\frac{dx^2}{dy^2} - \frac{3c}{2}\frac{dx^2}{dy^2} - \frac{3c}{4}\frac{dx^4}{dy^4} = c'; \tag{40}$$

where c' is also an arbitrary constant. And thus (40) becomes

$$2x - c\log\frac{dy}{dx} - c\frac{dx^2}{dy^2} - \frac{3c}{4}\frac{dx^4}{dy^4} = c'; \tag{41}$$

and either (39) or (40) is the equation to the required curve. The properties of the curve at the limits would be given by the integrated part of (37).

SECTION 2.—*Motion of particles on smooth inclined planes, under the action of the constant accelerating force of gravity.*

343.] As the problem of particles moving on smooth inclined planes under the action of a constant force, and which, to fix our thoughts, I will take to be the resolved part of gravity, is the most simple in which a constant force is resolved, it is convenient to treat of it in this part of our work: yet as it properly belongs to the theory of constrained motion, we are unable to give a complete solution of it, until the principles of such motion have been explained in a future Chapter.

Let the smooth plane be inclined to the horizon at the angle α: and let OA, AB, fig. 110, be the sections of the inclined and horizontal planes made by the plane of the paper, which is supposed to be vertical and perpendicular to the line of intersection of the two planes.

Let P be the place of the particle m at the time t, and suppose m to be under the action of gravity: let g, as in Section 3, Chapter VIII, represent the velocity-increment impressed by the earth in one second of time, so that mg represents the earth's impressed momentum on m due to a second of time in its own vertical line of action: therefore the component of it along the plane OA is $mg \sin \alpha$; let OP $= x$, and suppose m to be moving down the plane, then the expressed momentum-increment of m along the plane in an unit of time is $m\dfrac{d^2x}{dt^2}$; and as the plane and m are smooth, there is no friction, and the impressed momentum-increment along the plane is equal to the expressed momentum-increment: therefore

$$m\frac{d^2x}{dt^2} = mg \sin \alpha, \tag{42}$$

$$\frac{d^2x}{dt^2} = g \sin \alpha; \tag{43}$$

$g \sin \alpha$ being positive, because both x and the velocity of m increase as t increases. Let the velocity of m be u when $t = 0$, therefore

$$\frac{dx}{dt} - u = gt \sin \alpha;$$

$$\frac{dx}{dt} = u + gt \sin \alpha; \tag{44}$$

whereby the velocity due to the time t is known.

Also let $x = a$, when $t = 0$, therefore

$$x - a = ut + \frac{gt^2 \sin \alpha}{2}; \tag{45}$$

$$x = a + ut + \frac{gt^2 \sin \alpha}{2}; \tag{46}$$

whereby the distance due to the time t is given.

If m moves from rest, when $t = 0$, and from O, where $x = 0$, then (44) and (46) become

$$\left.\begin{aligned} \frac{dx}{dt} &= gt \sin \alpha, \\ x &= \frac{gt^2 \sin \alpha}{2}. \end{aligned}\right\} \tag{47}$$

Again, multiplying both sides of (43) by $2\,dx$, and integrating for the limits corresponding to $t = t$ and to $t = 0$, we have

$$\frac{2\,dx\,d^2x}{dt^2} = 2g\,dx\sin a;$$

$$\therefore\quad \frac{dx^2}{dt^2} - u^2 = 2g(x-a)\sin a; \tag{48}$$

and thus the velocity is given in terms of the space described.

If m is at rest when $t = 0$, and also at O, which is the origin of distance, then

$$\frac{dx^2}{dt^2} = 2gx\sin a. \tag{49}$$

Thus if OA, the length of the plane, is equal to l, and OB, the vertical projection of l, $= h$, then

$$\begin{aligned}(\text{the velocity due to the plane})^2 &= 2gl\sin a\\ &= 2gh;\end{aligned} \tag{50}$$

but $2gh$, see Art. 274, equation (49), is equal to the square of the velocity acquired by m in falling down the altitude OB; therefore the velocity acquired by m in falling down the plane depends only on the vertical projection of the length of the plane, and not separately on its length or its angle of inclination; that is, depends only on the distance through which the force has acted in its own line of action. Therefore the velocity acquired by m in falling down a plane is the same for all planes, the vertical heights of which are equal.

This is a particular instance of the law of work, see Art. 259; gravity acts through the same distance *in its own direction*, whether the particle falls freely through the vertical height or down the length of the plane, and the work done $= mgh = \frac{mv^2}{2}$.

If m is projected up the plane, and x is measured from the bottom of the plane, and thus in the direction contrary to that in which the resolved part of gravity acts, so that, in fig. 110, AP $= x$, then

$$\frac{d^2x}{dt^2} = -g\sin a;$$

and if $t=0$, when m is at A, and if the velocity of projection $= u$, then

$$\frac{dx}{dt} = u - gt\sin a;$$

$$\frac{dx^2}{dt^2} = u^2 - 2gx\sin a;$$

$$x = ut - \frac{gt^2\sin a}{2};$$

so that m ascends until $\frac{dx}{dt} = 0$, in which case,

$$t = \frac{u}{g\sin a}; \qquad x = \frac{u^2}{2g\sin a}.$$

344.] If a circle is placed in a vertical plane, the times of descent down all chords drawn from the highest point are the same.

Let O, fig. 111, be the highest point of the circle OQA, which is supposed to be in a vertical plane: let a = the radius, AOQ $= \theta$, therefore OQ $= 2a \cos\theta$; OP $= r$: then

$$\frac{d^2r}{dt^2} = g\cos\theta; \qquad \frac{dr}{dt} = gt\cos\theta;$$

$$\therefore\ r = 2a\cos\theta = \frac{gt^2\cos\theta}{2}; \qquad \therefore\ t = 2\left(\frac{a}{g}\right)^{\frac{1}{2}};$$

which is independent of θ, and is therefore the same, whatever is the inclination of OQ to the vertical line OCA. Therefore the times of descent down all chords drawn from O, the highest point, are the same.

By reason of this property the circle is called *the synchronous curve* of all straight lines in a vertical plane passing through O.

Similarly it may be shewn that the times of descent down all chords drawn to A, the lowest point, are equal; that is, the time down QA is equal to that down OA.

If the plane of the circle is inclined to the horizon at an angle i, a similar property is true; for the resolved part of gravity along the diameter OA becomes $g\sin i$, of which the resolved part along OG is $g\sin i\cos\theta$. Therefore using the same notation as in the preceding problem,

$$\frac{d^2r}{dt^2} = g\sin i\cos\theta;$$

$$\therefore\ r = 2a\cos\theta = \frac{gt^2}{2}\sin i\cos\theta;$$

$$\therefore\ t = 2\left(\frac{a}{g\sin i}\right)^{\frac{1}{2}};$$

which is independent of θ, and is therefore the same for all chords drawn from O, the highest point of the circle, to the circumference.

Similarly it may be shewn that the times down all chords from any point Q on the circle to the lowest point A are equal: the circle therefore is the synchronous curves for a pencil of lines drawn, (1) from a given point O, (2) to a given point A, on an inclined plane.

345.] By help of the preceding property of the circle, whether in a vertical, or on an inclined plane, may many problems be solved, which involve the determination of planes drawn from given points and lines to other points and lines, and which are such that the times of descent down them may be maxima or minima. Some examples are subjoined, and the principle contained in them is equally applicable to all similar problems.

Ex. 1. To determine the plane of quickest descent from (1) a given point to a given straight line, (2) a given straight line to a given point.

(1) Let A, fig. 112, be the given point, and BC the given straight line: the solution of the problem depends on the construction of a circle which passes through A, the highest point of the vertical diameter, and which touches the given straight line.

Through A draw the horizontal line AB: bisect the angle ABC by BO, which intersects in O the vertical line drawn through A: from O draw OP perpendicular to BC: then OP is manifestly equal to OA, and therefore the circle described from the centre O and with the radius OA or OP will touch the line BC at P; join AP: AP is the required line of quickest descent.

For since the time is the same down all chords of the circle drawn from A, it is manifest that the time down any line other than AP from A to the line BC is longer than that down AP.

(2) Let A be the given point, fig. 113, and BC the given straight line: through A draw the vertical line AO, and the horizontal line AB; bisect the angle ABC by BO, meeting AO in O; from O draw OP at right angles to BC, and describe a circle from O as a centre with the radius equal to either OA or OP, which are evidently equal to each other: join PA: AP is manifestly the line of quickest descent from any point in BC to the point A.

Ex. 2. To determine the line of quickest descent (1) from a point within a circle to the circle: (2) from a circle to a point without it.

(1) Let BPD, fig. 114, be the given circle, C its centre, and A the given point within it; through A draw the vertical line AO, and draw the vertical diameter BCD: join BA, and produce it to meet the circle in P: join CP, which intersects AO in O: then OA is manifestly equal to OP, and therefore the circle described from O as a centre with the radius OA or OP will touch the given circle at P: and AP is manifestly the line of quickest descent.

(2) Let BPD, fig. 115, be the given circle, C its centre, and A the point without it: draw the vertical diameter BCD of the

circle; and join BA cutting the circle in the point P: through A draw the vertical line AO, and draw the line CPO. From the geometry it is plain that OP = OA; and therefore the circle described from O as a centre, with the radius OP or OA, will touch the given circle in the point P: and thus PA is manifestly the straight line of quickest descent.

Ex. 3. To find the straight line of longest descent from a circle to a point without it, and which lies below the circle.

Let BPD be the circle, C its centre, fig. 116, D the lowest point of its vertical diameter BCD, and A the given point; join ADP, PC; and produce PC so as to intersect a vertical line through A in the point O: then the circle described from the centre O with the radius OA or OP manifestly touches the given circle at P, and the line AP is evidently that of the longest descent.

346.] Illustrative examples of the motion of a particle on an inclined plane.

Ex. 1. Of a parabola, whose axis is vertical and vertex downwards, to find that focal radius-vector the time of descent down which is a minimum.

Let $4a$ = the latus rectum: and let θ be the angle between r and the shortest focal distance: so that

$$r = \frac{2a}{1+\cos\theta}.$$

Now by (47), $$r = \frac{gt^2\cos\theta}{2};$$

$$\therefore \quad \frac{gt^2}{4a} = \frac{1}{\cos\theta\,(1+\cos\theta)};$$

$$\therefore \quad \frac{gt}{2a}\,dt = \frac{\sin\theta\,(1+2\cos\theta)}{\{\cos\theta(1+\cos\theta)\}^2}\,d\theta = 0;$$

therefore $\sin\theta = 0$, and the sign of $\frac{dt}{d\theta}$ changes from $-$ to $+$; therefore t is a minimum: so that the line from the focus to the vertex is that of quickest descent: also $\frac{dt}{d\theta} = 0$ when $\cos\theta = -\frac{1}{2}$, that is, when $\theta = 120°$: the radius-vector corresponding to which is the line of quickest descent from the parabola to the vertex.

With reference to these and similar problems it may be observed, that we have here determined the position of that plane down which, of all drawn from a given point or line to another given line or point, the time of descent is the least or greatest. It must not however be hence inferred that a *straight* line is that for which of all lines, straight or curved, joining two given points or two given curved lines, the time of descent is the least: we shall hereafter shew that the cycloid is the curve which, in vacuo and under the action of gravity, possesses this property of Brachistochronism, as it is called; and that the cycloid required cuts each of the given curves at right angles.

Ex. 2. To determine the inclination to the horizon of a smooth inclined plane, so that the time of descent of a particle m down the length may be n times that down the height of the plane.

Let $\theta =$ the inclination of the plane to the horizon,
$c =$ the length of the plane,
$b =$ the height of the upper end of the plane.

Therefore $b = c \sin\theta$. Now from (47),

$$\text{the time down the length of the plane} = \left(\frac{2c}{g\sin\theta}\right)^{\frac{1}{2}};$$

$$\text{and the time down the height} = \left(\frac{2b}{g}\right)^{\frac{1}{2}} = \left(\frac{2c\sin\theta}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad \left(\frac{2c}{g\sin\theta}\right)^{\frac{1}{2}} = n\left(\frac{2c\sin\theta}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad \sin\theta = \frac{1}{n}.$$

Ex. 3. It is required to shew that the times of descent down all the radii of curvature of the cycloid, (fig. 105, Differential Calculus,) are equal; that is, the time down $\mathrm{P\Pi}$ is equal to that down BC.

Employing our usual notation,

$$x = a \operatorname{versin}^{-1}\frac{y}{a} - (2ay-y^2)^{\frac{1}{2}};$$

$$\frac{dx}{y^{\frac{1}{2}}} = \frac{dy}{(2a-y)^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}};$$

$$\therefore \quad \rho = \mathrm{P\Pi} = 2(2ay)^{\frac{1}{2}};$$

$$\sin \mathrm{PGO} = \frac{dx}{ds} = \left(\frac{y}{2a}\right)^{\frac{1}{2}};$$

$$\therefore \quad (\text{the time down } \mathrm{P\Pi})^2 = \frac{4(2ay)^{\frac{1}{2}}}{g(y)^{\frac{1}{2}}}(2a)^{\frac{1}{2}}$$

$$= \frac{8a}{g} = (\text{the time down } \mathrm{BO})^2.$$

Ex. 4. To determine the form of a surface so that the times of descent to any point in it from two given points in the same vertical line may be equal.

It is evident that the surface is one of revolution about the given vertical line; we may therefore determine the curve by the revolution of which the surface is generated: and let us suppose the curve to be in the plane of (x, z): let the given vertical line be the axis of z; and let the two given points A and A′, fig. 117, on it be at a distance $2a$ apart: let O, the middle point of AA′, be the origin, and P be any point so that the time down AP is equal to that down A′P: then

$$\mathrm{OA} = \mathrm{OA}' = a; \qquad \mathrm{OM} = z, \qquad \mathrm{MP} = x,$$

$$(\text{time down AP})^2 = \frac{2.\mathrm{AP}}{g \sin \mathrm{APM}} = \frac{2.\mathrm{AP}^2}{g.\mathrm{AM}};$$

$$(\text{time down A}'\text{P})^2 = \frac{2.\mathrm{A'P}}{g.\sin \mathrm{A'PM}} = \frac{2.\mathrm{A'P}^2}{g.\mathrm{A'M}};$$

therefore by the conditions of the problem,

$$\frac{\mathrm{AP}^2}{\mathrm{AM}} = \frac{\mathrm{A'P}^2}{\mathrm{A'M}};$$

$$\therefore \quad \{x^2+(z+a)^2\}(z-a) = \{x^2+(z-a)^2\}(z+a);$$

$$\therefore \quad z^2-x^2 = a^2;$$

which is the equation to the equilateral hyperbola. And therefore the surface required is that which is generated by the revolution of an equilateral hyperbola about its transverse axis. The lower sheet is that to any point in which all straight lines drawn from A and A′ are lines down which the times of descent are equal: and the upper sheet is that from any point in which the lines drawn to A and A′ are those down which the times of descent are equal.

347.] Two smooth inclined planes, the inclinations of which to the horizon are respectively a and a', have a common vertex: on these are placed two smooth particles m and m', connected by a perfectly flexible and inextensible string which passes over a small pulley placed at the common vertex of the planes: it is required to determine the motion of m and m'.

Let the section of the two planes by the plane of the paper, which is supposed to be vertical and to pass through the pulley and to be perpendicular to the line of intersection of the two planes, be represented in fig. 118: let us suppose the pulley at

C to be so small that we may consider it to be (approximately) a point, and so that the strings CP and CP′ are parallel to the respective planes. Let CAA′ $= \alpha$, CA′A $= \alpha'$, CP $= x$, CP′ $= x'$, P and P′ being the places of m and m' at the time t: and to fix our thoughts let us suppose m to be descending. Because the string is inextensible,

$$x + x' = \text{a constant};$$

$$\therefore \quad \frac{dx}{dt} + \frac{dx'}{dt} = 0, \qquad \frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} = 0;$$

that is, $\frac{d^2x'}{dt^2} = -\frac{d^2x}{dt^2}$: which result is also manifest by general reasoning. Now $m+m'$ is the whole mass moved: and $\frac{d^2x}{dt^2}$ is the velocity-increment expressed in the motion of each:

$$\therefore \quad (m+m')\frac{d^2x}{dt^2} = \text{the momentum-increment expressed.}$$

And $mg \sin \alpha$, and $m'g \sin \alpha'$ are the respective impressed momentum-increments along the planes: but as these act in opposite directions,

$$mg \sin \alpha - m'g \sin \alpha' = \text{the momentum-increment impressed};$$

$$\therefore \quad (m+m')\frac{d^2x}{dt^2} = (m \sin \alpha - m' \sin \alpha')g; \tag{51}$$

$$\frac{d^2x}{dt^2} = \frac{m \sin \alpha - m' \sin \alpha'}{m+m'} g. \tag{52}$$

Similarly for the equation of motion of m', we have

$$\frac{d^2x'}{dt^2} = \frac{m' \sin \alpha' - m \sin \alpha}{m'+m} g. \tag{53}$$

If when $t = 0$, m and m' are at rest,

$$\frac{dx}{dt} = \frac{m \sin \alpha - m' \sin \alpha'}{m+m'} gt; \tag{54}$$

whereby the velocity acquired during the time t is known.

Also multiplying both sides of (52) by $2dx$, and supposing the limits of the integral to be such that the velocity $= 0$, when $x = a$,

$$\frac{dx^2}{dt^2} = \frac{m \sin \alpha - m' \sin \alpha'}{m+m'} 2g(x-a). \tag{55}$$

And integrating (54) again, and taking the limits of integration such that $x = a$, when $t = 0$, we have

$$x - a = \frac{m \sin \alpha - m' \sin \alpha'}{m+m'} \frac{gt^2}{2}. \tag{56}$$

If the velocity, with which m and m' begin to move, is u, then if $x = a$, when $t = 0$,

$$\frac{dx}{dt} - u = \frac{m\sin\alpha - m'\sin\alpha'}{m+m'}gt, \tag{57}$$

$$x - a = ut + \frac{m\sin\alpha - m'\sin\alpha'}{m+m'}\frac{gt^2}{2}, \tag{58}$$

$$\frac{dx^2}{dt^2} - u^2 = \frac{m\sin\alpha - m'\sin\alpha'}{m+m'}2gx. \tag{59}$$

As to the initial velocity u; suppose m and m' respectively to have the velocities v and v' down the corresponding planes; then if u is the common velocity with which the two particles by reason of their connexion by means of the string begin to move, we have from the equality of the expressed and the impressed momenta,
$$(m+m')u = mv - mv',$$

$$\therefore \quad u = \frac{mv - m'v'}{m+m'}. \tag{60}$$

The preceding formulae are also applicable, whatever are the inclinations of the planes. Thus suppose the plane CA′ to be horizontal, then $\alpha' = 0$, and

$$\frac{d^2x}{dt^2} = \frac{mg\sin\alpha}{m+m'},$$

that is, m' has no impressed momentum-increment; and if $\alpha = 90°$, m is then moving vertically downwards: this case is that of m hanging by a string over the edge of a horizontal table, and drawing another body m' which is on the table at the other end of the string.

If $\alpha = \alpha' = 90°$, we have the same formulae as those which were investigated in Art. 276.

348.] Examples in illustration :

Ex. 1. A small ball m descending vertically draws an equal ball 25 feet in 2.5 seconds up a plane inclined at 30° to the horizon, by means of a string passing over a pulley at the top: it is required to determine the force of gravity.

Here $\alpha = 90°$, $\alpha' = 30°$, $m = m'$;

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{g}{4}, \qquad \therefore \quad \frac{dx}{dt} = \frac{g}{4}t,$$

$$x = \frac{g}{8}t^2, \qquad 25 = \frac{g}{8}(2.5)^2, \qquad g = 32 \text{ feet}.$$

Ex. 2. Two smooth inclined planes are placed as in fig. 119: BC is inclined at 30°, and CA at 60° to the horizontal line OA: it is required to determine the distance CP through which m moves in t'', m' being equal to m.

The equation of motion becomes

$$2m\frac{d^2x}{dt^2} = mg(\sin 30° + \sin 60°);$$

$$\therefore \quad x = \frac{3^{\frac{1}{2}}+1}{8}gt^2.$$

SECTION 3.—*The determination of the curvilinear paths described by particles moving in vacuo under the action of given accelerating forces.*

349.] In the preceding section the effects of resolved forces have been considered, when the path taken by the particle in consequence of them is straight; we have now to investigate the effects of resolved forces in a more general way: and I shall take first the simple case of a particle moving in vacuo under the action of gravity which is a constant accelerating force, and the line of action of which is always vertical. The projected body or particle is called *a projectile*, and the problem is in this case that of the motion of a projectile in vacuo.

Let m be the mass of the projectile; and let us prove, in the first place, that the particle during the motion is always in one and the same plane.

Let the horizontal plane passing through the point, whence the particle is projected, be the plane of (x, y), so that the axis of z is parallel to the line of action of gravity: also let the velocity of projection of the particle be u; and let it be projected in a line which makes an angle β with the axis of z, and of which the projection on the plane of (x, y) makes an angle α with the axis of x; so that the three components of u along the three rectangular axes are

$$u \sin\beta \cos\alpha, \qquad u \sin\beta \sin\alpha, \qquad u \cos\beta.$$

Now the components of the impressed accelerating force give the following equations:

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = 0, \qquad \frac{d^2z}{dt^2} = -g; \qquad (61)$$

therefore integrating the first two, and taking for the limits of integration the values which correspond to $t=t$ and to $t=0$, we have from (61),

$$\frac{dx}{dt}-u\sin\beta\cos a=0; \qquad \frac{dy}{dt}-u\sin\beta\sin a=0; \tag{62}$$

$$\therefore \quad x=ut\sin\beta\cos a, \qquad y=ut\sin\beta\sin a; \tag{63}$$

$$\therefore \quad \frac{x}{\cos a}=\frac{y}{\sin a};$$

which is the equation to a plane perpendicular to the plane of (x, y), and containing the axis of z; therefore the moving particle is always in the vertical plane, which is inclined at the angle a to the plane of (x, z).

350.] Let us take the plane in which the particle moves to be that of (x, y): let the point of projection, fig. 121, be the origin: let the axis of x be horizontal, that of y vertical: let the velocity of projection $= u$, and let the line of projection be inclined at an angle a to the axis of x, so that $u\cos a$ and $u\sin a$ are the resolved parts of the velocity of projection along the coordinate axes of x and y.

Let P be the position of m at the time t, OM $= x$, MP $= y$; $g =$ the accelerating force of gravity which acts parallel to the axis of y; therefore mg is the impressed momentum-increment; so that the equations of motion are

$$m\frac{d^2x}{dt^2}=0, \qquad m\frac{d^2y}{dt^2}=-mg;$$

the latter being affected with a negative sign, because the tendency of gravity is to make the velocity increase, and y decrease, as t increases. Therefore

$$\frac{d^2x}{dt^2}=0, \qquad \frac{d^2y}{dt^2}=-g. \tag{64}$$

Now integrating these, and taking the limits corresponding to $t=t$ and to $t=0$, we have

$$\frac{dx}{dt}-u\cos a=0; \qquad \frac{dy}{dt}-u\sin a=-gt; \tag{65}$$

$$x=ut\cos a; \qquad y=ut\sin a-\frac{gt^2}{2}. \tag{66}$$

whence eliminating t, we have

$$y=x\tan a-\frac{gx^2}{2u^2(\cos a)^2}. \tag{67}$$

And this equation is that of a parabola; whence it follows that a parabola is the trajectory of the particle. And (67) may be put into the form

$$\left(x - \frac{u^2 \cos a \sin a}{g}\right)^2 = \frac{2u^2 (\cos a)^2}{g} \left\{ \frac{u^2 (\sin a)^2}{2g} - y \right\}; \qquad (68)$$

so that we have

$$(1) \quad \text{the abscissa to the vertex} = \frac{u^2 \sin a \cos a}{g}; \qquad (69)$$

$$(2) \quad \text{the ordinate to the vertex} = \frac{u^2 (\sin a)^2}{2g}; \qquad (70)$$

$$(3) \quad \text{the latus rectum} = \frac{2u^2 (\cos a)^2}{g}. \qquad (71)$$

Also the form of the equation (68) shews that the parabola is placed with its axis vertical, as in the figure, and that the vertex is the highest point of the curve.

The distance OB between the point of projection and the point where the projectile strikes the horizontal plane is called *the Range* on the horizontal plane, and is the value of x when $y=0$; that is, putting $y=0$ in (67),

$$\text{the range} = \text{OB} = \frac{u^2 \sin 2a}{g}; \qquad (72)$$

also, as is geometrically manifest, OB $= 2$OC; that is, the range is equal to twice the abscissa to the vertex.

From (72) it appears that for a given velocity of projection, the range is the greatest when $a = 45°$, in which case the range $= \frac{u^2}{g}$; and the focus of the parabola in this case lies in the horizontal line drawn through the point of projection.

Also from the value of the range (72) it appears, that if u is the same, the range is unaltered when a is replaced by its complementary angle: so that the range is the same for two different angles which are complementary of each other, if the velocity of projection is the same: hence if $a = 45°$, these two angles become identical, and the range is a maximum.

CA is called *the altitude* or the greatest height of the projectile, and is the value of y when $\frac{dy}{dx} = 0$; therefore from (67),

$$\text{the greatest height} = \frac{u^2 (\sin a)^2}{2g}. \qquad (73)$$

Also from (66), $\qquad x = ut \cos a; \qquad (74)$

that is, the abscissa uniformly increases along OM. Hence if we substitute the range for x, we shall have an expression for the time which a particle takes in passing from O to B, and which is called *the time of flight:* and thus

$$\text{the time of flight} = \frac{2u \sin a}{g}. \tag{75}$$

351.] Again multiplying (64) respectively by $2dx$ and $2dy$, and integrating, and taking the limits corresponding to $t = t$ and to $t = 0$, we have

$$\frac{dx^2}{dt^2} = u^2 (\cos a)^2; \qquad \frac{dy^2}{dt^2} = u^2 (\sin a)^2 - 2gy; \tag{76}$$

$$\therefore \quad \frac{ds^2}{dt^2} = (\text{the velocity})^2 = u^2 - 2gy. \tag{77}$$

Now this result deserves notice. Let the directrix of the parabolic trajectory be drawn as in the figure: then

$$\text{AD} = \frac{1}{4}\text{th of the latus rectum} = \frac{u^2 (\cos a)^2}{2g};$$

and since by (70), $\text{OA} = \dfrac{u^2 (\sin a)^2}{2g}$, therefore $\text{CD} = \dfrac{u^2}{2g}$; that is, see equation (49), Art. 274, CD is the vertical height through which a particle falling in vacuo will acquire the velocity with which the particle m moves at its projection from O.

Let $\text{CD} = h$; therefore $u^2 = 2gh$; and substituting in (77),

$$\frac{ds^2}{dt^2} = 2g(h-y); \tag{78}$$

that is, the velocity at any point P on the curve is that which would be acquired by a heavy particle falling freely in vacuo down a vertical height equal to $h-y$, that is, to SP. Hence it follows that the velocity of m at any point P in its path is that which would be acquired by a particle falling freely from the directrix to the curve. The directrix of the parabola is therefore determined by the velocity of projection, and is at a vertical distance above the point of projection equal to that down which a particle falling would have the velocity of projection. Hence also the vis viva of the projectile is at every point of the path the same as that of an equal particle acquired in falling from the directrix to the point of the curve.

352.] The equation to the path of the projectile may also be found by the following process: and as the result of simultaneous velocities taking place in combination is well exemplified by it, I do not hesitate to insert it.

Let the particle m be projected from O, fig. 121, with a velocity u, in the line OQ making an angle a with the horizontal line; then if no force acted to impress velocity on m, it would in the time t describe a space OQ $= ut$, and its coordinates at the time t would be

$$x = ut\cos a, \qquad y = ut\sin a. \tag{79}$$

But as gravity is a constant force, and acts in a line parallel to the axis of y, and tends to diminish y according to the arrangement which we have assumed in the figure, y will by it be diminished in the time t by a quantity equal to $\frac{gt^2}{2}$; see equation (49), Art. 274; so that at the end of the time t we have

$$x = ut\cos a, \qquad y = ut\sin a - \frac{gt^2}{2}; \tag{80}$$

which values are the same as (66); and therefore it appears, that if P is the place of m in the parabolic path at the time t, QP $= \frac{gt^2}{2}$.

353.] A particle m is projected from a given point on an inclined plane in a given line; it is required to determine the point at which it will strike the plane.

Let the angle of inclination of the plane to the horizon be i: let $a =$ the angle between the line of projection and the horizontal line: $u =$ the velocity of projection: then the equations to the inclined plane and to the path of the projectile respectively are

$$y = x\tan i; \qquad y = x\tan a - \frac{gx^2}{2u^2(\cos a)^2};$$

whence eliminating y, we have

$$\begin{aligned} x &= \frac{2u^2}{g}(\cos a)^2(\tan a - \tan i) \\ &= \frac{2u^2\cos a\sin(a-i)}{g\cos i}; \end{aligned}$$

$$\therefore \quad y = \frac{2u^2\cos a\tan i\sin(a-i)}{g\cos i};$$

which give the point on the plane at which the projectile strikes it: and the range on the plane is equal to $x\sec i$, that is,

$$\text{the range} = \frac{2u^2\cos a\sin(a-i)}{g(\cos i)^2}. \tag{81}$$

Also the range is the greatest when

$$a = \frac{1}{2}\left\{\frac{\pi}{2} + i\right\}, \quad \text{and} \quad a - i = \frac{1}{2}\left\{\frac{\pi}{2} - i\right\}; \tag{82}$$

which latter value gives the angle between the plane and the line of projection for which the range is the greatest: and in this case

$$\text{the greatest range} = \frac{u^2}{g(1+\sin\iota)}. \tag{83}$$

Hence it follows that if from a given point a system of straight lines is drawn in the same vertical plane, and particles are projected with a given velocity u, and in such lines that the ranges on the planes are the greatest, the locus of the extreme points of these ranges is given by the equation (83); and therefore if u is the velocity of projection, r = the range, θ = the angle between the plane and the vertical line through the point of projection, then from (83),

$$r = \frac{u^2}{g(1+\cos\theta)}; \tag{84}$$

and if h is the vertical distance to which u is due

$$r = \frac{2h}{1+\cos\theta}; \tag{85}$$

which is the equation to a parabola, the focus of which is at the origin of coordinates, whose axis is vertical, and of which $4h$ is the latus rectum.

354.] It is required to determine the angle of projection for a given velocity so that a projectile may pass through a given point.

Let the given point be (x_1, y_1): then these coordinates satisfy the equation of the path of the projectile, and we have

$$y_1 = x_1 \tan a - \frac{g x_1^2}{2u^2}\{1+(\tan a)^2\};$$

$$\therefore \quad \tan a = \frac{u^2}{g x_1} \pm \frac{1}{g x_1}\{u^4 - 2u^2 g y_1 - g^2 x_1^2\}^{\frac{1}{2}}; \tag{86}$$

therefore two different values of a satisfy the condition, if

$$u^4 \text{ is greater than } 2u^2 g y_1 + g^2 x_1^2;$$

only one value of a satisfies it, if

$$u^4 = 2u^2 g y_1 + g^2 x_1^2;$$

$$\text{that is, if} \quad x_1^2 = \frac{2u^2}{g}\left\{\frac{u^2}{2g} - y_1\right\}; \tag{87}$$

and the projectile cannot reach the point, if

$$u^4 \text{ is less than } 2u^2 g y_1 + g^2 x_1^2.$$

Now (87) is the equation to a parabola of which o, fig. 121, is the focus, $\frac{2u^2}{g}$ is the latus rectum, and $\frac{u^2}{2g}$ or h, see Art. 351, is the distance to the highest point; all points therefore on this parabola are the farthest which the projectile can reach; all points without it are beyond the reach; and all points within it may be reached by two different angles of projection. The same result may also evidently be proved by the following process:

It is required to find the envelope of all parabolas described by particles projected with a given velocity u from a given point in the same vertical plane.

The equation to the path of one is

$$y = x \tan a - \frac{gx^2}{2u^2} \{1 + (\tan a)^2\}; \tag{88}$$

therefore differentiating by making $\tan a$ to vary, we have

$$0 = \left(x - \frac{gx^2}{u^2} \tan a\right) d . \tan a;$$

$$\therefore \quad \tan a = \frac{u^2}{gx};$$

so that (88) becomes

$$x^2 = \frac{2u^2}{g} \left\{ \frac{u^2}{2g} - y \right\};$$

which result is of course the same as (87).

355.] Problems in illustration of the preceding equations:

Ex. 1. To determine the angle of projection so that the area contained between the path of the projectile and the horizontal line may be a maximum.

Since the area of a parabola is two-thirds of that of the circumscribed rectangle, if A represents the required area,

$$\begin{aligned} \text{A} &= \frac{2}{3} \text{ the range} \times \text{the greatest altitude} \\ &= \frac{2u^4}{3g^2} (\sin a)^3 \cos a; \end{aligned}$$

$$\therefore \quad \frac{d\text{A}}{da} = \frac{2u^4}{3g^2} (\sin a)^2 \{3 (\cos a)^2 - (\sin a)^2\} = 0,$$

if $\tan a = 3^{\frac{1}{2}}$, and changes sign from + to − : therefore the area is a maximum and $= \frac{u^4 \, 3^{\frac{1}{2}}}{8g^2}$, if $a = 60°$.

Ex. 2. It is required to compare the areas of the two parabolas described by projectiles, of which the horizontal ranges are the same, and the angles of projection are therefore complementary to each other.

Let A_1 and A_2 be the areas: then as the ranges are equal, these are to each other as the greatest altitudes: therefore

$$\frac{A_1}{A_2} = \frac{(\sin a)^2}{(\cos a)^2} = (\tan a)^2.$$

Ex. 3. From the top of a tower two particles are projected at angles a and β to the horizon with the same velocity u, and both strike the horizontal plane passing through the bottom of the tower at the same point; it is required to find the height of the tower.

Let h = the height of the tower: u = the velocity of projection: then if the particles are projected from the edge of the top of the tower, and x is the distance from the bottom of the tower to the point where they strike the horizontal plane,

$$-h = x \tan a - \frac{gx^2}{2u^2}\{1 + (\tan a)^2\}, \tag{89}$$

$$-h = x \tan \beta - \frac{gx^2}{2u^2}\{1 + (\tan \beta)^2\}; \tag{90}$$

therefore by subtraction,

$$x = \frac{2u^2}{g(\tan a + \tan \beta)} = \frac{2u^2 \cos a \cos \beta}{g \sin (a+\beta)};$$

substituting which in either (89) or (90), we have

$$h = \frac{2u^2 \cos a \cos \beta \cos (a+\beta)}{g\{\sin (a+\beta)\}^2}.$$

Ex. 4. Particles are projected with a given velocity in all lines in a vertical plane from the point O: it is required to find the locus of them at a given time t.

From (66) we have

$$x = ut \cos a, \qquad y = ut \sin a - \frac{gt^2}{2};$$

$$\therefore \quad ut \cos a = x, \qquad ut \sin a = y + \frac{gt^2}{2};$$

therefore squaring and adding, we have

$$x^2 + \left(y + \frac{gt^2}{2}\right)^2 = u^2t^2; \tag{91}$$

the equation to a circle of which the radius is ut, and the centre is on the axis of y at a distance $\frac{gt^2}{2}$ below the origin.

Ex. 5. Particles are projected from O with a given velocity in all lines in a vertical plane: it is required to find the locus of their highest points.

Let x and y be the coordinates to the highest point: then from (69) and (70),

$$x = \frac{u^2 \sin a \cos a}{g}, \qquad y = \frac{u^2 (\sin a)^2}{2g};$$

$$\therefore \quad (\sin a)^2 = \frac{2gy}{u^2}, \qquad (\cos a)^2 = \frac{gx^2}{2u^2 y};$$

therefore adding, $$4y^2 + x^2 = \frac{2u^2 y}{g}; \tag{92}$$
which is the equation to an ellipse, of which the major axis $= \frac{u^2}{g}$, and the minor axis $= \frac{u^2}{2g}$; and the origin is at the extremity of the minor axis.

The preceding investigations into the motion of projectiles would explain the theory of gunnery, if it were allowable to neglect the resistance of the air; but as the velocity with which balls and shells traverse their paths is very great, much of their momentum is lost by the resistance of the medium; and the ratio of the vertical and horizontal velocities is so much altered, that the form of the trajectory is very different from that of a parabolic path. In the last Section of the present Chapter we shall investigate, as far as it is possible, this path, and shall take account of the loss of momentum which is due to the resistance of the medium.

We proceed now to other cases of curvilinear motion in vacuo; and I would observe, once for all, that if a particle is projected with a given velocity in a plane, and if the lines of action of the forces, which act on the particle, are in that plane, the particle is during its motion in that plane: this is evident by the principle of sufficient reason.

356.] From the vertex of a parabola a particle m is projected with a velocity u at right angles to the principal axis, and is acted on by an attractive force which is perpendicular to that axis and varies directly as the distance of the particle from it. It is required to determine the law of force acting parallel to the axis of x so that the particle may move in the parabola, and the other circumstances of motion.

Let the vertex be the origin, and the principal axis and the tangent at the vertex be the axes of x and y; and let P, (x, y), fig. 120, be the position of m at the time t, so that the equation to the parabola is $$y^2 = 4ax;$$

$$\therefore \quad \frac{dy}{dx} = \frac{2a}{y} = \left(\frac{a}{x}\right)^{\frac{1}{2}}.$$

By the conditions of the problem,

$$\frac{d^2x}{dt^2} = \text{X}, \text{ which is to be determined;}$$

$$\frac{d^2y}{dt^2} = -\mu y; \tag{93}$$

therefore multiplying (93) by $2\,dy$, integrating, and taking the limits corresponding to $t = t$ and to $t = 0$,

$$\frac{dy^2}{dt^2} - u^2 = -\mu y^2;$$

$$\therefore \quad \frac{dy^2}{dt^2} = u^2 - \mu y^2 \tag{94}$$

$$= u^2 - 4\mu a x;$$

$$\therefore \quad \frac{dx^2}{dt^2} = \frac{u^2 x - 4\mu a x^2}{a}; \tag{95}$$

$$\therefore \quad \frac{d^2x}{dt^2} = \text{X} = \frac{u^2}{2a} - 4\mu x; \tag{96}$$

therefore the required force which is parallel to the axis of x partly is constant and repulsive, and partly varies as the abscissa and is attractive towards the axis of x.

Also from (94) and (95) $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both vanish, when

$$y = \pm \frac{u}{\mu^{\frac{1}{2}}}, \quad \text{and} \quad x = \frac{u^2}{4\mu a}; \tag{97}$$

so that at this point, say B in the figure, m comes to rest; and afterwards under the action of the forces returns to the vertex of the parabola, through which it passes with the original velocity u, and comes to rest again at B′, which is equidistant with B from the vertex: thus the motion is oscillatory. Also from (94)

$$\frac{dy}{(u^2 - \mu y^2)^{\frac{1}{2}}} = dt,$$

taking the positive sign, as we will consider the original motion from O to B. Therefore integrating, and taking the limits corresponding to $t = t$ and to $t = 0$,

$$\sin^{-1}\frac{y(\mu)^{\frac{1}{2}}}{u} = \mu^{\frac{1}{2}}\,t;$$

$$\therefore \quad y = \frac{u}{\mu^{\frac{1}{2}}}\sin \mu^{\frac{1}{2}}\,t; \tag{98}$$

therefore the time from O to B is $\frac{\pi}{2\mu^{\frac{1}{2}}}$; and the time of an oscillation, viz. from B to B′, $= \dfrac{\pi}{\mu^{\frac{1}{2}}}$, which is independent of the velocity of projection from O, and depends on only the absolute force parallel to the axis of y.

I have, for the sake of simplicity, taken the parabola for the example whereby the process may be illustrated, but the method is the same in all cases. Thus in the ellipse, if m is projected from the extremity of the major axis with the velocity u,

$$\mathrm{X} = \frac{a^2 \mu y^2 (a^2+x^2)}{b^2 x^3} - \frac{a^4 u^2}{b^2 x^3}; \tag{99}$$

and the coordinates of the point B to which m passes are

$$y = \frac{u}{\mu^{\frac{1}{2}}}, \qquad x = a - \frac{a}{b}\Big(b^2 - \frac{u^2}{\mu}\Big)^{\frac{1}{2}}; \tag{100}$$

therefore m comes to rest at the extremity of the minor axis, if $u = \mu^{\frac{1}{2}} b$.

357.] From a given point in the axis of y a particle is projected with a given velocity in a line parallel to the axis of x, and is acted on by an attractive force parallel to the axis of y and which varies as the distance from the axis of x: it is required to determine the circumstances of motion.

Let $b =$ the distance from the origin of the point of projection,
$u =$ the velocity of projection;

so that the equations of motion are

$$\frac{d^2x}{dt^2} = 0; \qquad \frac{d^2y}{dt^2} = -\mu y;$$

$$\therefore \quad \frac{dx}{dt} = u; \qquad \frac{dy^2}{dt^2} = \mu(b^2-y^2);$$

$$x = ut; \qquad \frac{-dy}{(b^2-y^2)^{\frac{1}{2}}} = \mu^{\frac{1}{2}}\,dt;$$

$$\therefore \quad \cos^{-1}\frac{y}{b} = \mu^{\frac{1}{2}} t;$$

$$y = b\cos\mu^{\frac{1}{2}} t$$

$$= b\cos\frac{\mu^{\frac{1}{2}} x}{u}; \tag{101}$$

which is the equation to the companion to the cycloid.

If the force is repulsive, we have $x = ut$, and

$$\frac{dy^2}{dt^2} = \mu(y^2-b^2);$$

$$\therefore \quad y = \frac{b}{2}\left\{e^{\mu^{\frac{1}{2}}t} + e^{-\mu^{\frac{1}{2}}t}\right\} \tag{102}$$

$$= \frac{b}{2}\left\{e^{\frac{\mu^{\frac{1}{2}}}{u}x} + e^{-\frac{\mu^{\frac{1}{2}}}{u}x}\right\}; \tag{103}$$

and consequently the path is the catenary.

358.] From a given point in the axis of y a particle is projected with a given velocity u along a line parallel to the axis of x, and is under the action of an attractive force parallel to the axis of y which varies inversely as the cube of the ordinate: determine the other circumstances of motion.

$$\frac{d^2x}{dt^2} = 0; \qquad \frac{d^2y}{dt^2} = -\frac{\mu}{y^3};$$

$$\therefore \quad \frac{dx}{dt} = u; \qquad \frac{dy^2}{dt^2} = \mu\left(\frac{1}{y^2} - \frac{1}{b^2}\right);$$

$$\therefore \quad x = ut; \qquad \frac{-y\,dy}{(b^2-y^2)^{\frac{1}{2}}} = \frac{\mu^{\frac{1}{2}}}{b}\,dt;$$

$$\therefore \quad (b^2-y^2)^{\frac{1}{2}} = \frac{\mu^{\frac{1}{2}}}{b}\,t$$

$$= \frac{\mu^{\frac{1}{2}}}{bu}\,x;$$

$$\therefore \quad \frac{\mu x^2}{b^2u^2} + y^2 = b^2, \tag{104}$$

which is the equation of an ellipse, whose centre is the origin. If the force parallel to the axis of y is repulsive, the path is a hyperbola.

If the initial circumstances had been the same, and the force had been attractive and varied inversely as the square of the ordinate, then the equation to the path would be

$$(b^2-y^2)^{\frac{1}{2}} + \frac{b}{2}\left(\pi - \text{versin}^{-1}\frac{2y}{b}\right) = \frac{(2\mu)^{\frac{1}{2}}}{bu}\,x. \tag{105}$$

359.] From a given point in the axis of y a particle is projected with a velocity u in a line parallel to the (rectangular) axis of x, and is attracted by a force the intensity of which varies directly as the distance, and which has its source in the origin of coordinates: it is required to find the equation of the path of the particle, and to define the circumstances of motion.

Let b be the distance on the axis of y from the origin of the point whence the particle is projected with the velocity u: let P, fig. 122, be the place of m at the time t, B its place when $t = 0$; OM $= x$, MP $= y$, OP $= r$, OB $= b$; and let μ be the absolute force of the attraction at O. Then the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -\mu r \frac{x}{r} = -\mu x, \\ \frac{d^2y}{dt^2} &= -\mu r \frac{y}{r} = -\mu y. \end{aligned}\right\} \tag{106}$$

Now multiplying these equations respectively by $2dx$ and $2dy$, and integrating, and taking the limits corresponding to $t = t$ and to $t = 0$, we have

$$\frac{dx^2}{dt^2} - u^2 = -\mu x^2; \tag{107}$$

$$\frac{dy^2}{dt^2} = -\mu(y^2 - b^2); \tag{108}$$

therefore adding,

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = \frac{ds^2}{dt^2} = u^2 - \mu(x^2 + y^2 - b^2); \tag{109}$$

which gives the velocity of m at any point of its path. Also from (107) and (108),

$$\frac{dx}{\left(\frac{u^2}{\mu} - x^2\right)^{\frac{1}{2}}} = \mu^{\frac{1}{2}}\, dt = \frac{-dy}{(b^2 - y^2)^{\frac{1}{2}}}; \tag{110}$$

$$\therefore \quad x = \frac{u}{\mu^{\frac{1}{2}}} \sin \mu^{\frac{1}{2}} t; \qquad y = b \cos \mu^{\frac{1}{2}} t; \tag{111}$$

whence we have
$$\frac{\mu x^2}{u^2} + \frac{y^2}{b^2} = 1; \tag{112}$$

which is the equation to an ellipse whose x-axis is $\frac{2u}{\mu^{\frac{1}{2}}}$, and whose y-axis is $2b$, and whose centre is at the origin, that is, at the source of the force.

From the preceding values we have

$$\frac{ds^2}{dt^2} = (\text{velocity})^2 = u^2 (\cos \mu^{\frac{1}{2}} t)^2 + \mu b^2 (\sin \mu^{\frac{1}{2}} t)^2; \tag{113}$$

and from (111), the time from B to A $= \frac{\pi}{2\mu^{\frac{1}{2}}}$; therefore the whole periodic time $= \frac{2\pi}{\mu^{\frac{1}{2}}}$; and is independent of the velocity and distance of projection, and involves only the absolute force of the impressed force.

If the force at O had been repulsive, then the sign of μ would be changed, and the equations of motion would be

$$\frac{d^2x}{dt^2} = \mu x, \qquad \frac{d^2y}{dt^2} = \mu y; \tag{114}$$

$$\therefore \quad \frac{dx^2}{dt^2} - u^2 = \mu x^2, \qquad \frac{dy^2}{dt^2} = \mu(y^2 - b^2); \tag{115}$$

$$\frac{dx}{\left(\frac{u^2}{\mu} + x^2\right)^{\frac{1}{2}}} = \mu^{\frac{1}{2}} dt = \frac{dy}{(y^2 - b^2)^{\frac{1}{2}}}; \tag{116}$$

$$\therefore \quad \log \frac{x\mu^{\frac{1}{2}} + (u^2 + \mu x^2)^{\frac{1}{2}}}{u} = \mu^{\frac{1}{2}} t = \log \frac{y + (y^2 - b^2)^{\frac{1}{2}}}{b}; \tag{117}$$

$$\left.\begin{aligned} \therefore \quad \frac{2x\mu^{\frac{1}{2}}}{u} &= e^{\mu^{\frac{1}{2}}t} - e^{-\mu^{\frac{1}{2}}t}, \\ \frac{2y}{b} &= e^{\mu^{\frac{1}{2}}t} + e^{-\mu^{\frac{1}{2}}t}; \end{aligned}\right\} \tag{118}$$

whence squaring, and subtracting the former from the latter, we have

$$\frac{y^2}{b^2} - \frac{\mu x^2}{u^2} = 1, \tag{119}$$

which is the equation to a hyperbola with its centre at the origin; and which might have been deduced from (112) by affecting μ with a negative sign.

If, in the case of the force being attractive, the velocity and distance of projection are such that $u = b\mu^{\frac{1}{2}}$, the path which m describes is a circle, and the velocity in it is constant and equal to that of projection.

360.] A particle m is projected from a given point with a given velocity, and is acted on by a force which varies inversely as the square of the distance from a given point which is its source: it is required to determine the path of the particle, and the other circumstances of motion.

Let u be the velocity of projection, $a =$ the angle between the line of projection and the axis of x, so that the components of the velocity of projection along the axes are $u \cos a$ and $u \sin a$; let (a, b) be the point of projection, $\mu =$ the absolute force, and let $a^2 + b^2 = c^2$: let r be the distance of m at the time t from the centre of force, which we will take to be the origin, and let (x, y) be the place of m at the time t; then the equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{r^3}, \qquad \frac{d^2y}{dt^2} = -\frac{\mu y}{r^3}; \tag{120}$$

which are simultaneous differential equations, and from which the solution of the problem is to be obtained. Multiply (120) respectively by y and x, and subtract; then

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0, \tag{121}$$

and adding and subtracting $\frac{dx}{dt}\frac{dy}{dt}$, and integrating, and taking definite integrals with limits corresponding to $t = t$ and to $t = 0$, we have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = u\,(a\sin\alpha - b\cos\alpha) \tag{122}$$

$$= h \text{ (say)}. \tag{123}$$

Again, from (123) and the first of (120), we have

$$\begin{aligned} h\frac{d^2x}{dt^2} &= -\frac{\mu x}{r^3}\,\frac{x\,dy - y\,dx}{dt} \\ &= \frac{-\mu}{r^3}\,\frac{x^2\,dy - xy\,dx}{dt}; \end{aligned}$$

but $x^2 + y^2 = r^2$; therefore $x\,dx + y\,dy = r\,dr$;

$$\begin{aligned} \therefore\quad h\frac{d^2x}{dt^2} &= -\frac{\mu}{r^3}\,\frac{x^2\,dy - y\,(r\,dr - y\,dy)}{dt} \\ &= -\frac{\mu}{r^3}\,\frac{r^2\,dy - ry\,dr}{dt} \\ &= -\mu\frac{d}{dt}\frac{y}{r}; \end{aligned}$$

$$\therefore\quad h\frac{dx}{dt} - hu\cos\alpha = -\frac{\mu y}{r} + \frac{\mu b}{c}; \tag{124}$$

similarly, from the second of (120),

$$h\frac{dy}{dt} - hu\sin\alpha = \frac{\mu x}{r} - \frac{\mu a}{c}. \tag{125}$$

Multiplying (125) by x, and (124) by y, and subtracting, we have

$$h\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) - hu\,(x\sin\alpha - y\cos\alpha) = \mu r - \frac{\mu}{c}(ax + by);$$

therefore by (123),

$$h^2 - hu\,(x\sin\alpha - y\cos\alpha) = \mu r - \frac{\mu}{c}(ax + by), \tag{126}$$

which is of the form $\qquad r = \mathrm{A}x + \mathrm{B}y + \mathrm{C};$ $\qquad$ (127)

and as r is a rational function of the coordinates x and y, the equation is that of a conic of which the focus is the origin. A conic therefore is the trajectory, with the source of the impressed force in the focus; and the constants A, B, C are given in terms of the velocity, the direction of the line of action, and the coordinates of the point, of projection.

Also from (120), multiplying respectively by $2\,dx$ and by $2\,dy$, and adding and integrating with the limits assigned above,

$$\frac{ds^2}{dt^2} - u^2 = \frac{2\mu}{r} - \frac{2\mu}{c}, \tag{128}$$

whereby the velocity is given at any point of the curve.

361.] In some cases oblique coordinates may be used with advantage. Thus suppose, as in Art. 359, a particle m to be projected with a given velocity u in a given line from a given point, and to be attracted by a force the intensity of which varies directly as the distance, and which has its source in a given point: it is required to determine the path which it describes.

Let the given source of the force be the origin; and let the line passing through it and the point of projection be the axis of y: and let the axis of x be drawn parallel to the line of projection; let the distance from the origin to the point of projection be b_1: then the equations of motion are

$$\frac{d^2x}{dt^2} = -\mu x, \qquad \frac{d^2y}{dt^2} = -\mu y;$$

and by a process similar to that of Art. 359, we shall have

$$\frac{\mu x^2}{u^2} + \frac{y^2}{b_1^{\,2}} = 1,$$

which is the equation to an ellipse, referred to oblique coordinates, whose centre is at the origin, and of which the angle of ordination is (say) ω, where ω is the angle between the line of projection and the line joining the point of projection and the centre of force. If a and b are the principal semi-axes, then by the properties of such axes we have

$$a^2+b^2 = \frac{u^2}{\mu}+b_1^{\,2},$$

$$ab = \frac{ub_1}{\mu^{\frac{1}{2}}}\sin\omega;$$

$$\therefore \quad a = \frac{1}{2}\left\{\left(\frac{u^2}{\mu}+b_1^{\,2}+\frac{2ub_1\sin\omega}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{2}} + \left(\frac{u^2}{\mu}+b_1^{\,2}-\frac{2ub_1\sin\omega}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right\},$$

$$b = \frac{1}{2}\left\{\left(\frac{u^2}{\mu}+b_1^{\,2}+\frac{2ub_1\sin\omega}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{2}} + \left(\frac{u^2}{\mu}+b_1^{\,2}-\frac{2ub_1\sin\omega}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right\}.$$

362.] Lagrange, in the "Mécanique Analytique," second part, Sect. VII, Chap. III, Art. 83, remarks that a conic, say, an ellipse, which would be described by a particle under the action of a force varying inversely as the square of the distance and tending to the focus of the ellipse, or under the action of a force varying directly as the distance and having its source in the centre of the ellipse, may also be described under the action of three similar forces which have their sources in the two foci and in the centre of the ellipse respectively; and he makes this remark, after he has proved that such forces yield a particular integral of the differential equation which expresses the motion of a particle under the action of two central forces, which vary inversely as the square of the distance, and whose centres are in two given points. This fact however is only a special application of the following more general law:

If many particles $m_1, m_2, \dots m_n$ which are respectively under the action of the force $\mathrm{P}_1, \mathrm{P}_2, \dots \mathrm{P}_n$ are projected from a given point with the velocities respectively $\mathrm{V}_1, \mathrm{V}_2, \dots \mathrm{V}_n$ along the same line and in the same direction; and if each of these particles describes the same path; then one particle M, projected with the velocity V from the same point, along the same line, and in the same direction as the m's, will describe the same path, if the initial vis viva of M is equal to the sum of the initial vires vivae of the m's; that is, if

$$\mathrm{M}\mathrm{V}^2 = m_1 \mathrm{V}_1^2 + m_2 \mathrm{V}_2^2 + \dots + m_n \mathrm{V}_n^2.$$

Let (x, y, z) be the position of M at the time t; so that its expressed momentum-increments along the coordinate axes are

$$\mathrm{M}\frac{d^2x}{dt^2}, \qquad \mathrm{M}\frac{d^2y}{dt^2}, \qquad \mathrm{M}\frac{d^2z}{dt^2};$$

and let $\mathrm{X}_1, \mathrm{Y}_1, \mathrm{Z}_1, \mathrm{X}_2, \mathrm{Y}_2, \mathrm{Z}_2, \dots \mathrm{X}_n, \mathrm{Y}_n, \mathrm{Z}_n$ be the components of the impressed momentum-increments of the several forces $\mathrm{P}_1, \mathrm{P}_2, \dots \mathrm{P}_n$; and let N be a certain normal force, the direction angles of the line of action of which are α, β, γ, and which is such that M under the action of it and the P's describes the required path. Then the equations of motion of M are

$$\left.\begin{aligned}
\mathrm{M}\frac{d^2x}{dt^2} &= \mathrm{X}_1 + \mathrm{X}_2 + \dots + \mathrm{X}_n + \mathrm{N}\cos\alpha = \Sigma.\mathrm{X} + \mathrm{N}\cos\alpha,\\
\mathrm{M}\frac{d^2y}{dt^2} &= \mathrm{Y}_1 + \mathrm{Y}_2 + \dots + \mathrm{Y}_n + \mathrm{N}\cos\beta = \Sigma.\mathrm{Y} + \mathrm{N}\cos\beta,\\
\mathrm{M}\frac{d^2z}{dt^2} &= \mathrm{Z}_1 + \mathrm{Z}_2 + \dots + \mathrm{Z}_n + \mathrm{N}\cos\gamma = \Sigma.\mathrm{Z} + \mathrm{N}\cos\gamma.
\end{aligned}\right\} \quad (129)$$

Multiplying these respectively by $2dx$, $2dy$, $2dz$, and supposing the velocities of M, $m_1, m_2, \dots m_n$ at the time t to be $v, v_1, v_2, \dots v_n$, we have

$$d.\mathrm{M}v^2 = 2dx\,\Sigma.\mathrm{X} + 2dy\,\Sigma.\mathrm{Y} + 2dz\,\Sigma.\mathrm{Z}, \qquad (130)$$

because $dx\cos\alpha + dy\cos\beta + dz\cos\gamma = 0$.

But the equations of motion of $m_1, m_2, \dots m_n$ under the action of their respective forces yield the following equations:

$$\left.\begin{aligned}
d.m_1 v_1^2 &= 2(\mathrm{X}_1\,dx + \mathrm{Y}_1\,dy + \mathrm{Z}_1\,dz),\\
d.m_2 v_2^2 &= 2(\mathrm{X}_2\,dx + \mathrm{Y}_2\,dy + \mathrm{Z}_2\,dz),\\
&\cdots\cdots\cdots\cdots\\
d.m_n v_n^2 &= 2(\mathrm{X}_n\,dx + \mathrm{Y}_n\,dy + \mathrm{Z}_n\,dz);
\end{aligned}\right\}$$

so that (130) becomes

$$d.\mathrm{M}v^2 = \Sigma.d.mv^2 = d.\Sigma.mv^2;$$

and therefore taking definite integrals, with limits corresponding to $t=t$ and to $t=0$, we have

$$\mathrm{M}v^2 = \Sigma.mv^2;$$

and therefore at all points of the path of M, its vis viva is equal to the sum of the vires vivae of the m's in their separate motions.

Hence it follows that the normal force N assumed in equations (238) is zero, and consequently M, under the action of the several impressed momenta which act on $m_1, m_2, \dots m_n$, will describe the same path as each of the m's. This general proposition is due to M. Ossian Bonnet, and is given in the notes appended by M. Bertrand to the edition of the "Mécanique Analytique" of M. Lagrange, Vol. II, 1855.

363.] In illustration of the process of tangential and normal resolution we will consider the simple case of the motion of a projectile in vacuo under the action of gravity.

A particle of mass m is projected from a given point, in a given line, and with a given velocity; and moves subject to the action of gravity: it is required to determine the curvilinear path.

Let the point of projection be taken as the origin of coordinates; and let the vertical and horizontal lines drawn through it be the axes of y and x respectively: let u be the velocity of projection, and let a be the angle between the line of projection and the axis of x: and let (x, y) be the position of m at the time t: then because the vertical line, in which gravity acts, makes with the tangent to the curve at the point (x, y) $\tan^{-1}\frac{dx}{dy}$, we have the following equations of motion:

$$\frac{d^2s}{dt^2} = -g\frac{dy}{ds}, \qquad \frac{v^2}{\rho} = g\frac{dx}{ds}. \tag{131}$$

From (131) we have

$$\frac{2\,ds\,d^2s}{dt^2} = -2g\,dy;$$

$$\therefore \quad \frac{ds^2}{dt^2} - u^2 = -2gy,$$

$$\frac{ds^2}{dt^2} = u^2 - 2gy, \tag{132}$$

which result is the same as (77) Art. 351. Also from (131), since

$$\rho = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}},$$

$$-\frac{d^2y}{dx^2}(u^2-2gy) = g\frac{dx}{ds}\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}$$

$$= g\left(1 + \frac{dy^2}{dx^2}\right);$$

$$\therefore \quad \frac{d.\frac{dy^2}{dx^2}}{1+\frac{dy^2}{dx^2}} = \frac{-2g\,dy}{u^2-2gy},$$

$$\log \frac{1+\frac{dy^2}{dx^2}}{(\sec \alpha)^2} = \log \frac{u^2-2gy}{u^2},$$

$$\frac{dy}{\{u^2(\sin\alpha)^2-2gy\}^{\frac{1}{2}}} = \frac{dx}{u\cos\alpha},$$

$$-\frac{1}{g}\{u^2(\sin\alpha)^2-2gy\}^{\frac{1}{2}} + \frac{u\sin\alpha}{g} = \frac{x}{u\cos\alpha};$$

$$\therefore \quad y = x\tan\alpha - \frac{gx^2}{2u^2(\cos\alpha)^2}, \qquad (133)$$

which result is the same as that obtained by the method of coordinate resolution in Art. 350.

364.] A particle m describes a helix with a constant velocity: it is required to determine the laws of the accelerating forces which act on it parallel to the three coordinate axes.

Let the equations to the curve be

$$\left.\begin{array}{l} x = a\cos\phi, \\ y = a\sin\phi, \\ z = ka\phi; \end{array}\right\} \qquad \therefore \quad \left.\begin{array}{l} dx = -a\sin\phi\,d\phi, \\ dy = a\cos\phi\,d\phi, \\ dz = ka\,d\phi; \end{array}\right\} \qquad (134)$$

$$\therefore \quad ds^2 = a^2(1+k^2)\,d\phi^2. \qquad (135)$$

But since the velocity along the curve is constant, $ds = c\,dt$;

$$\therefore \quad \frac{d\phi}{dt} = \frac{c}{a(1+k^2)^{\frac{1}{2}}};$$

$$\therefore \quad \frac{dx}{dt} = -\frac{c\sin\phi}{(1+k^2)^{\frac{1}{2}}}, \qquad \frac{dy}{dt} = \frac{c\cos\phi}{(1+k^2)^{\frac{1}{2}}}, \qquad \frac{dz}{dt} = \frac{kc}{(1+k^2)^{\frac{1}{2}}};$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{c^2x}{a^2(1+k^2)}, \qquad \frac{d^2y}{dt^2} = -\frac{c^2y}{a^2(1+k^2)}, \qquad \frac{d^2z}{dt^2} = 0.$$

Hence we infer that the accelerating forces parallel to the axes of x and y vary directly as x and y respectively, and are attractive, and have the same absolute force; that is, the resultant force of these two forces will have a line of action always passing through the axis of z, and will be constant. Also the accelerating force parallel to the axis of z vanishes, the velocity of m parallel to that axis being constant.

365.] The following are problems in relative motion, wherein the place, velocity and velocity-increment of a particle is referred to the moving place of another particle, the directions of the coordinate axes being parallel throughout the motion. We shall have other examples hereafter in which the directions of the axes move. The following are applications of (77), Art. 332:

Two material particles m and m' attract each other with a force varying directly as their masses and inversely as the square of the distance: it is required to determine the motion of m relatively to m'.

Let, in reference to m' placed at the origin O, which is moveable, x, y, z be the coordinates to m at the time t, and let r be the distance of m from O; then since the impressed velocity-increment which acts on m' and attracts it towards m along the line r is $\frac{m}{r^2}$, the components of the impressed velocity-increments on m' at the origin are

$$\frac{mx}{r^3}, \qquad \frac{my}{r^3}, \qquad \frac{mz}{r^3}; \tag{136}$$

and the components of the impressed velocity-increments on m are

$$-\frac{m'x}{r^3}, \qquad -\frac{m'y}{r^3}, \qquad -\frac{m'z}{r^3}. \tag{137}$$

Therefore by (77) in Art. 332,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{(m+m')\,x}{r^3} &= 0, \\ \frac{d^2y}{dt^2} + \frac{(m+m')\,y}{r^3} &= 0, \\ \frac{d^2z}{dt^2} + \frac{(m+m')\,z}{r^3} &= 0; \end{aligned}\right\} \tag{138}$$

and the equations of motion of m' relatively to m will be similar in form: and therefore m will describe relatively to m' a curve similar to that which m' will describe relatively to m.

And to determine the path of m relatively to m': multiplying the second of (138) by z, and the third by y, and subtracting, we have

$$z\frac{d^2y}{dt^2} - y\frac{d^2z}{dt^2} = 0;$$

therefore integrating,

$$\left.\begin{aligned} z\frac{dy}{dt} - y\frac{dz}{dt} &= h_1; \\ \text{similarly,}\qquad x\frac{dz}{dt} - z\frac{dx}{dt} &= h_2; \\ y\frac{dx}{dt} - x\frac{dy}{dt} &= h_3; \end{aligned}\right\} \tag{139}$$

where h_1, h_2, h_3 are arbitrary constants: now multiplying these severally by x, y, z, and adding,

$$h_1 x + h_2 y + h_3 z = 0; \tag{140}$$

which is the equation to a plane passing through the origin, that is, through m'; and therefore the motion of the particles is wholly in one plane.

We may if we choose take this plane to be that of (x, y), and thus reduce the inquiry to one of two dimensions. We will however proceed with the most general case.

From (138) and (139), and writing μ for $m+m'$, we have

$$h_2\frac{d^2z}{dt^2} - h_3\frac{d^2y}{dt^2} = -\left(x\frac{dz}{dt} - z\frac{dx}{dt}\right)\frac{\mu z}{r^3} + \left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)\frac{\mu y}{r^3};$$

$$d\left(h_2\frac{dz}{dt} - h_3\frac{dy}{dt}\right) = \frac{\mu}{r^3}\{(x^2+y^2+z^2)\,dx - x\,(x\,dx + y\,dy + z\,dz)\}$$

$$= \mu\frac{r\,dx - x\,dr}{r^2} = \mu d.\frac{x}{r},$$

since $x^2+y^2+z^2 = r^2$, and $x\,dx + y\,dy + z\,dz = r\,dr$;

$$\left.\begin{aligned} \therefore\quad h_2\frac{dz}{dt} - h_3\frac{dy}{dt} &= \frac{\mu x}{r} + f_1; \\ \text{similarly,}\qquad h_3\frac{dx}{dt} - h_1\frac{dz}{dt} &= \frac{\mu y}{r} + f_2; \\ h_1\frac{dy}{dt} - h_2\frac{dx}{dt} &= \frac{\mu z}{r} + f_3; \end{aligned}\right\} \tag{141}$$

where f_1, f_2, f_3 are three undetermined constants: now multiplying these severally by x, y, z, and adding, and observing (139), we have

$$\mu r + f_1 x + f_2 y + f_3 z = h_1^2 + h_2^2 + h_3^2; \tag{142}$$

and if r is replaced by $(x^2+y^2+z^2)^{\frac{1}{2}}$, and the equation is cleared of radical quantities, it is that to a surface of the second order; and as the intersection of it by the plane (140) is a conic, it follows that the path of m relatively to m' is a conic: and similarly m' describes a conic of the same species relatively to m.

In reference to (142) I would observe, that if (x, y, z) is a point in the orbit of m, $f_1 x + f_2 y + f_3 z - (h_1^2 + h_2^2 + h_3^2)$ is proportional to the perpendicular from (x, y, z) to the plane

$$f_1 x + f_2 y + f_3 z - (h_1^2 + h_2^2 + h_3^2) = 0; \tag{143}$$

and r is the distance from the origin to the same point: therefore from (142) it appears that the distance from the origin to any point on the surface has a constant ratio to the perpendicular distance from the point on a fixed plane: the surface therefore is one of revolution of a conic about its axis, which is perpendicular to the given plane (143), the origin being the focus of the conic, and the given plane being generated by the revolution of the directrix. Hence also the direction-cosines of the axis of the surface are proportional to f_1, f_2, f_3: and since by reason of (14),

$$h_1 f_1 + h_2 f_2 + h_3 f_3 = 0, \tag{144}$$

it appears that the plane (140) passes through the axis: therefore the conic in which m moves is a principal section of the surface (142); and m' is placed in the focus: m therefore describes relatively to m' a conic with m' in the focus: and similarly m' describes relatively to m a conic about m in its focus.

Also multiplying the equations (138) severally by $2\,dx$, $2\,dy$, and $2\,dz$, and adding and integrating, we have

$$\frac{ds^2}{dt^2} = \frac{2\mu}{r} + c; \tag{145}$$

where c is another undetermined constant; but it is to be observed that all the undetermined constants may be found in terms of the initial velocity, the direction-angles of its line of motion, and the coordinates of the point of projection.

The eccentricity of the conic may thus be found. Since the ratio of the focal radius of a conic to the perpendicular from a point on it to the directrix is that of e to 1: and since from (143) the perpendicular from the point (x, y, z) on the directrix is

$$\pm \frac{f_1 x + f_2 y + f_3 z - (h_1^2 + h_2^2 + h_3^2)}{(f_1^2 + f_2^2 + f_3^2)^{\frac{1}{2}}} = p \text{ (say)};$$

$$\begin{aligned}
\therefore \quad \frac{r}{p} &= \pm \frac{r (f_1^2 + f_2^2 + f_3^2)^{\frac{1}{2}}}{f_1 x + f_2 y + f_3 z - (h_1^2 + h_2^2 + h_3^2)}; \\
&= \frac{r (f_1^2 + f_2^2 + f_3^2)^{\frac{1}{2}}}{\mu r} \\
&= \frac{(f_1^2 + f_2^2 + f_3^2)^{\frac{1}{2}}}{\mu};
\end{aligned}$$

$$\therefore \quad e = \frac{(f_1^2+f_2^2+f_3^2)^{\frac{1}{2}}}{\mu}. \tag{146}$$

If the path of m is an ellipse, the equations of the major axis are

$$\frac{x}{f_1} = \frac{y}{f_2} = \frac{z}{f_3} = \frac{\pm r}{(f_1^2+f_2^2+f_3^2)^{\frac{1}{2}}}; \tag{147}$$

and substituting for these values in (142), and taking r_1 and r_2 to be the greatest and least values of r, and substituting

$$h^2 = h_1^2+h_2^2+h_3^2, \quad f^2 = f_1^2+f_2^2+f_3^2, \quad \text{we have}$$

$$r_1 = \frac{h^2}{\mu-f}, \qquad r_2 = \frac{h^2}{\mu+f};$$

$$\therefore \quad r_1+r_2 = 2a = \frac{2h^2\mu}{\mu^2-f^2};$$

$$\therefore \quad a = \frac{h^2\mu}{\mu^2-f^2}$$

$$= \frac{\mu(h_1^2+h_2^2+h_3^2)}{\mu^2-(f_1^2+f_2^2+f_3^2)}; \tag{148}$$

and if i is the inclination to the plane of (x, y) of the plane of motion, then from (140),

$$\cos i = \frac{h_3}{(h_1^2+h_2^2+h_3^2)^{\frac{1}{2}}}; \tag{149}$$

and if Ω is the angle between the axis of x and the line of intersection of the plane of m's motion with the plane of (x, y), then

$$\tan \Omega = -\frac{h_2}{h_1}; \tag{150}$$

and thus the plane of the motion is completely determined.

And thus (149) and (150) give the position of the plane in which m moves: (147) give the direction-angles of the major axis of the ellipse, and therefore assign the position of the ellipse; and (146) and (148) give the dimensions of the elliptic path.

This problem is manifestly the astronomical one of two bodies m and m' moving relatively to each other, and under the action of their mutual attractions, and on this account I have considered it at greater length than would otherwise be necessary. The determination of the other incidents of the motion requires data which it would be out of place here to enter upon.

366.] If two particles m and m' move subject to their mutual attractions, the centre of gravity of them either remains at rest or moves in a straight line.

Let the positions of the particles m and m' at the time t be respectively (x, y, z), and (x', y', z'): and let r be the distance between them, and, to fix our thoughts, let us suppose m farther from the origin than m'; then the equations of motion of the two relatively to the fixed origin are

$$\frac{d^2x}{dt^2} = \frac{-m'(x-x')}{r^3}, \quad \frac{d^2y}{dt^2} = \frac{-m'(y-y')}{r^3}, \quad \frac{d^2z}{dt^2} = \frac{-m'(z-z')}{r^3}; \qquad (151)$$

$$\frac{d^2x'}{dt^2} = \frac{m(x-x')}{r^3}, \quad \frac{d^2y'}{dt^2} = \frac{m(y-y')}{r^3}, \quad \frac{d^2z'}{dt^2} = \frac{m(z-z')}{r^3}; \qquad (152)$$

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = 0, \quad m\frac{d^2y}{dt^2} + m'\frac{d^2y'}{dt^2} = 0, \quad m\frac{d^2z}{dt^2} + m'\frac{d^2z'}{dt^2} = 0. \qquad (153)$$

Let $(\bar{x}, \bar{y}, \bar{z})$ be the position of the centre of gravity of m and m' at the time t: therefore

$$\left.\begin{aligned} (m+m')\bar{x} &= mx + m'x', \\ (m+m')\bar{y} &= my + m'y', \\ (m+m')\bar{z} &= mz + m'z'; \end{aligned}\right\} \qquad (154)$$

and differentiating these twice we have, by reason of (153),

$$(m+m')\frac{d^2\bar{x}}{dt^2} = (m+m')\frac{d^2\bar{y}}{dt^2} = (m+m')\frac{d^2\bar{z}}{dt^2} = 0; \qquad (155)$$

$$\therefore \quad \frac{d^2\bar{x}}{dt^2} = \frac{d^2\bar{y}}{dt^2} = \frac{d^2\bar{z}}{dt^2} = 0. \qquad (156)$$

Suppose α, β, γ to be the components of the velocity of the centre of gravity, and (a, b, c) to be its position when $t = 0$; then integrating (156),

$$\frac{d\bar{x}}{dt} = \alpha, \qquad \frac{d\bar{y}}{dt} = \beta, \qquad \frac{d\bar{z}}{dt} = \gamma;$$

$$\bar{x} - a = \alpha t, \qquad \bar{y} - b = \beta t, \qquad \bar{z} - c = \gamma t; \qquad (157)$$

$$\therefore \quad \frac{\bar{x} - a}{\alpha} = \frac{\bar{y} - b}{\beta} = \frac{\bar{z} - c}{\gamma}; \qquad (158)$$

which are the equations to the rectilinear path of the centre of gravity of m and m'; if $\alpha = \beta = \gamma = 0$, so that the centre of gravity is at rest when $t = 0$, then for all values of t we have

$$\bar{x} = a, \qquad \bar{y} = b, \qquad \bar{z} = c; \qquad (159)$$

and the centre of gravity remains in the same position.

The equations of motion of m and m' relatively to their centre of gravity may be calculated as follows:

Let (ξ, η, ζ) (ξ', η', ζ') be the positions of m and m' relatively to the centre of gravity as origin; let r be the distance between m and m'; ρ and ρ' the distances of m and m' from their centre of gravity, so that

$$\frac{m+m'}{r} = \frac{m'}{\rho} = \frac{m}{\rho'}.$$

Now the x-component of the expressed velocity-increment of m is as follows:

$$\frac{d^2\xi}{dt^2} = -\left(\frac{m'}{r^2} + \frac{m}{r^2}\right)\frac{\xi}{\rho}$$

$$= -\frac{m'^3}{m+m'}\,\frac{\xi}{\rho^3};$$

similarly, $\dfrac{d^2\eta}{dt^2} = -\dfrac{m'^3}{m+m'}\,\dfrac{\eta}{\rho^3}$; $\quad \dfrac{d^2\zeta}{dt^2} = -\dfrac{m'^3}{m+m'}\,\dfrac{\zeta}{\rho^3}$.

By a similar process we find

$$\frac{d^2\xi'}{dt^2} = -\frac{m^3}{m+m'}\,\frac{\xi'}{\rho'^3}, \quad \frac{d^2\eta'}{dt^2} = -\frac{m^3}{m+m'}\,\frac{\eta'}{\rho'^3}, \quad \frac{d^2\zeta'}{dt^2} = -\frac{m^3}{m+m'}\,\frac{\zeta'}{\rho'^3}.$$

The identity of form of these equations shews that the paths which m and m' describe about their centre of gravity are similar; and as the form is the same as that of the equations (138), it follows that the paths are conics, of which the foci are in the centre of gravity of the particles.

367.] There is another important problem of the same kind, the differential equations of which it is desirable to insert.

It is required to calculate the motion of m relatively to M, when M and m are acted on by another particle m', the law of attraction of all three being that of gravitation.

Let (x, y, z), (x', y', z') be the positions of m and of m' relatively to M at the time t; and let r and r' be the distances of m and of m' from M at the same time: and, to fix our thoughts, let us suppose m' to be farther from M than m: then by virtue of the principle of Article 332, the velocity-increment of M due to the attraction of m and of m' is to be impressed on M in a direction the opposite of that along which the attractions of m and m' act; therefore, for the x-component of the velocity-increment of m, we have

$$\frac{d^2x}{dt^2} = -\frac{mx}{r^3} - \frac{m'x'}{r'^3} - \frac{\mathrm{M}x}{r^3} + \frac{m'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}$$

$$= -\frac{(m+\mathrm{M})x}{r^3} - \frac{m'x'}{r'^3} + \frac{m'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}};$$

similarly,

$$\frac{d^2y}{dt^2} = -\frac{(m+\text{M})y}{r^3} - \frac{m'y'}{r'^3} + \frac{m'(y'-y)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}},$$

$$\frac{d^2z}{dt^2} = -\frac{(m+\text{M})z}{r^3} - \frac{m'z'}{r'^3} + \frac{m'(z'-z)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}.$$

Let

$$\frac{m'(xx'+yy'+zz')}{r'^3} - \frac{m'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}} = \text{R}; \quad (160)$$

$$\therefore \quad \left(\frac{d\text{R}}{dx}\right) = \frac{m'x'}{r'^3} - \frac{m'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}; \quad (161)$$

and with similar values for $\left(\frac{d\text{R}}{dy}\right)$ and $\left(\frac{d\text{R}}{dz}\right)$.

So that the equations become

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{(m+\text{M})x}{r^3} + \left(\frac{d\text{R}}{dx}\right) &= 0, \\ \frac{d^2y}{dt^2} + \frac{(m+\text{M})y}{r^3} + \left(\frac{d\text{R}}{dy}\right) &= 0, \\ \frac{d^2z}{dt^2} + \frac{(m+\text{M})z}{r^3} + \left(\frac{d\text{R}}{dz}\right) &= 0; \end{aligned}\right\} \quad (162)$$

R is called *the disturbing function*, because it alone involves m', which is the mass of the body which disturbs the otherwise conic path of m relatively to M.

The problem is manifestly that of the moon moving about the earth, the sun being the disturbing body: or of a planet moving about the sun, another planet being the disturbing body.

SECTION 4.—*The curvilinear motion of a particle in a resisting medium.*

368.] When a particle describes a curvilinear path in a resisting medium, the momentum of the molecules of the medium, which they have on account of their displacement and on account of the particle passing amongst them, is withdrawn from the moving particle, and from it in the line in which it moves: the medium therefore has no effect in diminishing the velocity of the particle in the line which is normal to its path; and the loss takes place along the tangent to the curvilinear path: we proceed to consider the circumstances of a particle moving in such a resisting medium.

Let us consider the moving particle to be spherical in form, so that an equal surface is presented to the medium, whatever is the line in which the particle moves: and to take the general case, whatever is the law of the resistance of the medium, let R represent the velocity-increment (or decrement) which the resisting medium withdraws from the velocity of the particle m in the line of its motion, that is, along the tangent of its curvilinear path in an unit of time; then if X, Y, Z are the three impressed velocity-increments, the equations of motion, referred to three rectangular axes, of a particle moving in space, are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mathrm{X} - \mathrm{R}\frac{dx}{ds}, \\ \frac{d^2y}{dt^2} &= \mathrm{Y} - \mathrm{R}\frac{dy}{ds}, \\ \frac{d^2z}{dt^2} &= \mathrm{Z} - \mathrm{R}\frac{dz}{ds}; \end{aligned}\right\} \tag{163}$$

since $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are the direction-cosines of the tangent of the curvilinear path, that is, of the line of action of R: now multiplying these severally by dx, dy, and dz, and adding, we have

$$\frac{dx\,d^2x + dy\,d^2y + dz\,d^2z}{dt^2} = \mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz - \mathrm{R}\,ds; \tag{164}$$

let $v_0 =$ the velocity, when $t = 0$; and using the symbols indicative of definite integration according to the following form, so that the limits may be those corresponding to $t=t$ and to $t=0$, we have

$$\frac{v^2}{2} - \frac{v_0^{\,2}}{2} = \int_0^t (\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz) - \int_0^t \mathrm{R}\,ds. \tag{165}$$

Also because

$$\frac{d^2s}{dt^2} = \frac{dx}{ds}\frac{d^2x}{dt^2} + \frac{dy}{ds}\frac{d^2y}{dt^2} + \frac{dz}{ds}\frac{d^2z}{dt^2};$$

$$\therefore \quad \frac{d^2s}{dt^2} = \mathrm{X}\frac{dx}{ds} + \mathrm{Y}\frac{dy}{ds} + \mathrm{Z}\frac{dz}{ds} - \mathrm{R}; \tag{166}$$

that is, the expressed velocity-increment along the curve is that due to the impressed forces less that due to the resistance.

Also if P is the resultant of the impressed velocity-increments; then

$$\begin{aligned}
\mathrm{F}^2 &= \mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2 \\
&= \left(\frac{d^2x}{dt^2}+\mathrm{R}\frac{dx}{ds}\right)^2+\left(\frac{d^2y}{dt^2}+\mathrm{R}\frac{dy}{ds}\right)^2+\left(\frac{d^2z}{dt^2}+\mathrm{R}\frac{dz}{ds}\right)^2 \\
&= \frac{(d^2x)^2+(d^2y)^2+(d^2z)^2}{dt^4}+2\mathrm{R}\frac{dx\,d^2x+dy\,d^2y+dz\,d^2z}{ds\,dt^2}+\mathrm{R}^2 \\
&= \left(\frac{v^2}{\rho}\right)^2+\left(\frac{d^2s}{dt^2}\right)^2+2\mathrm{R}\frac{d^2s}{dt^2}+\mathrm{R}^2 \\
&= \left(\frac{v^2}{\rho}\right)^2+\left(\frac{d^2s}{dt^2}+\mathrm{R}\right)^2, \qquad (167)
\end{aligned}$$

if ρ is the radius of absolute curvature of the path; that is, the resultant of the impressed velocity-increments consists of two components, the lines of action of which are at right angles to each other; and of which one acts along the principal normal to the curvilinear path, and is equal to $\frac{v^2}{\rho}$, and the other acts along the tangent and is the sum of the tangential expressed velocity-increment and the resistance. This resolution is therefore the tangential and the normal one; and if T = the tangential impressed velocity-increment, and N = the normal impressed velocity-increment,

$$\mathrm{T} = \frac{d^2s}{dt^2}+\mathrm{R}; \qquad \mathrm{N} = \frac{v^2}{\rho}. \qquad (168)$$

369.] If the motion is wholly in one plane we may take that plane to be the plane of (x, y); and if we resolve along the rectangular axes of x and y, the equations of motion are

$$\frac{d^2x}{dt^2} = \mathrm{X}-\mathrm{R}\frac{dx}{ds}, \qquad \frac{d^2y}{dt^2} = \mathrm{Y}-\mathrm{R}\frac{dy}{ds}. \qquad (169)$$

And if we take the tangential and normal components, we have

$$\frac{d^2s}{dt^2} = \mathrm{T}-\mathrm{R}, \qquad \frac{v^2}{\rho} = \mathrm{N}; \qquad (170)$$

$$\text{or,}\quad \frac{d^2s}{dt^2} = \mathrm{X}\frac{dx}{ds}+\mathrm{Y}\frac{dy}{ds}-\mathrm{R}, \qquad \frac{v^2}{\rho} = \mathrm{X}\frac{dy}{ds}-\mathrm{Y}\frac{dx}{ds}; \qquad (171)$$

$$\text{and}\quad \frac{v^2}{2}-\frac{v_0^2}{2} = \int_0^t(\mathrm{X}\,dx+\mathrm{Y}\,dy)-\int_0^t\mathrm{R}\,ds \qquad (172)$$

$$= \int_0^t(\mathrm{T}-\mathrm{R})\,ds. \qquad (173)$$

If the motion is referred to a system of polar coordinates, and if F [illegible] radial and transversal components of the impressed velocity-increments, then because $\frac{dr}{ds}$ and $\frac{r\,d\theta}{ds}$ are respectively the sine and cosine of the angle between r and the normal to the curve,

$$\left.\begin{aligned}\frac{d^2r}{dt^2}-r\frac{d\theta^2}{dt^2} &= \mathrm{P}-\mathrm{R}\frac{dr}{ds},\\ 2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2} &= \mathrm{Q}-\mathrm{R}\frac{r\,d\theta}{ds}.\end{aligned}\right\} \qquad (174)$$

Similarly by tangential and normal resolution,

$$\left.\begin{aligned}\frac{d^2s}{dt^2} &= \mathrm{P}\frac{dr}{ds}+\mathrm{Q}\frac{r\,d\theta}{ds}-\mathrm{R},\\ \frac{v^2}{\rho} &= \mathrm{Q}\frac{dr}{ds}-\mathrm{P}\frac{r\,d\theta}{ds}.\end{aligned}\right\} \qquad (175)$$

The first of these last two being multiplied by ds and integrated gives

$$\frac{v^2}{2}-\frac{v_0^{\,2}}{2}=\int_0^t(\mathrm{P}\,dr+\mathrm{Q}\,r\,d\theta-\mathrm{R}\,ds); \qquad (176)$$

and these general formulae are sufficient for the solution of all problems relating to motion in a resisting medium.

370.] A particle moves in a resisting medium under the action of forces parallel to the axes of x and y: it is required to determine the law of resistance so that a given plane curve may be described.

Let the impressed forces parallel to the axes of x and y be X and Y; and let R be the resistance; then by (171),

$$\mathrm{R}=\mathrm{X}\frac{dx}{ds}+\mathrm{Y}\frac{dy}{ds}-\frac{d^2s}{dt^2},$$

$$\frac{ds^2}{dt^2}=\rho\frac{\mathrm{X}\,dy-\mathrm{Y}\,dx}{ds};$$

$$\therefore\quad \frac{d^2s}{dt^2}=\frac{1}{2}\frac{d}{ds}\cdot\frac{\rho(\mathrm{X}\,dy-\mathrm{Y}\,dx)}{ds};$$

$$\therefore\quad \mathrm{R}=\frac{\mathrm{X}\,dx+\mathrm{Y}\,dy}{ds}-\frac{1}{2}\frac{d}{ds}\cdot\frac{\rho(\mathrm{X}\,dy-\mathrm{Y}\,dx)}{ds}; \qquad (177)$$

into which expression t, which has been equicrescent, does not enter; we are therefore free to make any other variable equicrescent: and the expression will hereby become somewhat less complicated.

If, according to the law investigated in Art. 267, the resistance varies as the density and as the square of the velocity, and if the density also varies, and the law of its variation is to be discovered so that a given curve may be described, then if s represents the varying density, $R = s \times v^2$;

$$\therefore \quad s = \frac{R}{v^2} = \frac{1}{\rho}\,\frac{X\,dx + Y\,dy}{X\,dy - Y\,dx} - \frac{1}{2}\,\frac{d}{ds}\log\frac{\rho(X\,dy - Y\,dx)}{ds}. \qquad (178)$$

Ex. 1. A particle describes a parabola under the action of a constant force parallel to its principal axis: it is required to determine the law of resistance.

Let the equation to the parabola be $y^2 = 4ax$; so that

$$\frac{dy}{2a} = \frac{dx}{y} = \frac{ds}{(y^2+4a^2)^{\frac{1}{2}}} = \frac{ds}{(4ax+4a^2)^{\frac{1}{2}}},$$

$$\rho = \frac{2(x+a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

In this case $Y = 0$, $X =$ a constant $= k$ (say); therefore by (177),

$$R = k\frac{dx}{ds} - k\frac{d}{ds}(x+a) = 0;$$

that is, in vacuo only does a particle moving under the action of a constant force parallel to the axis of x describe a parabola.

Ex. 2. A particle moves in a circle under the action of a constant force in parallel lines, and the resistance of the medium varies as the density and the square of the velocity: it is required to determine the law of variation of the density of the medium.

Let the equation to the circle be $x^2+y^2 = a^2$; therefore

$$\frac{dx}{y} = -\frac{dy}{x} = \frac{ds}{a};$$

and let the line of action of the constant force be parallel to the axis of y: so that $X = 0$, $Y =$ a constant $= -k$ (say); therefore from (178), $\quad s = \dfrac{3x}{2ay}.$

371.] A particle moves in a resisting medium under the action of a central force P: it is required to find the law of resistance, so that a given curve may be described.

From (175), if $Q = 0$, we have

$$\frac{d^2s}{dt^2} = P\frac{dr}{ds} - R; \qquad \frac{ds^2}{dt^2} = -P\rho\frac{r\,d\theta}{ds} = -Pp\frac{dr}{dp};$$

since $\quad \rho = \dfrac{r\,dr}{dp}$, and $\dfrac{r\,d\theta}{ds} = \dfrac{p}{r}$;

$$\therefore \quad \frac{d^2s}{dt^2} = -\frac{1}{2}\,\frac{d}{ds}\Big(Pp\frac{dr}{dp}\Big); \qquad (179)$$

$$\therefore \quad R = P\frac{dr}{ds} + \frac{1}{2}\,\frac{d}{ds}\Big(Pp\frac{dr}{dp}\Big) \qquad (180)$$

$$\text{R} = \frac{1}{2p^2}\frac{d}{ds}\Big(\text{P}p^3\frac{dr}{dp}\Big); \tag{181}$$

$$\therefore \quad \text{P} = \frac{dp}{p^3\,dr}\int 2p^2\,\text{R}\,ds; \tag{182}$$

whereby, if the law of resistance and the equation of the curve are given, the central force P may be found.

Again, if the resistance varies as the density and as the square of the velocity, and if the density also varies, and the law of its variation is to be investigated, so that a given curve may be described, then if S represents the density, $\text{R} = \text{S} \times v^2$;

$$\therefore \quad \text{S} = \frac{\text{R}}{v^2} = -\frac{1}{2}\frac{d}{ds}\log\Big(\text{P}p^3\frac{dr}{dp}\Big); \tag{183}$$

and if the density of the medium is given, and the central force is to be discovered, then from this last equation we have

$$\text{P} = \frac{h^2}{p^3}\frac{dp}{dr}e^{-2\int \text{S}\,ds}, \tag{184}$$

where h^2 is a constant introduced in integration.

And (184) may be put under the following form:

$$\text{Let } u = \frac{1}{r}; \qquad \therefore \quad \frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2};$$

$$\therefore \quad -\frac{2\,dp}{p^3} = 2\Big(u + \frac{d^2u}{d\theta^2}\Big)du$$

$$= -2\Big(u + \frac{d^2u}{d\theta^2}\Big)\frac{dr}{r^2};$$

$$\therefore \quad \frac{dp}{p^3\,dr} = \Big(u + \frac{d^2u}{d\theta^2}\Big)u^2;$$

$$\therefore \quad \text{P} = h^2u^2\Big(u + \frac{d^2u}{d\theta^2}\Big)e^{-2\int \text{S}\,ds}. \tag{185}$$

Ex. 1. A particle moves in the circumference of a circle under the attraction of a central force whose origin is a point in the circumference, and the law of which varies as the nth power of the distance: it is required to determine the law of the density and the resistance of the medium.

Let the radius of the circle be a, and let the pole be at the centre of force, then

$$r^2 = 2ap, \quad \text{and} \quad \text{P} = -\mu r^n; \quad \text{also} \quad r\,dr = (r^2 - p^2)^{\frac{1}{2}}\,ds;$$

therefore from (181),

$$\text{R} = \frac{-\mu(n+5)\,r^{n+4}}{16a^2p^2}\frac{dr}{ds}$$

$$= \frac{-\mu(n+5)\,r^n(4a^2 - r^2)^{\frac{1}{2}}}{8a}; \tag{186}$$

and from (183), $s = -\frac{(n+5)(4a^2-r^2)^{\frac{1}{2}}}{4ar}$. (187)

It appears therefore that if $n = -5$, that is, if the central force varies inversely as the fifth power of the distance, $R = 0$ and $s = 0$; that is, the particle must move in vacuo; also if $r = 2a$, whatever n is, the resistance and the density vanish.

372.] To determine the motion of a projectile under the action of gravity in a medium of which the resistance varies directly as the velocity and of which the density is uniform.

Let the velocity of projection be u, and let the point of projection be taken as the origin: let the horizontal plane through it be that of (x, y) and let the axis of z be measured in a direction contrary to that of the action of gravity: let the resistance be $k\frac{ds}{dt}$, of which the components along the coordinate axes are

$$k\frac{dx}{dt}, \qquad k\frac{dy}{dt}, \qquad k\frac{dz}{dt};$$

so that the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -k\frac{dx}{dt}, \\ \frac{d^2y}{dt^2} &= -k\frac{dy}{dt}, \\ \frac{d^2z}{dt^2} &= -g-k\frac{dz}{dt}; \end{aligned}\right\} \tag{188}$$

from the first and second of which we have

$$dy\, d^2x - dx\, d^2y = 0;$$

$$\therefore \quad \frac{d^2x}{dx} = \frac{d^2y}{dy}; \qquad \therefore \quad \log\frac{dx}{a} = \log\frac{dy}{b};$$

$$\therefore \quad \frac{x}{a} = \frac{y}{b}, \tag{189}$$

which is the equation to a plane passing through the axis of z, in which therefore the motion of the particle takes place.

Since then the motion takes place in one plane, let us assume that to be the plane of (x, y); and let the point of projection be the origin, and let the horizontal line through the origin be the axis of x; and let the angle between the line of projection and the axis of $x = a$: and let the axis of y be taken in a direction the opposite of that of gravity: so that the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -k\frac{dx}{dt}, \\ \frac{d^2y}{dt^2} &= -g-k\frac{dy}{dt}; \end{aligned}\right\} \tag{190}$$

from the former of which we have by definite integration

$$\frac{dx}{dt} - u\cos a = -kx; \tag{191}$$

$$\therefore \quad \frac{dx}{u\cos a - kx} = dt, \qquad \log\frac{u\cos a - kx}{u\cos a} = -kt;$$

$$\therefore \quad x = \frac{u\cos a}{k}\{1 - e^{-kt}\}, \tag{192}$$

the limits of integration being the values corresponding to $t = t$ and to $t = 0$: and from the latter of (190) we have

$$\frac{dy}{dt} - u\sin a = -gt - ky,$$

$$\frac{dy}{dt} + ky = u\sin a - gt; \tag{193}$$

$$\begin{aligned} \therefore \quad y &= \left(\frac{d}{dt} + k\right)^{-1}(u\sin a - gt) \\ &= e^{-kt}\int_0^t e^{kt}(u\sin a - gt)\,dt \\ &= e^{-kt}\left[\frac{(u\sin a - gt)}{k}e^{kt} + \frac{g}{k^2}e^{kt}\right]_0^t; \end{aligned}$$

$$\therefore \quad ky + gt = \left(u\sin a + \frac{g}{k}\right)(1 - e^{-kt}); \tag{194}$$

and eliminating t by means of (192) and (194) we have

$$y = x\tan a + \frac{gx}{ku\cos a} + \frac{g}{k^2}\log\left(1 - \frac{kx}{u\cos a}\right); \tag{195}$$

which is the equation to the path of the projectile.

373.] If we expand the logarithmic term in the preceding expression, the equation becomes

$$\begin{aligned} y &= x\tan a + \frac{gx}{ku\cos a} - \frac{g}{k^2}\left\{\frac{kx}{u\cos a} + \frac{k^2x^2}{2u^2(\cos a)^2} + \frac{k^3x^3}{3u^3(\cos a)^3} + \dots\right\} \\ &= x\tan a - \frac{gx^2}{2u^2(\cos a)^2} - \frac{gkx^3}{3u^3(\cos a)^3} - \dots, \end{aligned} \tag{196}$$

of which series all terms, except the first two, contain k (the coefficient of resistance); and if $k = 0$, the equation is that to the parabolic path which is the trajectory of a projectile in vacuo, see Article 350; and thus the terms on the right-hand side of (196), after the first two, express the excess of the ordinate of the parabola described in vacuo over the ordinate of the curve whose equation is (196): the trajectory (196) therefore is of a form somewhat parabolic, but the curve recedes from the directrix further than a parabola.

Also by reason of (191), $\frac{dx}{dt} = 0$ if $x = \frac{u\cos\alpha}{k}$; for this value of x therefore the horizontal velocity vanishes, and the projectile moves in a vertical path; and therefore a vertical line, at a distance $= \frac{u\cos\alpha}{k}$ from the point of projection, is an asymptote to the curve.

Also from (195) $\frac{dy}{dx} = 0$, that is, the projectile comes to its highest point when

$$0 = \tan\alpha + \frac{g}{ku\cos\alpha} - \frac{g}{k(u\cos\alpha - kx)}, \tag{197}$$

viz. when $x = \frac{u^2\sin\alpha\cos\alpha}{g + ku\sin\alpha}$, $y = \frac{u\sin\alpha}{k} + \frac{g}{k^2}\log\frac{g}{g+ku\sin\alpha}$.

Also from (197), if $x = -\infty$, $\frac{dy}{dx} = \tan\alpha + \frac{g}{ku\cos\alpha}$; that is, the curve being produced backwards through the origin continually approaches to a certain definite angle with the axis of x.

374.] But the most important application of this theory is that of gunnery, in which the motion takes place in air, the resistance of which (at least approximately) varies as the square of the velocity; and thus the following problem offers itself:

A particle, or a spherical homogeneous ball, acted on by gravity, is projected with a given velocity u in a given line, in a medium of which the resistance varies as the square of the velocity and the density is uniform: it is required to determine the circumstances of motion.

It may be shewn, by a process similar to that at the commencement of Art. 372, that the motion takes place in one plane; and therefore we may assume that plane to be the plane of (x, y).

Let $u =$ the velocity of projection; and let a horizontal and a vertical line through the point of projection be the axes respectively of x and y: let the axis of y be taken upwards: let α be the angle between the axis of x and the line of projection: and let the resistance of the medium be $k\frac{ds^2}{dt^2}$; then, as the line of action of this is the tangent of the curve, its components are

$$k\frac{ds}{dt}\frac{dx}{dt}, \text{ and } k\frac{ds}{dt}\frac{dy}{dt};$$

and as k, see Art. 294, varies directly as the density of the medium, as the surface which the moving ball presents to the medium, and inversely as the mass of the ball, then, as the ball is spherical and the density is uniform, k is constant. Thus the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -k\frac{ds}{dt}\frac{dx}{dt}, \\ \frac{d^2y}{dt^2} &= -g-k\frac{ds}{dt}\frac{dy}{dt}. \end{aligned}\right\} \tag{198}$$

Integrating the first of these, and taking the limits corresponding to $t = t$ and to $t = 0$, we have

$$\frac{\frac{d^2x}{dt^2}}{\frac{dx}{dt}} = -k\frac{ds}{dt}, \qquad \therefore \quad \log\frac{\frac{dx}{dt}}{u\cos\alpha} = -ks;$$

$$\therefore \quad \frac{dx}{dt} = u\cos\alpha e^{-ks}. \tag{199}$$

Again from (198), and transforming the equations so that t is not equicrescent, we have

$$\frac{d^2y\,dx - d^2x\,dy}{dt^2} = -g\,dx;$$

therefore from (199),

$$\frac{d^2y\,dx - d^2x\,dy}{dx^2} = -\frac{g}{u^2(\cos\alpha)^2}e^{2ks}dx; \tag{200}$$

$$\therefore \quad d.\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = -\frac{g}{u^2(\cos\alpha)^2}e^{2ks}ds; \tag{201}$$

and integrating, and taking the limits corresponding to $t = t$ and to $t = 0$, we have

$$\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} + \log\left\{\frac{dy}{dx} + \left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\}$$

$$-\tan\alpha\sec\alpha - \log(\tan\alpha+\sec\alpha) = -\frac{g}{ku^2(\cos\alpha)^2}(e^{2ks}-1); \tag{202}$$

and for convenience let us substitute

$$\tan\alpha\sec\alpha + \log(\tan\alpha+\sec\alpha) + \frac{g}{ku^2(\cos\alpha)^2} = \text{c}; \tag{203}$$

so that (202) becomes

$$\begin{aligned} \frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} + \log\left\{\frac{dy}{dx} + \left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\} &= \text{c} - \frac{g}{ku^2(\cos\alpha)^2}e^{2ks} \\ &= \text{c} + \frac{1}{k}\frac{d}{dx}\left(\frac{dy}{dx}\right); \end{aligned} \tag{204}$$

$$\therefore \quad \frac{d.\frac{dy}{dx}}{\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} + \log\left\{\frac{dy}{dx} + \left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\} - \text{c}} = k\,dx, \tag{205}$$

$$\frac{\frac{dy}{dx}\,d.\frac{dy}{dx}}{\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}+\log\left\{\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\}-\mathrm{C}} = k\,dy; \quad (206)$$

from which equations, were it possible to integrate them, x and y might be found in terms of $\frac{dy}{dx}$; and if $\frac{dy}{dx}$ were eliminated from the two integrals, the resulting equation in terms of x and y would be that of the required trajectory.

But as the equations are not integrable, we must deduce from them in their present forms such results as are possible.

Equating the values of e^{ks} which are given in equations (199) and (204), we have

$$dt = -\frac{1}{(kg)^{\frac{1}{2}}}\frac{d.\frac{dy}{dx}}{\left\{\mathrm{C}-\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}-\log\left(\frac{dy}{dx}+\left\{1+\frac{dy^2}{dx^2}\right\}^{\frac{1}{2}}\right)\right\}^{\frac{1}{2}}}; \quad (207)$$

whence might the time be found in terms of $\frac{dy}{dx}$: also squaring (205) and (206), and dividing by the square of (207), we have

$$\frac{ds^2}{dt^2} = \frac{g}{k}\,\frac{1+\frac{dy^2}{dx^2}}{\mathrm{C}-\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}-\log\left\{\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\}}; \quad (208)$$

which equation gives the velocity in terms of $\frac{dy}{dx}$.

(204) is the intrinsic equation to the path of the projectile.

375.] When $\frac{dy}{dx} = -\infty$,

$$\mathrm{C}-\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}-\log\left\{\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\} \text{ becomes } -\frac{dy^2}{dx^2};$$

so that (205) becomes

$$k\,dx = -\frac{d.\frac{dy}{dx}}{\frac{dy^2}{dx^2}}; \quad (209)$$

$\therefore\ \ k(x-a) = \frac{dx}{dy}$, where a is an arbitrary constant: therefore if $\frac{dy}{dx} = -\infty$, $x = a =$ a constant; that is, the line, whose equation is $x = a$, is an asymptote to the curve. Also under the same supposition from (208), we have

$$(\text{vel.})^2 = \frac{g}{k}; \tag{210}$$

that is, the velocity of the projectile, as it falls down the descending branch of the trajectory, approaches to the constant limit $\left(\frac{g}{k}\right)^{\frac{1}{2}}$.

And at the highest point of the path, when $\frac{dy}{dx} = 0$, $(\text{vel.})^2 = \frac{g}{k\,\mathrm{c}}$.

Thus the path of the projectile is a curve of the form delineated in fig. 124, where OA $= a$.

376.] The case however frequently occurs in practice wherein the angle of projection is very small; and where the ball rises very little above the horizontal line; and therefore $\frac{dy}{dx}$ being very small, we may throughout the path on the upper side of the axis of x neglect powers of $\frac{dy}{dx}$ higher than the first. In this case then

$$ds = dx; \qquad \therefore \quad s = x:$$

so that (200) becomes

$$d.\frac{dy}{dx} = -\frac{g}{u^2(\cos\alpha)^2}\,e^{2kx}\,dx;$$

$$\therefore \quad \frac{dy}{dx} - \tan\alpha = -\frac{g}{2ku^2(\cos\alpha)^2}(e^{2kx}-1);$$

$$\therefore \quad y = x\tan\alpha + \frac{gx}{2ku^2(\cos\alpha)^2} - \frac{g}{4k^2u^2(\cos\alpha)^2}(e^{2kx}-1); \tag{211}$$

and expanding in a series the last term, we have

$$y = x\tan\alpha - \frac{gx^2}{2u^2(\cos\alpha)^2} - \frac{gkx^3}{3u^2(\cos\alpha)^2} - \dots; \tag{212}$$

which equation, if the terms involving k are omitted, is that of a parabola, which is the path of the projectile in vacuo. It appears therefore that the ordinate of the actual curve is that of the parabola diminished by a quantity which is the sum of all the terms of the right-hand member of the last equation except the first two.

Also from (199), $\frac{dx}{dt} = u\cos\alpha e^{-kx}$;

$$\therefore \quad ku\cos\alpha\, t = e^{kx} - 1; \tag{213}$$

which gives the time in terms of the abscissa.

CHAPTER XI.

THE FREE MOTION OF PARTICLES, UNDER THE ACTION OF CENTRAL FORCES.

SECTION 1.—*General investigations; determination of laws of force and other circumstances of motion in given orbits.*

377.] A *central force* is that of which the source of influence is at a certain point, towards which it attracts or from which it repels any particle of matter within reach of its action; and according as the action of it is attraction or repulsion, so is it called an attractive or a repulsive force. The forces, whose effects will be considered, are supposed to be functions of the distance between their centres and the particle on which they act, and not to be functions explicitly of either the time or the velocity; the case in which the line of motion of the particle is coincident with that of the action of the force has been considered in Chapter VIII; and it remains for us now to discuss the case in which the line of motion of the particle is inclined at any angle to the line of action of the central force. The principles and equations of Chapter IX are sufficient for the inquiry, and have indeed been applied to the subject in Art. 359, and others, in the form of rectangular coordinate resolution: but as the method of resolution into radial and transversal components is more convenient, and as formulae different to any heretofore employed are deducible from them; and moreover as nature presents to us more instances of this than of any other kind of dynamical action, it is desirable to devote a separate Chapter to the inquiry: and in the course of it we shall take occasion to exhibit the first elements of celestial mechanics in the form of the simple elliptic orbit which a planet undisturbed would describe about its primary.

378.] Let m be the mass of the moving particle, and let (x, y, z) be its position at the time t; let the centre of force, which we suppose to be fixed, be the origin of coordinates: and let P represent the central force; that is, the impressed velocity-increment in an unit of time: let r be the distance of m from the centre of force at the time t: then we suppose P to be a function of r; let P be attractive, so that the equations of motion in terms of velocity-increments are

$$\frac{d^2x}{dt^2} = -\frac{\text{P}x}{r}, \qquad \frac{d^2y}{dt^2} = -\frac{\text{P}y}{r}, \qquad \frac{d^2z}{dt^2} = -\frac{\text{P}z}{r}. \qquad (1)$$

Now in the first place I shall shew that the moving particle is always in one and the same plane, and that a plane which passes through the centre of force.

From the second and third of (1) we have

$$y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2} = 0;$$

$$\left.\begin{aligned} \therefore \text{ integrating} \quad & y\frac{dz}{dt} - z\frac{dy}{dt} = h_1; \\ \text{similarly} \quad & z\frac{dx}{dt} - x\frac{dz}{dt} = h_2; \\ & x\frac{dy}{dt} - y\frac{dx}{dt} = h_3; \end{aligned}\right\} \qquad (2)$$

therefore multiplying these equations severally by x, y, z, and adding,

$$h_1x + h_2y + h_3z = 0; \qquad (3)$$

which is the equation to a plane passing through the origin, which is the centre of force, and in which therefore m always is. The orbit therefore of m, as the trajectory is called, is a plane curve.

This fact too is evident by the principle of sufficient reason: because every reason which can be urged why m should move out of the plane, which contains the centre and two consecutive points of the path, on one side may be shewn to be equally valid why it should leave the plane towards the other side.

We may therefore, without loss of generality, suppose the plane in which m moves to be that of (x, y).

379.] Let P represent the central force and be attractive; let the centre of force be the origin, (x, y) the position of m at the time t; r = the distance of m from the origin: then the equations of motion in terms of velocity-increments are

$$\frac{d^2x}{dt^2} = -\frac{\text{P}x}{r}, \qquad \frac{d^2y}{dt^2} = -\frac{\text{P}y}{r}; \qquad (4)$$

multiplying the former by y, and the latter by x, and subtracting, we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0; \qquad (5)$$

whence, adding and subtracting $\frac{dx}{dt}\frac{dy}{dt}$, and integrating, we have

$$x\frac{dy}{dt}-y\frac{dx}{dt}=h; \tag{6}$$

where h is an undetermined constant.

Now by Differential Calculus, Art. 219, (44), if p is the perpendicular from the origin on the tangent,

$$x\,dy-y\,dx=p\,ds;$$

therefore from (6),

$$\frac{ds}{dt}=\frac{h}{p}; \tag{7}$$

that is, the velocity at any point of the orbit varies inversely as the perpendicular on the tangent at that point from the centre of force.

Also since

$$\left.\begin{array}{ll} x=r\cos\theta, & dx=dr\cos\theta-r\sin\theta\,d\theta, \\ y=r\sin\theta; & dy=dr\sin\theta+r\cos\theta\,d\theta; \end{array}\right\} \tag{8}$$

$$\therefore\quad x\,dy-y\,dx=r^2\,d\theta; \tag{9}$$

therefore from (6),

$$r^2\,d\theta=h\,dt. \tag{10}$$

Now $r^2\,d\theta$ is twice the sectorial area which the radius-vector of m describes in the time dt, and as it is proportional to dt by (10), we infer that the sectorial areas described by the radius-vector of m are proportional to the times of describing them: or in other and equivalent words, equal sectorial areas are described in equal times. Hence also it appears that

h = twice the sectorial area described in one unit of time. (11)

Let us give a geometrical proof and interpretation of the theorems (7) and (10). Suppose S to be the centre of force, fig. 125, P to be the position of m at the time t, and the element PQ to be its path-element in the time dt; and let t be equicrescent; and let SY $= p$ be the perpendicular from S on PQ produced. Now let PQ be produced to R′ so that QR′ = PQ; then if no force acted, m would at the end of the second dt be at R′; but suppose, when m is at Q, the central force to act impulsively and to draw m over a distance Qr in the time dt; then at the end of dt, if R′R is equal and parallel to Qr, and the parallelogram rR′ is completed, m is at R; by a similar process and construction it may be shewn that m is at T at the end of the third dt, and so on: now since PQ = QR′, therefore the triangles SPQ, SQR′ are equal, and because SRQ, SR′Q are on the same base

SQ and between the same parallels, SRQ = SR′Q; therefore SPQ = SRQ: similarly it may be shewn that SRQ = STR = SUT, = ...; and thus the sectorial triangles which correspond to equal dt's are equal. And the same result is true in the limit, when the polygon which is drawn in the figure becomes a continuous curve, and the central force acts continuously; and thus under the action of a central force equal sectorial areas are described in equal times.

Also let h = twice the sectorial area described in an unit of time: let PQ = ds, SP $= r$, PSQ $= d\theta$, SY $= p$: then the triangle described in dt units of time is PSQ; and the area of PSQ $= \frac{r^2 d\theta}{2}$, see Integral Calculus, Art. 226, (12); and also is equal to

$$\frac{1}{2}\text{PQ} \times \text{SY} = \frac{p\,ds}{2};$$

$$\therefore \quad h\,dt = r^2 d\theta = p\,ds; \qquad \therefore \quad h = r^2\frac{d\theta}{dt} = p\frac{ds}{dt};$$

and therefore as equal sectorial areas are described in equal times, so does the velocity vary inversely as the perpendicular from the centre of force on the tangent.

Also since $\frac{d\theta}{dt} = \frac{h}{r^2}$, it appears that the angular velocity varies inversely as the square of the radius-vector at the point.

Also if t is the time during which the particle passes from a point in its orbit corresponding to θ_0 to another point corresponding to θ, then since

$$dt = \frac{r^2}{h}d\theta; \qquad \therefore \quad t = \frac{1}{h}\int_{\theta_0}^{\theta} r^2 d\theta; \tag{12}$$

whereby in a given orbit the time may be found in terms of the angle through which the radius-vector of the particle has moved. The means of determining h in a given orbit and under a given absolute force will be shewn hereafter.

Thus also if the orbit is a closed curve whose area is A, and if T is the whole time, or the periodic time, as it is called, in which m has described it,

$$\text{T} = \frac{2\text{A}}{h}. \tag{13}$$

380.] Again, multiplying the first of (4) by $2\,dx$, and the second by $2\,dy$, and adding, we have

$$\frac{2\,dx\,d^2x + 2\,dy\,d^2y}{dt^2} = -\frac{2\,\text{P}(x\,dx + y\,dy)}{r}; \tag{14}$$

but since $x^2+y^2 = r^2$; $\therefore$ $x\,dx+y\,dy = r\,dr$;

$$\therefore \quad \frac{2\,dx\,d^2x+2\,dy\,d^2y}{dt^2} = -2\,\text{P}\,dr. \tag{15}$$

Let V be the velocity at a given point, at which $r = \text{R}$ (say); then integrating (15), we have

$$\frac{ds^2}{dt^2}-\text{V}^2 = -2\int_{\text{R}}^{r} \text{P}\,dr; \tag{16}$$

$$\therefore \quad (\text{velocity})^2 = \text{V}^2-2\int_{\text{R}}^{r} \text{P}\,dr. \tag{17}$$

Let v be the velocity at the point to which r corresponds: and let m be the mass of the moving particle: then

$$\frac{mv^2-m\text{V}^2}{2} = -m\int_{\text{R}}^{r} \text{P}\,dr; \tag{18}$$

which is the equation of vis viva and of work, and is the form which (65), Art. 325, takes when there is no transversal force. Thus if P is a function of the distance of m from the centre of force, so that the right-hand member admits of integration, this equation shews that the increase of vis viva of the particle in passing from one point to another depends on the coordinates of the two points, and not on the path which the particle has described in the passage: that is, the change in vis viva, or the increase of work, depends only on the distance through which the force has acted in its own line of action.

From (17) it appears that the velocity is the same at all points which are equally distant from the centre; for if $r = \text{R}$, the velocity $=$ V: and thus if the orbit is a re-entering curve, the particle always in its successive revolutions passes through the same point with the same velocity.

Now equating the values of the velocity in (7) and (16), we have

$$\frac{h^2}{p^2} = \text{V}^2-2\int_{\text{R}}^{r} \text{P}\,dr; \tag{19}$$

therefore differentiating, we have

$$\frac{-2h^2}{p^3}\,dp = -2\,\text{P}\,dr;$$

$$\therefore \quad \text{P} = \frac{h^2}{p^3}\,\frac{dp}{dr}. \tag{20}$$

And if $r = \dfrac{1}{u}$, $dr = -\dfrac{du}{u^2}$;

and Differential Calculus, Art. 270, (19), $\dfrac{1}{p^2} = u^2 + \dfrac{du^2}{d\theta^2}$;

$$\therefore \quad \frac{-2\,dp}{p^3} = 2\,du\left(u + \frac{d^2u}{d\theta^2}\right);$$

and (20) becomes $$\text{P} = h^2u^2\left\{\frac{d^2u}{d\theta^2} + u\right\}; \qquad (21)$$

which is the result already found by a different process in Art. 312.

Thus from either (20) or (21) the law of central force may be determined, under the action of which a particle moves in a given curve. And from (12) or (13) the time, which is occupied by its passage through a given arc, or through the whole curve, if the curve is closed, may be found. And from (7) or (17) the velocity at any point in the orbit may be determined.

Also because that part of the radius-vector at any point of a curve referred to polar coordinates which is intercepted by the circle of curvature, or the chord of the circle of curvature, as it is called, see Differential Calculus, Art. 301, is equal to $2p\frac{dr}{dp}$, equation (20) gives

$$\frac{h^2}{p^2} = (\text{vel.})^2 = p\,\text{P}\frac{dr}{dp}$$

$$= 2\text{P} \times \frac{\text{chord of circle of curvature}}{4}; \qquad (22)$$

and comparing this with (32), Art. 268, it appears that a particle at rest on the curve, and moving from it towards the centre of force under the action of the force continuing constant, acquires the velocity which the particle has in its curvilinear course, when it has moved through one-fourth of the chord of the circle of curvature.

Hence if the orbit is a circle, having the centre of force in the centre, and R, P, V are respectively the radius, central force, and velocity, $\text{V}^2 = \text{P} \times \text{R}$.

A point in an orbit at which the curve is at right angles to the radius-vector is called *an apse*; the radius-vector at an apse is called *an aspidal distance;* and the angle between two consecutive apsidal distances is called *an apsidal angle* of the orbit. The analytical character of an apse is manifestly

$$\frac{du}{d\theta} = 0, \quad \text{or} \quad = \infty. \qquad (23)$$

381.] Examples illustrative of the preceding equations:

Ex. 1. It is required to find the law of force, the velocity, and the periodic time, in an elliptic orbit, when the centre of force is in the focus.

Let the equation to the ellipse, the focus being the pole, be

$$r = \frac{a(1-e^2)}{1+e\cos\theta}; \qquad \therefore \quad u = \frac{1+e\cos\theta}{a(1-e^2)}; \tag{24}$$

$$\frac{du}{d\theta} = -\frac{e\sin\theta}{a(1-e^2)}, \qquad \frac{d^2u}{d\theta^2} = \frac{-e\cos\theta}{a(1-e^2)};$$

$$\therefore \quad \mathrm{P} = h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = \frac{h^2u^2}{a(1-e^2)}$$

$$= \frac{h^2}{a(1-e^2)}\,\frac{1}{r^2}; \tag{25}$$

and the force varies inversely as the square of the distance, and is attractive, as appears by its sign and by the convention as to signs which was assumed in Art. 377. Let μ be the absolute force of the central force, then

$$\mathrm{P} = \frac{\mu}{r^2};$$

$$\text{and} \qquad \mu = \frac{h^2}{a(1-e^2)}; \qquad \therefore \quad h = \{\mu a(1-e^2)\}^{\frac{1}{2}}; \tag{26}$$

so that h is given in terms of the absolute force, which is the mass of the attracting body, or the sum of the masses of the attracting and attracted bodies, if the motion is relative, and of the quantities which determine the magnitude of the orbit.

$$\text{Also since} \qquad \frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2} = \frac{2au-1}{a^2(1-e^2)};$$

$$\therefore \quad (\text{the velocity})^2 = \frac{h^2}{p^2} = \frac{\mu(2au-1)}{a}. \tag{27}$$

Hence if S is the focus in which the force is, and H is the other focus, (the velocity)^2 varies as $\frac{\mathrm{HP}}{\mathrm{SP}}$.

If T is the periodic time in the elliptic orbit, then, as the area of the ellipse $= \pi ab = \pi a^2(1-e^2)^{\frac{1}{2}}$, by (13),

$$\mathrm{T} = \frac{2\pi a^2(1-e^2)^{\frac{1}{2}}}{\{\mu a(1-e^2)\}^{\frac{1}{2}}} = \frac{2\pi}{\mu^{\frac{1}{2}}}a^{\frac{3}{2}}; \tag{28}$$

thus the periodic time varies as the square root of the cube (as the sesquiplicate power) of the major axis. As these results will come under consideration hereafter, it is unnecessary now to comment on them.

Ex. 2. To find the law of force and the velocity in a parabola, the focus of which is the centre of force.

Let the equation to the parabola be

$$r = \frac{2a}{1+\cos\theta}; \tag{29}$$

where the line joining the focus and the vertex is the prime radius.

$$\begin{aligned} \mathrm{P} &= h^2u^2\Big(\frac{d^2u}{d\theta^2}+u\Big) = \frac{h^2u^2}{2a} \\ &= \frac{h^2}{2a}\,\frac{1}{r^2}; \end{aligned} \tag{30}$$

thus the central force varies inversely as the square of the distance, and is attractive. Let $\mu =$ the absolute force;

$$\therefore \quad \mu = \frac{h^2}{2a}; \qquad\qquad h^2 = 2a\mu; \tag{31}$$

$$\begin{aligned} (\text{the velocity})^2 &= \frac{h^2}{p^2} = h^2\Big(u^2+\frac{du^2}{d\theta^2}\Big) \\ &= \frac{2\mu}{r}. \end{aligned} \tag{32}$$

Also let t be the time during which the particle moves from a point corresponding to θ_0 to another point corresponding to θ_n; then by (12),

$$\begin{aligned} t &= \frac{(2a)^{\frac{3}{2}}}{(\mu)^{\frac{1}{2}}}\int_{\theta_0}^{\theta_n}\frac{d\theta}{(1+\cos\theta)^2} \\ &= \frac{a^{\frac{3}{2}}}{(2\mu)^{\frac{1}{2}}}\int_{\theta_0}^{\theta_n}\Big(\sec\frac{\theta}{2}\Big)^4 d\theta \\ &= \Big(\frac{2a^3}{\mu}\Big)^{\frac{1}{2}}\Big\{\tan\frac{\theta_n}{2}-\tan\frac{\theta_0}{2}+\frac{1}{3}\Big(\tan\frac{\theta_n}{2}\Big)^3-\frac{1}{3}\Big(\tan\frac{\theta_0}{2}\Big)^3\Big\}. \end{aligned} \tag{33}$$

And this value for t may be expressed in terms of r_n and r_0, the focal radii vectores corresponding to θ_n and to θ_0, and of the chord c (say) which joins their extremities.

For the sake of more convenient symbols, let $\tan\frac{\theta_n}{2} = t_n$, $\tan\frac{\theta_0}{2} = t_0$; so that (33) becomes

$$\begin{aligned} t &= \Big(\frac{2a^3}{\mu}\Big)^{\frac{1}{2}}\Big\{t_n-t_0+\frac{t_n^3-t_0^3}{3}\Big\} \\ &= \Big(\frac{2a^3}{\mu}\Big)^{\frac{1}{2}}(t_n-t_0)\Big(1+\frac{t_n^2+t_nt_0+t_0^2}{3}\Big). \end{aligned}$$

By a substitution due to Gauss, let

$$1+\frac{(t_n+t_0)^2}{4} = \eta^2;$$

$$\therefore \quad t = \left(\frac{2a^3}{\mu}\right)^{\frac{1}{2}}(t_n-t_0)\left(\eta^2+\frac{(t_n-t_0)^2}{12}\right)$$

$$= \frac{1}{3}\left(\frac{2a^3}{\mu}\right)^{\frac{1}{2}}\left\{\left(\eta+\frac{t_n-t_0}{2}\right)^3-\left(\eta-\frac{t_n-t_0}{2}\right)^3\right\}. \qquad (34)$$

But if c is the chord joining the extremities of r_n and r_0,

$$c^2 = r_n{}^2-2r_nr_0\cos(\theta_n-\theta_0)+r_0{}^2$$

$$= (r_n\cos\theta_n-r_0\cos\theta_0)^2+(r_n\sin\theta_n-r_0\sin\theta_0)^2.$$

Also $\qquad r_n = a(1+t_n{}^2), \qquad r_0 = a(1+t_0{}^2),$

and $\qquad \cos\theta_n = \dfrac{1-t_n{}^2}{1+t_n{}^2}, \qquad \sin\theta_n = \dfrac{2t_n}{1+t_n{}^2};$

and similar values are true for $\cos\theta_0$ and $\sin\theta_0$; therefore

$$c^2 = 4a^2(t_n-t_0)^2\left\{\frac{(t_n+t_0)^2}{4}+1\right\}$$

$$= 4a^2(t_n-t_0)^2\eta^2;$$

$$\therefore \quad c = 2a(t_n-t_0)\eta;$$

$$\therefore \quad r_n+r_0+c = 2a\left\{1+\frac{t_n{}^2+t_0{}^2}{2}+(t_n-t_0)\eta\right\}$$

$$= 2a\left\{\eta+\frac{t_n-t_0}{2}\right\}^2;$$

similarly, $\qquad r_n+r_0-c = 2a\left\{\eta-\dfrac{t_n-t_0}{2}\right\}^2;$

therefore substituting in (34),

$$t = \frac{1}{6\mu^{\frac{1}{2}}}\left\{(r_n+r_0+c)^{\frac{3}{2}}-(r_n+r_0-c)^{\frac{3}{2}}\right\}. \qquad (35)$$

This theorem is generally known by the name of Lambert's Theorem.

Hence the time through an arc of a parabolic orbit bounded by a focal chord varies as (the chord)$^{\frac{3}{2}}$.

Ex. 3. If the equation of a hyperbola, of which the focus is the pole, is

$$r = \frac{a(e^2-1)}{1+e\cos\theta};$$

then $\qquad \mathrm{P} = \dfrac{h^2}{a(e^2-1)}\dfrac{1}{r^2}; \qquad (\text{velocity})^2 = \dfrac{\mu(2au+1)}{a}.$

382.] Ex. 4. A particle moves in an ellipse about a centre of force in the centre: it is required to find the law of force, the velocity at any point of the orbit, and the periodic time.

The equation to the ellipse is

$$\frac{(\cos\theta)^2}{a^2}+\frac{(\sin\theta)^2}{b^2} = u^2; \qquad (36)$$

$$\therefore \quad u\frac{du}{d\theta} = \Big(\frac{1}{b^2} - \frac{1}{a^2}\Big)\sin\theta\cos\theta; \tag{37}$$

$$u\frac{d^2u}{d\theta^2} + \frac{du^2}{d\theta^2} = \Big(\frac{1}{b^2} - \frac{1}{a^2}\Big)\{(\cos\theta)^2-(\sin\theta)^2\}. \tag{38}$$

But from (36),

$$(\cos\theta)^2 = \frac{u^2 - \frac{1}{b^2}}{\frac{1}{a^2} - \frac{1}{b^2}}, \qquad (\sin\theta)^2 = \frac{u^2 - \frac{1}{a^2}}{\frac{1}{b^2} - \frac{1}{a^2}};$$

therefore substituting these values in (38), and also from (37), we shall find

$$\text{P} = \frac{h^2}{a^2b^2u} = \frac{h^2}{a^2b^2}r; \tag{39}$$

thus the force varies directly as the distance, and is attractive: and if $\mu =$ the absolute force, $h^2 = \mu a^2 b^2$. Also

$$\begin{aligned}(\text{the velocity})^2 &= \mu(a^2+b^2-r^2) \qquad (40)\\ &= \mu r'^2,\end{aligned}$$

if r' is the radius conjugate to r.

And if $\text{T} =$ the periodic time, by (13),

$$\text{T} = \frac{2\pi ab}{h} = \frac{2\pi}{\mu^{\frac{1}{2}}}; \tag{41}$$

that is, the periodic time is independent of the magnitude of the ellipse, and depends only on the absolute central force.

And the time in which the particle passes through an arc which subtends a given angle at the centre may thus be found. Let the arc begin at the extremity of the major axis; then if $t =$ the time required, from (12) we have

$$\begin{aligned}t &= \frac{ab}{\mu^{\frac{1}{2}}}\int_0^{\theta}\frac{d\theta}{a^2(\sin\theta)^2+b^2(\cos\theta)^2}\\ &= \frac{1}{\mu^{\frac{1}{2}}}\tan^{-1}\Big(\frac{a\tan\theta}{b}\Big); \qquad (42)\end{aligned}$$

and thus if $\theta = \frac{\pi}{2}$, $t = \frac{\pi}{2\mu^{\frac{1}{2}}}$; where t is one-fourth of the periodic time: and thus the whole time is the same as that given in (41).

Ex. 5. In the hyperbola described by a particle under the action of a central force in its centre, and of which the equation is

$$\frac{(\cos\theta)^2}{a^2} - \frac{(\sin\theta)^2}{b^2} = u^2,$$

$$\text{P} = -\frac{h^2}{a^2b^2}r = -\mu r; \tag{43}$$

that is, the central force varies directly as the distance, and is repulsive: also

$$(\text{velocity})^2 = \mu(r^2 - a^2 + b^2); \qquad (44)$$

and the time from the extremity of the transverse axis

$$= \frac{1}{2\mu^{\frac{1}{2}}} \log \frac{b + a\tan\theta}{b - a\tan\theta}. \qquad (45)$$

383.] Ex. 6. Let the particle move in a circle; and

(1) Let the centre of force be in the centre: let a = the radius: then the equation to the circle is

$$r = a;$$

$$\therefore \quad u = \frac{1}{a}; \qquad \frac{du}{d\theta} = \frac{d^2u}{d\theta^2} = 0; \qquad \therefore \quad \text{P} = \frac{h^2}{a^3};$$

$$(\text{the velocity})^2 = \frac{h^2}{a^2} = \text{V}^2 \text{ (say)};$$

therefore the central force is constant, but varies inversely as the cube of the radius as we pass from one circle to another: also the velocity is constant, and varies inversely as the radius: and if T = the periodic time,

$$\text{T} = \frac{2\pi a}{\text{V}} = \frac{2\pi a^2}{h}.$$

Also $\text{P} = \frac{\text{V}^2}{a}$; now as V^2 is the velocity and a is the radius of the circle, $\frac{\text{V}^2}{a}$, see Art. 326, is the centrifugal force: the central force is therefore equal to the centrifugal force in the circle; the central force, that is, draws the particle towards the centre over a space equal to that by which the centrifugal force (so to speak) removes it from the centre: and as the velocity is constant, no part of the central force acts either to increase or to diminish the velocity in the circular path.

(2) Let the centre of force be in the circumference of the circle: and let the equation be

$$r = 2a\cos\theta; \qquad \therefore \quad 2au = \sec\theta;$$

$$2a\frac{du}{d\theta} = \sec\theta\tan\theta; \qquad 2a\frac{d^2u}{d\theta^2} = \sec\theta(\tan\theta)^2 + (\sec\theta)^3;$$

$$\therefore \quad \frac{d^2u}{d\theta^2} = 8a^2u^3 - u;$$

$$\therefore \quad \text{P} = 8a^2h^2u^5 = \frac{8a^2h^2}{r^5}; \qquad (46)$$

that is, the force varies inversely as the fifth power of the distance, and is attractive; and if $\mu =$ the absolute force, $\mu = 8a^2 h^2$;

$$\therefore \quad h^2 = \frac{\mu}{8a^2}; \text{ and (the velocity)}^2 = \frac{\mu}{2r^4}.$$

And if T = the periodic time,

$$\text{T} = \frac{2^{\frac{5}{2}} \pi a^3}{\mu^{\frac{1}{2}}}. \tag{47}$$

And if t is the time of the motion of the particle from the extremity of the diameter to the point corresponding to θ, then

$$t = 4a^3 \left(\frac{2}{\mu}\right)^{\frac{1}{2}} \left(\theta + \frac{\sin 2\theta}{2}\right).$$

(3) Let the centre of force be at any point within or without the circle: and suppose it to be at S, see fig. 126; and let C be the centre of the circle, $\text{SC} = c$, $\text{CA} = a$, $\text{SP} = r$, $\text{SY} = p$: then because $\text{SC}^2 = \text{SP}^2 - 2\,\text{SP}.\text{CP} \cos \text{SPC} + \text{CP}^2$;

$$\therefore \quad c^2 = r^2 - 2ap + a^2; \tag{48}$$

$$\therefore \quad \frac{dr}{dp} = \frac{a}{r};$$

therefore, from (20),

$$\text{P} = \frac{8a^2 h^2 r}{(r^2 + a^2 - c^2)^3}. \tag{49}$$

384.] Ex. 7. It is required to find the law of force, the velocity, and the periodic time in (1) the Lemniscata of Bernoulli, (2) the Cardioid.

(1) $r^2 = a^2 \cos 2\theta$;

$$\therefore \quad a^2 u^2 = \sec 2\theta;$$

$$a^2 u \frac{du}{d\theta} = \sec 2\theta \tan 2\theta = a^2 u^2 \tan 2\theta;$$

$$\therefore \quad \frac{du}{d\theta} = u \tan 2\theta;$$

$$\frac{d^2 u}{d\theta^2} = \frac{du}{d\theta} \tan 2\theta + 2u (\sec 2\theta)^2;$$

$$\therefore \quad \text{P} = \frac{3h^2 a^4}{r^7}; \tag{50}$$

and thus the force varies inversely as the seventh power of the distance, and is attractive: and if $\mu =$ the absolute force, $\mu = 3h^2 a^4$; and thus

$$\text{(the velocity)}^2 = \frac{\mu}{3r^6}.$$

If $t =$ the time from an apse, then from (12),

$$t = \left(\frac{3}{\mu}\right)^{\frac{1}{2}} a^4 \int_0^\theta \cos 2\theta \, d\theta$$

$$= \left(\frac{3}{\mu}\right)^{\frac{1}{2}} \frac{a^4}{2} \sin 2\theta; \qquad (51)$$

and therefore the time to the node $= \left(\frac{3}{\mu}\right)^{\frac{1}{2}} \frac{a^4}{2}$; and the time of describing one loop $= \left(\frac{3}{\mu}\right)^{\frac{1}{2}} a^4$.

(2) The equation to the Cardioid is

$$r = a(1+\cos\theta) = 2a\left(\cos\frac{\theta}{2}\right)^2;$$

$$\therefore \quad 2au = \left(\sec\frac{\theta}{2}\right)^2;$$

$$2a\frac{du}{d\theta} = \left(\sec\frac{\theta}{2}\right)^2 \tan\frac{\theta}{2} = 2au\tan\frac{\theta}{2};$$

$$\therefore \quad \frac{du}{d\theta} = u\tan\frac{\theta}{2};$$

$$\frac{d^2u}{d\theta^2} = u\left(\tan\frac{\theta}{2}\right)^2 + \frac{u}{2}\left(\sec\frac{\theta}{2}\right)^2 = 3au^2 - u;$$

$$\therefore \quad \text{P} = \frac{3ah^2}{r^4}; \qquad (52)$$

and thus the force is attractive and varies inversely as the fourth power of the distance; and if μ is the absolute force, $\mu = 3ah^2$; and

$$(\text{the velocity})^2 = \frac{2\mu}{3r^3}.$$

If $t =$ the time from the apse to the point corresponding to $\theta = \theta$, then

$$t = \frac{\mu^{\frac{1}{2}} a^{\frac{5}{2}}}{3^{\frac{1}{2}}} \int_0^\theta (1+\cos\theta)^2 \, d\theta$$

$$= \frac{\mu^{\frac{1}{2}} a^{\frac{5}{2}}}{3^{\frac{1}{2}}} \left\{ \frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4} \right\}; \qquad (53)$$

and thus if $\theta = \pi$, the time from the apse to the pole $= \frac{(3\mu a^5)^{\frac{1}{2}}}{2}\pi$; and if $\text{T} =$ the periodic time,

$$\text{T} = (3\mu a^5)^{\frac{1}{2}} \pi.$$

385.] It is required to find the law of force in (1) the curve whose equation is $r = 2a\cos n\theta$, (2) the curve whose equation is $r = \frac{2a}{1 - e\cos n\theta}$.

(1) The equation of the first curve in terms of u is

$$2au = \sec n\theta;$$

$$\therefore\quad 2a\frac{du}{d\theta} = n\sec n\theta\tan n\theta;$$

$$2a\frac{d^2u}{d\theta^2} = n^2\sec n\theta(\tan n\theta)^2 + n^2(\sec n\theta)^3$$

$$= 2aun^2(8a^2u^2-1);$$

$$\therefore\quad \text{P} = \frac{8a^2h^2n^2}{r^5} + \frac{(1-n^2)h^2}{r^3}; \qquad (54)$$

and thus the force varies partly as the inverse fifth power, and partly as the inverse cube, of the distance.

And if $t =$ the time in which the particle moves from the point corresponding to $\theta = 0$, to that corresponding to $\theta = \theta$,

$$t = \frac{4a^2}{h}\int_0^\theta (\cos n\theta)^2\,d\theta$$

$$= \frac{2a^2}{h}\left(\theta + \frac{\sin 2n\theta}{2n}\right).$$

As to (54) it is to be observed that the second term of the equivalent of P disappears, if $n = 1$; and that in this case the central force varies as the inverse fifth power of the distance. Now if $n = 1$, the equation to the orbit is $r = 2a\cos\theta$; that is, the orbit is a circle, of which the pole is on the circumference and the prime radius passes through the centre: and when $\theta = 0$, $r = 2a$ in both the circle and the given curve. A process of tracing such a curve, and of representing the motion of a particle on it, is hereby suggested to us. In fig. 127 take a line SX for a prime radius; on it take SCA $= 2a$; and on SA as a diameter describe a semicircle SQA: then since $r = 2a$, when $\theta = 0$, in the equation to the orbit, the point A is common to the circle and to the orbit: let us suppose n to be less than unity: and let AP be the curve of the orbit: on it take any point P; join SP: then as PSX $= \theta$, SP $= 2a\cos n.$PSX; let the angle PSA′ $= n.$PSX, and make SA′ $=$ SA $= 2a$; on SA′ as a diameter describe a semicircle; then the semicircle will pass through P, because by the property of the semicircle SP $=$ SA′cos PSA′ $= 2a\cos n\theta$, and thus SP is the same for both the semicircle and the curve of the orbit. In the same manner may every point P be shewn to be on a semicircle, the diameter of which has a varying position, and as PSA $= \theta$, PSA′ $= n\theta$, therefore ASA′ $= (1-n)\theta$: therefore while P has moved over an arc of the orbit subtending an angle θ at S, SA′ has moved through an angle $(1-n)\theta$: and therefore the ratio of the angular velocity of P, to that of the revolving diameter, is as

$1 : 1-n$; and, according to our system, SA′ revolves in the same direction as P moves. If however n is greater than 1, SA′ revolves in a direction the opposite of that in which SP revolves. In either case the moving particle m may be represented as moving in a circle, the diameter of which revolves about the pole S with an angular velocity bearing a constant ratio to the angular velocity of m; and the orbit of m is for this reason called *a revolving circle*.

(2) Again in the second curve which is given,

$$2au = 1 - e\cos n\theta;$$

$$2a\frac{d^2u}{d\theta^2} = en^2\cos n\theta;$$

$$\therefore \quad P = \frac{h^2n^2}{2a}\,\frac{1}{r^2} + \frac{(1-n^2)h^2}{r^3}; \tag{55}$$

therefore the central force is compounded of two different forces, of which one varies inversely as the square of the distance, and the other varies inversely as the cube of the distance.

Now the second term of the right-hand member of (55) disappears if $n = 1$, and in that case the central force varies inversely as the square of the distance; but if $n = 1$, the equation to the orbit becomes

$$r = \frac{2a}{1-e\cos\theta}; \tag{56}$$

the equation to a conic, of which the focus is the pole, and the prime radius is the principal axis: and to fix our thoughts I will suppose e less than unity, so that the conic is an ellipse: now if $\theta = 0$, $r = \frac{2a}{1-e}$, in both the conic and the orbit curve; so that if SX, fig. 128, is a prime radius, and the line SA is taken on it such that SA $= \frac{2a}{1-e}$, then A is a point common to the conic and the orbit. And to fix our thoughts, let us also suppose n to be less than unity; and let us suppose the curve APQ to be that of the orbit, of which P is any point, and PSX $= \theta$: take an angle PSA′ $= n\theta$, and let SA′ $=$ SA, and produce A′S to B′, so that SB′ $= \frac{2a}{1+e}$; and on A′B′ as a major axis, with S as the focus, describe an ellipse; then the radius-vector of the ellipse corresponding to the angle PSA′

$$= \frac{2a}{1-e\cos \text{PSA}'} = \frac{2a}{1-e\cos n\theta};$$

therefore the radius-vector of the ellipse is equal to that of the orbit, and therefore the point P which is on the orbit is also on the ellipse. In a similar way it may be shewn that every point on the orbit is on an ellipse, the major axis of which has a varying position: and as PSA $= \theta$, PSA$' = n\theta$, therefore A$'$SA $= (1-n)\theta$; that is, as m has passed over an arc subtending an angle θ at S, so has the major axis of the ellipse passed through an angle $(1-n)\theta$; the angular velocity therefore of the former is to that of the latter as $1 : 1-n$: and SA$'$ revolves in a direction the same as that of m if n is less than 1, but in an opposite direction if n is greater than 1. In either case the moving particle may be represented as moving in an ellipse, the major axis of which revolves about the centre of force with an angular velocity bearing a constant ratio to that of the moving particle: the orbit of m is for this reason called *a revolving ellipse.*

Also since in the orbit

$$u = \frac{1-e\cos n\theta}{2a}; \qquad \therefore \quad \frac{du}{d\theta} = \frac{ne\sin n\theta}{2a};$$

now if $\frac{du}{d\theta} = 0$, the corresponding point in the orbit is an apse, and the line drawn from the pole to an apse is the apsidal distance: therefore the orbit has an apse, whenever $\sin n\theta = 0$: that is, when

$$\theta = 0, \qquad \theta = \frac{\pi}{n}, \qquad \theta = \frac{2\pi}{n}, \qquad \dots;$$

therefore the angle between two successive apsidal distances $= \frac{\pi}{n}$.

386.] By processes similar to those employed above let it be shewn that in the orbits whose equations are the following, viz.

$$(1) \quad r = \frac{2a}{e^{n\theta} \pm e^{-n\theta}}; \qquad (2) \quad r = a^{\theta};$$

$$(3) \quad r = \frac{a}{\theta}; \qquad (4) \quad r = a\sec n\theta;$$

the central force varies inversely as the cube of the distance: that in the lituus, whose equation is $a^2u^2 = \theta$, the force varies partly directly as the distance, and partly inversely as the cube of the distance: and that in the involute of the circle, of which the equation is $r^2 = a^2 + p^2$,

$$\mathrm{P} = \frac{h^2 r}{(r^2-a^2)^2}.$$

387.] As an accurate comprehension of the expressions for P, the central force, is of great importance for the complete understanding of the phaenomena of the action of a central force, it is desirable to insert the following proof which is founded on first principles:

Let m be the mass of a particle moving in a certain orbit under the action of a central force, the impressed velocity-increment of which along the radius-vector is represented by P: let P, fig. 129, be the place of m at the time t, and let $\text{PQ} = ds$ be the length-element described by m in the time dt; and let QS be the length-element described in the succeeding dt. Now if no central force acted, m would in dt pass through QR in the line PQ produced, instead of passing through QS: but if a force, whose source is in O, acts on m at Q, and causes it to pass through QV in the line QO in the time dt; then at the end of the second dt, m is at S; QS being the diagonal of the parallelogram of which QV and QR are two containing and adjacent sides; it is our purpose to estimate the effect of the central force as expressed in the deflexion of m from its rectilinear path.

We may consider the central force P to be constant during the time dt of its action on m, whereby it draws m over the space QV; and therefore by (31), Art. 268,

$$2.\text{QV} = \text{P}\,dt^2. \tag{57}$$

Let QΠ be the radius of curvature of the curve at Q, and let QN be the projection of QV on it: therefore, as it has been proved in Art. 303, if $\rho = \text{Q}\Pi$,

$$ds^2 = 2\rho \times \text{QN}.$$

Now
$$\begin{aligned}\text{QV} &= \text{QN} \sec \text{OQ}\Pi \\ &= \frac{ds^2}{2\rho}\,\frac{r}{p} = \frac{ds^2\,dp}{2\,p\,dr};\end{aligned}$$

therefore from (57),
$$\text{P} = \frac{ds^2}{dt^2}\,\frac{dp}{p\,dr} = \frac{h^2}{p^2}\,\frac{dp}{dr}, \tag{58}$$

because by reason of (7), $\dfrac{ds^2}{dt^2} = \dfrac{h^2}{p^2}$: and this expression is the same as that before deduced analytically in Art. 380. The preceding process is nearly identical with that employed by Newton in Prop. VI, Section 2, Book I, of the Principia.

SECTION 2.—*The determination of orbits, and of their dimensions and position, when the laws of central force and other circumstances of motion are given.*

388.] In the previous Section the law of force and other circumstances of motion have been determined, when the equation to the orbit and the position of the centre of force have been given: it is our purpose now to inquire into the converse problem; and let it be observed, that for a complete determination of the orbit, when the law and centre of force are given, four constants, or what are, by means of the limits of the integral or otherwise, equivalent to four constants, are required: this is evident from the form of the differential equations (4), which are two simultaneous differential equations of the second order, and the complete integral of each of which requires two constants: or again the equation (21) contains an undetermined constant h; and being of the second order, two more undetermined constants will be introduced during the process of integration: and one other constant will be required in the integral of (12), by means of which the time at which m is at any point of the orbit may be found. The conditions which will for the most part be given in the following examples are, (1) the distance from the centre of force of the point where m is at a given time; (2) the line in which m is moving at the time, and the inclination of that line to the corresponding radius-vector; and (3) the velocity with which m is moving at the given time: the time at which all these circumstances are given is called the epoch; and in terms of them, the constants, or the limits of the integrals, can always be expressed.

389.] A particle m is projected with a given velocity, in a given line, from a given point, and moves under the action of a central force, which varies directly as the distance and is attractive: it is required to determine the equation to the orbit, and the circumstances of motion.

The plane in which m moves is manifestly that passing through the centre of force and the point of projection, and which contains the line of projection.

Let the centre of force be the origin; let $\mathrm{R} =$ the distance of the point of projection from the centre of force; $\mathrm{V} =$ the velocity of projection; and let us suppose m to be projected from an apse, so that the line of V is perpendicular to R: then since generally the velocity $= \frac{h}{p}$,

$$\mathrm{V} = \frac{h}{\mathrm{R}}; \qquad \therefore \quad h = \mathrm{VR}. \tag{59}$$

Let $t=0$, when m is projected with the velocity V: let $\mu =$ the absolute force of the central force: then since the force varies directly as the distance and is attractive,

$$\mathrm{P} = \mu r = \frac{\mu}{u}. \tag{60}$$

Now from (21) we have

$$\mathrm{P} = h^2 u^2 \left\{ \frac{d^2 u}{d\theta^2} + u \right\};$$

therefore in this case $\quad \dfrac{d^2 u}{d\theta^2} + u = \dfrac{\mu}{h^2 u^3};$ $\hfill$ (61)

multiplying by $2du$, and integrating, and taking the limits corresponding to $t=t$ and to $t=0$, and observing that

$$(\text{the velocity})^2 = h^2 \left(u^2 + \frac{du^2}{d\theta^2}\right),$$

we have $\quad \dfrac{du^2}{d\theta^2} + u^2 - \dfrac{\mathrm{V}^2}{h^2} = -\dfrac{\mu}{h^2 u^2} + \dfrac{\mu \mathrm{R}^2}{h^2};$ $\hfill$ (62)

$$\begin{aligned} \therefore \quad (\text{the velocity})^2 &= \mathrm{V}^2 + \mu \mathrm{R}^2 - \frac{\mu}{u^2} \\ &= \mathrm{V}^2 + \mu(\mathrm{R}^2 - r^2); \end{aligned} \tag{63}$$

and therefore the velocity is the greatest and least, according as r is the least or the greatest.

And replacing h in (62) by its value from (59), we have

$$\frac{du^2}{d\theta^2} + u^2 - \frac{1}{\mathrm{R}^2} = -\frac{\mu}{\mathrm{V}^2 \mathrm{R}^2 u^2} + \frac{\mu}{\mathrm{V}^2};$$

$$\therefore \quad u^2 \frac{du^2}{d\theta^2} + \left(u^2 - \frac{\mathrm{V}^2 + \mu \mathrm{R}^2}{2\mathrm{V}^2 \mathrm{R}^2}\right)^2 = \left(\frac{\mathrm{V}^2 - \mu \mathrm{R}^2}{2\mathrm{V}^2 \mathrm{R}^2}\right)^2;$$

$$\frac{2u\,du}{\left\{\left(\dfrac{\mathrm{V}^2 - \mu \mathrm{R}^2}{2\mathrm{V}^2 \mathrm{R}^2}\right)^2 - \left(u^2 - \dfrac{\mathrm{V}^2 + \mu \mathrm{R}^2}{2\mathrm{V}^2 \mathrm{R}^2}\right)^2\right\}^{\frac{1}{2}}} = 2\,d\theta;$$

therefore integrating, and taking the limits corresponding to $t=t$ and to $t=0$, and assuming that the prime radius coincides with R, we have

$$\sin^{-1} \frac{2\mathrm{V}^2 \mathrm{R}^2 u^2 - (\mathrm{V}^2 + \mu \mathrm{R}^2)}{\mathrm{V}^2 - \mu \mathrm{R}^2} - \sin^{-1} 1 = 2\theta;$$

$$\therefore \quad 2\mathrm{V}^2 \mathrm{R}^2 u^2 - (\mathrm{V}^2 + \mu \mathrm{R}^2) = (\mathrm{V}^2 - \mu \mathrm{R}^2) \cos 2\theta; \tag{64}$$

and changing to rectangular coordinates, we have

$$2\text{V}^2\text{R}^2-(\text{V}^2+\mu\text{R}^2)(x^2+y^2) = (\text{V}^2-\mu\text{R}^2)(x^2-y^2);$$

$$\therefore \quad \frac{x^2}{\text{R}^2}+\frac{\mu y^2}{\text{V}^2} = 1; \tag{65}$$

the equation to an ellipse, of which the centre is the origin; $\text{R} =$ the semi-axis parallel to the axis of x, $\frac{\text{V}}{\mu^{\frac{1}{2}}} =$ the semi-axis parallel to the axis of y; and which is the orbit in which m moves.

From (63) it appears that the velocity $= \text{V}$, when $r = \text{R}$; and $= \text{R}\mu^{\frac{1}{2}}$, when $r = \frac{\text{V}}{\mu^{\frac{1}{2}}}$: these therefore are the velocities at the extremities of the principal axes of the ellipse.

The point of projection is the extremity of the major or minor axes of the ellipse according as R^2 is greater or less than $\frac{\text{V}^2}{\mu}$; or, in other words, as

$$\text{V}^2 \text{ is less than, or greater than, } \mu\text{R}^2. \tag{66}$$

Now suppose m to be placed at a point at a distance R from the centre of force and to move in a rectilinear course towards it, then, as appears from Art. 279, the velocity of m when it arrives at the centre is $\text{R}\mu^{\frac{1}{2}}$; and therefore as the velocity of projection is greater than or less than this, so is the point of projection the extremity of the minor or the major axis.

And if the velocity of projection is equal to that which would be acquired by m moving in the rectilinear path into the centre, then

$$\text{V}^2 = \mu\text{R}^2, \tag{67}$$

and the orbit is a circle.

The periodic time is, by Art. 382, (41), equal to $\frac{2\pi}{\mu^{\frac{1}{2}}}$, and is independent of the magnitude of the ellipse. And if $t =$ the time from the point of projection to the point on the orbit corresponding to θ, then as in (42),

$$t = \frac{1}{\mu^{\frac{1}{2}}}\tan^{-1}\left(\frac{\text{R}\mu^{\frac{1}{2}}}{\text{V}}\tan\theta\right); \tag{68}$$

and thus the circumstances of the orbit are completely determined.

I may observe that if the central force is repulsive, the sign of μ will be changed throughout the preceding investigations;

and that the orbit will be a hyperbola referred to the centre as pole, of which the equation will be

$$\frac{x^2}{R^2} - \frac{\mu y^2}{V^2} = 1; \tag{69}$$

and

$$(\text{the velocity})^2 = V^2 + \mu(r^2 - R^2); \tag{70}$$

$$t = \frac{1}{2\mu^{\frac{1}{2}}} \log \frac{V + R\mu^{\frac{1}{2}} \tan\theta}{V - R\mu^{\frac{1}{2}} \tan\theta}. \tag{71}$$

390.] In Article 280, (2), it is said that according to the principles of the undulatory theory of light, the force acting on a displaced molecule of ether, and bringing it back to its original position of rest, varies directly as the distance through which the molecule has been displaced; and it is also said that *generally* the line of motion of the particle, when it is brought within the action of this force, is not in the line joining the particle and its original position of rest. Here then are the circumstances required in the preceding Article; a particle moves in a given line, with a given velocity, and is acted on by an attractive force varying directly as the distance: its orbit therefore is an ellipse; and the periodic time in the ellipse is independent of the magnitude of it, and depends only on the absolute force in the centre; therefore the ethereal molecules move in ellipses which are in planes perpendicular to the line of propagation of a ray, and if the absolute force of the central force is the same for all the ellipses the periodic time is the same, but if it varies, the periodic time varies inversely as its square root. Now in this case, as in that of rectilinear motion, the intensity of light is supposed to depend on the magnitude of the ellipse, and the colour of light on the periodic time of the orbit: hence it appears that the variations of the intensity and of the colour are independent of each other; and this fact is in accordance with observation.

Hence also we have a physical explanation of other kinds of polarised light: if all the major axes (say) of the ellipses of ethereal motion are parallel to each other, the light is said to be elliptically-polarised: if the ellipses become circles, we have circularly-polarised light: if the azimuth of the major axes of the ellipses rotates uniformly, we have another modification of the ethereal vibrations. The further investigations however of such changes must be sought for in treatises where the phaenomena of light are specially inquired into.

391.] A particle m is projected with a given velocity from a given point and in a given line; and moves under the action of a central attractive force, which varies inversely as the square of the distance: it is required to determine the equation to the orbit, and the other circumstances of motion.

The plane in which m moves is manifestly that which contains the point of projection, the centre of force, and the line in which m is projected. Let the centre of force be the origin; $\text{V} =$ the velocity of projection; $\text{R} =$ the distance of the point of projection from the origin; $a =$ the angle between R and the line of projection; then since generally the velocity $= \frac{h}{p}$, and at the point of projection $p = \text{R} \sin a$;

$$\therefore \quad \text{V} = \frac{h}{\text{R} \sin a}; \quad \text{and} \quad h = \text{VR} \sin a; \tag{72}$$

and let $t = 0$, when m is projected from the given point.

Let $\mu =$ the absolute force of the central force: then since the central force P varies inversely as the square of the distance, and is attractive,

$$\text{P} = \frac{\mu}{r^2} = \mu u^2; \tag{73}$$

so that (21) becomes
$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2}. \tag{74}$$

Multiplying through by $2\,du$, integrating, taking the limits corresponding to $t = t$ and to $t = 0$, and observing that

$$(\text{the velocity})^2 = h^2 \left(u^2 + \frac{du^2}{d\theta^2}\right),$$

we have
$$\frac{du^2}{d\theta^2} + u^2 - \frac{\text{V}^2}{h^2} = \frac{2\mu u}{h^2} - \frac{2\mu}{h^2 \text{R}}; \tag{75}$$

therefore $(\text{the velocity})^2 = \text{V}^2 + 2\mu u - \frac{2\mu}{\text{R}}$

$$= \text{V}^2 + 2\mu\left(\frac{1}{r} - \frac{1}{\text{R}}\right); \tag{76}$$

and thus the velocity is the greatest or least according as r is the least or the greatest.

In (75) replacing h by its value in (72) we have

$$\frac{du^2}{d\theta^2} + u^2 - \frac{1}{(\text{R} \sin a)^2} = \frac{2\mu u}{(\text{RV} \sin a)^2} - \frac{2\mu}{\text{R}^3 (\text{V} \sin a)^2}. \tag{77}$$

To express this in a simpler form, let

$$\frac{\mu}{(\text{RV} \sin a)^2} = b, \qquad \frac{\text{V}^2 \text{R} - 2\mu}{\text{R}^3 \text{V}^2 (\sin a)^2} + \frac{\mu^2}{(\text{RV} \sin a)^4} = c^2; \tag{78}$$

so that (77) becomes

$$\frac{du^2}{d\theta^2} = c^2-(u-b)^2;$$

$$\therefore \quad \frac{-du}{\{c^2-(u-b)^2\}^{\frac{1}{2}}} = d\theta, \tag{79}$$

the negative sign being taken in the extraction of the square root, because we will assume that r and θ simultaneously increase and decrease. Therefore integrating, and leaving the position of the prime radius undetermined, so that it may be determined by means of subsequent considerations, and thus introducing an arbitrary constant γ, we have

$$\cos^{-1}\frac{u-b}{c} = \theta-\gamma; \tag{80}$$

$$\therefore \quad u = b+c\cos(\theta-\gamma); \tag{81}$$

and restoring the equivalents of b and c,

$$u = \frac{\mu}{(\mathrm{R}\mathrm{V}\sin a)^2} + \left\{\frac{\mathrm{V}^2\mathrm{R}-2\mu}{\mathrm{R}^2\mathrm{V}^2(\sin a)^2} + \frac{\mu^2}{(\mathrm{R}\mathrm{V}\sin a)^4}\right\}^{\frac{1}{2}}\cos(\theta-\gamma); \tag{82}$$

which is the equation to a conic, of which the focus is the pole. For if e is the eccentricity of a conic, r is the focal radius-vector, and ϕ = the angle between r and the principal axis, and measured, say, from that point of a conic which is nearest to the focus,

$$\frac{1}{r} = u = \frac{1+e\cos\phi}{2k}; \tag{83}$$

and comparing this with (82), it appears that

$$(1) \quad e^2 = 1+\frac{(\mathrm{V}^2\mathrm{R}-2\mu)\,\mathrm{R}\mathrm{V}^2(\sin a)^2}{\mu^2}; \tag{84}$$

$$(2) \quad \theta-\gamma = \phi. \tag{85}$$

Now from (84) e is less than, equal to, or greater than, unity according as $\mathrm{V}^2\mathrm{R}-2\mu$ is negative, zero, or positive; the orbit therefore is an ellipse, a parabola, or a hyperbola, with the centre of force at the focus, according as V^2 is less than, equal to, or greater than, $\frac{2\mu}{\mathrm{R}}$. The interpretation of this discriminating condition is as follows:

Suppose a particle to move from rest at an infinite distance in a rectilinear path towards the centre of force which varies as the inverse square of the distance, then

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}, \tag{86}$$

where x = the distance of the particle from the centre of force at the time t. Multiplying both sides of the equation by $2dx$, and integrating, we have

$$\frac{dx^2}{dt^2} = \frac{2\mu}{x}; \tag{87}$$

and therefore the square of the velocity at a distance R from the centre $= \frac{2\mu}{\text{R}}$. Thus it appears that according as the velocity with which the particle is projected at a distance R from the centre of force is less than, equal to, or greater than, that which would be acquired by the particle moving from an infinite distance to that point under the action of the central force, so will the orbit be an ellipse, a parabola, or a hyperbola, with the centre of force in the focus. The species of conic therefore does not depend on the position of the line in which the particle is projected, but on the velocity of projection in reference to the distance of the point of projection from the centre of force. I may also observe that, by reason of (17), according as the velocity of projection is less than, equal to, or greater than, that from an infinite distance at the point of projection, so will it be at all points of the orbit. Thus if a particle moves in a parabola with the centre of force in the focus, the velocity at every point of the orbit is equal to that which would be acquired by the particle moving in a rectilinear path from an infinite distance under the action of the central force to the point on the orbit.

And with respect to γ, the undetermined constant which is introduced at the integration of (79); from (85) it appears that $\theta-\gamma$ is the angle between the focal radius-vector r and that part of the principal axis which is between the focus and the point of the orbit which is nearest to the focus; therefore γ is the angle between the prime radius and that part of the principal axis; and therefore if $\gamma = 0$, the principal axis is the prime radius. For the present I shall suppose this to be the case; and I shall consider each of the three conics separately.

392.] Let us in the first place consider the ellipse in which V^2 is less than $\frac{2\mu}{\text{R}}$; so that by reason of (84), if $e =$ the eccentricity,

$$e^2 = 1 - \frac{(2\mu - \text{V}^2\text{R})\,\text{R}\,\text{V}^2 (\sin a)^2}{\mu^2}. \tag{88}$$

Now the equation of an ellipse, where $r =$ the focal radius-vector, θ is measured from the shorter segment of the major axis, $2a =$ the major axis, $e =$ the eccentricity, is

$$r = \frac{a(1-e^2)}{1+e\cos\theta};$$

$$\therefore \quad u = \frac{1}{a(1-e^2)} + \frac{e}{a(1-e^2)}\cos\theta; \tag{89}$$

comparing which with (82), we have e as in (88), and

$$a = \frac{\mu \mathrm{R}}{2\mu - \mathrm{V}^2 \mathrm{R}}; \tag{90}$$

and thus the major axis of the ellipse is also independent of the angle between the line in which the particle is projected and the line joining the point of projection and the centre of force.

Let $\beta =$ the angle between the major axis of the ellipse and R; then from (82), $u = \frac{1}{\mathrm{R}}$, when $\theta - \gamma = \beta$, and we have

$$\sin\beta = \frac{\mathrm{R}\mathrm{V}^2 \sin\alpha\cos\alpha}{\{\mu^2 - (2\mu - \mathrm{V}^2\mathrm{R})\mathrm{R}\mathrm{V}^2(\sin\alpha)^2\}^{\frac{1}{2}}}; \tag{91}$$

which equation determines the position of the major axis of the ellipse with reference to the given line R.

Thus the position and the dimensions of the elliptic orbit are completely determined, on the supposition that the initial circumstances are given. Fig. 130 explains the several quantities which we have introduced. Let B be the point of projection; SB $=$ R; BY the line along which the particle is projected with the velocity V; SBY$=\alpha$, the angle of projection; BSA$=\beta$; SP$=r$; PSA$=\theta$; SY$=$SB sin SBY$=$R sin α; if $\alpha = 90°$, the particle is projected from an apse, that is, from one or other of the extremities of the major axis of the ellipse.

The time during which the moving particle passes from one to another point in the orbit may thus be found; let us suppose $t = 0$, when m is at that extremity of the major axis which is nearest to the focus. Now since the quantities which determine the magnitude of the ellipse have been expressed in terms of the initial circumstances of motion, we may assume that a and e are known in the equation

$$r = \frac{a(1-e^2)}{1+e\cos\theta}; \tag{92}$$

and by Art. 381, (26),

$$h = \{\mu a(1-e^2)\}^{\frac{1}{2}}; \tag{93}$$

therefore by (10), $$dt = \frac{r^2}{h}\,d\theta = \frac{r^2\,d\theta}{\{\mu a(1-e^2)\}^{\frac{1}{2}}}; \tag{94}$$

but from (92),
$$d\theta = \frac{a(1-e^2)^{\frac{1}{2}}\,dr}{r\{a^2e^2-(r-a)^2\}^{\frac{1}{2}}}; \tag{95}$$

$$\therefore\quad dt = \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \frac{r\,dr}{\{a^2e^2-(r-a)^2\}^{\frac{1}{2}}}; \tag{96}$$

$$\begin{aligned}\therefore\quad t &= \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \int_{a(1-e)}^{r} \frac{r\,dr}{\{a^2e^2-(r-a)^2\}^{\frac{1}{2}}}\\ &= \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \Big[a \sin^{-1}\frac{r-a}{ae} - \{a^2e^2-(r-a)^2\}^{\frac{1}{2}}\Big]_{a(1-e)}^{r}\\ &= \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \Big\{a \cos^{-1}\frac{a-r}{ae} - \{a^2e^2-(a-r)^2\}^{\frac{1}{2}}\Big\};\end{aligned} \tag{97}$$

which gives the value of t corresponding to any value of r. At the farther extremity of the major axis, $r = a(1+e)$, the time corresponding to which is the semi-periodic time: therefore

$$\text{the semi-periodic time} = \frac{\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}; \tag{98}$$

and at the extremity of the minor axis, $r = a$; in which case

$$t = \frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \left\{\frac{\pi}{2} - e\right\};$$

and therefore the time from the extremity of the minor axis to the farther extremity of the major axis is

$$\frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \left\{\frac{\pi}{2} + e\right\}.$$

And thus

$$\text{the periodic time} = \frac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}; \tag{99}$$

the same result as (28).

393.] The expression for the time given in (97) admits of the following simplification: let

$$\cos^{-1}\frac{a-r}{ae} = u;$$

$$\therefore\quad r = a(1-e\cos u); \tag{100}$$

u being an auxiliary angle, the geometrical meaning of which we shall presently investigate: then substituting in (97), we have

$$t = \frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \{u - e\sin u\}. \tag{101}$$

For convenience of notation, let
$$\frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} = \frac{1}{n}; \tag{102}$$

$$\therefore\quad nt = u - e\sin u. \tag{103}$$

Hence when u is increased by 2π, that is, when the particle has passed through the whole orbit, t is increased by $\frac{2\pi}{n}$; $\frac{2\pi}{n}$ therefore is the periodic time of the orbit.

As r is given in terms of θ and of u, there is also a relation between θ and u; equating the values of r given in (92) and in (100), we have

$$\frac{1-e^2}{1+e\cos\theta} = 1-e\cos u;$$

$$\therefore \quad 1+\cos\theta = \frac{(1-e)(1+\cos u)}{1-e\cos u};$$

$$1-\cos\theta = \frac{(1+e)(1-\cos u)}{1-e\cos u};$$

$$\therefore \quad \frac{1-\cos\theta}{1+\cos\theta} = \frac{1+e}{1-e}\,\frac{1-\cos u}{1+\cos u};$$

$$\tan\frac{\theta}{2} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\tan\frac{u}{2}. \tag{104}$$

The value of t in terms of θ is so complicated that it is unnecessary to insert it. The geometrical meaning of these several quantities will be seen by the aid of fig. 131. On the major axis ACA′ of the elliptic orbit, describe a semicircle: let P be the point on the ellipse corresponding to which are $\text{SP} = r$, $\text{PSA} = \theta$; let the ordinate PM be drawn, and let it be produced so as to cut the circle in Q: draw QC to C the centre of the circle. Let $\text{CA} = \text{CQ} = \text{CA}' = a$: then by a property of the ellipse,

$$\text{SP} = a - e\,\text{CM};$$

$$\therefore \quad r = a - ae\cos \text{QCM}; \tag{105}$$

therefore comparing this with (100), $\text{QCM} = u$.

When a particle moves in an elliptic orbit under the action of a force in the focus, it is evident that as equal areas are described in equal times, the angles abutting at S are not described uniformly; and for this reason angles measured from SA are called *anomalies; θ* is called the *true anomaly, u* the *eccentric anomaly*, and nt the *mean anomaly*.

From (99) it appears that the periodic time of m in the elliptic orbit is independent of the eccentricity of the ellipse, and is therefore the same as that in a circle whose radius is a; but in this latter case $e = 0$, $r = a$, $\theta = u = nt$; thus equal angles are described in equal times, and the particle moves uniformly in the circle; hence nt represents the arc of the circle AQA′, which would be described *uniformly* by a particle in the same time as that in which the elliptic arc is described, both the particles being together at A; and therefore also at A′, because then $u = \pi$ and $\sin u = 0$, and thus from (103) the time from A to $\text{A}' = \frac{\pi}{n}$; n is called the *mean motion* of m: now as these two particles start from A simultaneously and in the same direction, one along the circular and the other along the elliptic orbit, the latter is before its mean place from A to A′, because $\sin u$ is positive in the first two quadrants, and therefore u is greater than nt; and is behind its mean place from A′ to A, because $\sin u$ is negative in the third and fourth quadrants. It is for this reason that nt is called the mean anomaly, and also that u which depends on the eccentricity is called the eccentric anomaly. Also as the velocity varies inversely as the perpendicular from the centre of force on the tangent, the velocity in the elliptic orbit is the greatest at A, and is the least at A′: the particle therefore leaves A with a velocity greater than its mean velocity, and thus precedes its mean place, but at A′ is at its mean time: and as it leaves A′ with a velocity less than its mean velocity, it is behind its mean place, until its velocity increasing it arrives at A at its mean time and with a velocity greater than its mean velocity.

One case of elliptic motion under the action of a force varying inversely as the square of the distance requires notice. Suppose $e = 0$, that is, suppose the orbit to be a circle; then from (88) we have

$$\{\mu - \text{R}\text{V}^2 (\sin a)^2\}^2 + \text{R}^2\text{V}^4 (\sin a \cos a)^2 = 0;$$

which can be satisfied only when $a = 90°$, and $\text{V}^2 = \frac{\mu}{\text{R}}$: in which case m is projected at an apse; and if $\text{P}' =$ the central force at the point of projection, $\text{P}' = \frac{\mu}{\text{R}^2}$: and therefore

$$\text{P}' = \frac{\text{V}^2}{\text{R}};$$

but by (66), Art. 326, $\frac{\text{V}^2}{\text{R}}$ is the centrifugal force at the point of projection; therefore at that point the central force is equal to it: if therefore m is projected from an apse with a velocity such that the centrifugal force is equal to the central force, the particle continues throughout its motion at the same distance from the centre of force, and the orbit is a circle. We shall hereafter point out the general cause of this circumstance.

394.] Let us in the next place consider the parabola in which, see Art. 391,

$$\text{V}^2 = \frac{2\mu}{\text{R}}. \tag{106}$$

Now the equation to the parabola, of which $4a$ is the latus rectum, and the focus is the pole, and the line from the focus to the vertex is the prime radius, is

$$r = \frac{2a}{1+\cos\theta}; \tag{107}$$

and comparing this with the form which (82) takes, when $\mathrm{V}^2\mathrm{R} = 2\mu$, we have

$$u = \frac{1}{2\mathrm{R}(\sin\alpha)^2}(1+\cos\theta); \tag{108}$$

$$\therefore \quad a = \mathrm{R}(\sin\alpha)^2; \tag{109}$$

and if β is the angle between the vertex and R, then $u = \frac{1}{\mathrm{R}}$ when $\theta = \beta$; in which case, from (108), $\beta = 180° - 2\alpha$. Thus the position and latus rectum of the parabola are completely determined. If $\alpha = 90°$, the particle is projected from an apse, which is the vertex of the parabola.

The time has been found in terms of θ in Art. 381.

396.] And lastly let us consider the hyperbola: in which

$$\mathrm{V}^2 \text{ is greater than } \frac{2\mu}{\mathrm{R}};$$

and if e is the eccentricity of the curve,

$$e^2 = 1 + \frac{(\mathrm{V}^2\mathrm{R} - 2\mu)\,\mathrm{R}\mathrm{V}^2(\sin\alpha)^2}{\mu^2}. \tag{110}$$

Now the equation of the hyperbola, if θ is measured from that extremity of the transverse axis which is nearest to the focus where the pole is, is

$$r = \frac{a(e^2-1)}{1+e\cos\theta}; \tag{111}$$

$$u = \frac{1}{a(e^2-1)} + \frac{e}{a(e^2-1)}\cos\theta; \tag{112}$$

comparing which with (82), we have

$$a = \frac{\mu\mathrm{R}}{\mathrm{V}^2\mathrm{R} - 2\mu}; \tag{113}$$

and if β is the angle between the transverse axis and R, β may be determined by a process similar to that by which equation (91) is found.

Thus the position and size of the orbit are completely determined.

And the time may thus be found. The equation to the hyperbola being (111), we have, as in Art. 381,

$$h = \{\mu a(e^2-1)\}^{\frac{1}{2}}.$$

Therefore by (12), Art. 379, if t is the time from the vertex of the hyperbola,

$$t = \int_0^\theta \frac{r^2\,d\theta}{\{\mu a(e^2-1)\}^{\frac{1}{2}}}$$

$$= \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \int_{a(e-1)}^{r} \frac{r\,dr}{\{(r+a)^2-a^2e^2\}^{\frac{1}{2}}}$$

$$= \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \left\{\{(r+a)^2-a^2e^2\}^{\frac{1}{2}} - a\log\frac{r+a+\{(r+a)^2-a^2e^2\}^{\frac{1}{2}}}{ae}\right\}. \quad (114)$$

To simplify this expression, let ϵ be the Napierian base, and let

$$\frac{r+a+\{(r+a)^2-a^2e^2\}^{\frac{1}{2}}}{ae} = \epsilon^u;$$

$$\therefore \quad t = \frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}\left\{\frac{e}{2}(\epsilon^u-\epsilon^{-u})-u\right\}. \quad (115)$$

Hence, in conclusion, it appears that if a particle is projected with a given velocity, and moves under the action of a central force which varies inversely as the square of the distance, the path which it describes will be either an ellipse, a parabola, or a hyperbola: and that the species of curve depends on the velocity of projection. The physical application of the preceding results is deferred to Section 3 of the present Chapter.

396.] A particle is moving under the action of a central force, which varies inversely as the cube of the distance: it is required to determine the nature of the several orbits which it can describe.

Let us suppose the force to be attractive, and μ to be the absolute central force: so that for the central force we have

$$\text{P} = \frac{\mu}{r^3} = \mu u^3. \quad (116)$$

Let $\text{V} =$ the velocity of m at a point whose distance from the pole is R; and let $\alpha =$ the angle between R and the line in which m is moving: so that

$$h = \text{VR}\sin\alpha; \quad (117)$$

and thus h is given in terms of known quantities. Substituting (116) in (21), we have

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}u.$$

And multiplying by $2du$ and integrating, and taking the limits corresponding to $r = r$ and to $r = \text{R}$, we have

$$h^2\left(\frac{du^2}{d\theta^2}+u^2\right) - \text{V}^2 = \mu u^2 - \frac{\mu}{\text{R}^2};$$

$$\therefore \quad \frac{du^2}{d\theta^2} = \frac{\mu - h^2}{h^2} u^2 + \frac{\mathrm{V}^2 \mathrm{R}^2 - \mu}{h^2 \mathrm{R}^2}. \tag{118}$$

But if a particle moves in a rectilinear path from an infinite distance, under the action of a central force which varies inversely as the cube of the distance, towards the centre of force, then if V is the velocity at a distance R from the centre,

$$\mathrm{V}^2 = \frac{\mu}{\mathrm{R}^2};$$

so that $\mathrm{V}^2\mathrm{R}^2 - \mu$ is positive, zero, or negative, according as the velocity corresponding to R is greater than, equal to, or less than, that acquired in moving from an infinite distance.

Now (118) admits of different cases, according as the coefficients in its right-hand member are positive, zero, or negative; and replacing h^2 by its value in (117), we have

$$\mu - h^2 = \mu - \mathrm{V}^2\mathrm{R}^2 (\sin a)^2. \tag{119}$$

(1) Let $\mu - h^2$ be a positive quantity; and let

$$\frac{\mu - h^2}{h^2} = n^2:$$

(α) and suppose $\mathrm{V}^2\mathrm{R}^2$ to be greater than μ; and let

$$\frac{\mathrm{V}^2\mathrm{R}^2 - \mu}{h^2 \mathrm{R}^2} = n^2 c^2;$$

so that (118) becomes

$$\frac{du^2}{d\theta^2} = n^2 (u^2 + c^2);$$

$$\therefore \quad u = \frac{1}{r} = \frac{c}{2} \{ e^{n(\theta - \gamma)} - e^{-n(\theta - \gamma)} \}: \tag{120}$$

(β) suppose $\mathrm{V}^2\mathrm{R}^2 = \mu$; then $\mu - h^2 = \mathrm{V}^2\mathrm{R}^2 (\cos a)^2$; and (118) becomes

$$\frac{du^2}{d\theta^2} = (\cot a)^2 u^2;$$

$$\therefore \quad r = a e^{\pm \theta \cot a}: \tag{121}$$

(γ) let $\mathrm{V}^2\mathrm{R}^2$ be less than μ; and let

$$\frac{\mathrm{V}^2\mathrm{R}^2 - \mu}{h^2 \mathrm{R}^2} = -n^2 c^2;$$

so that (118) becomes

$$\frac{du^2}{d\theta^2} = n^2 (u^2 - c^2);$$

$$\therefore \quad u = \frac{1}{r} = \frac{c}{2} \{ e^{n(\theta - \gamma)} + e^{-n(\theta - \gamma)} \}. \tag{122}$$

(2) Let $\mu - h^2 = 0$; so that from (119) $\mu = \mathrm{V}^2\mathrm{R}^2 (\sin a)^2$; and therefore

$$\frac{\mathrm{V}^2\mathrm{R}^2 - \mu}{h^2 \mathrm{R}^2} = \frac{(\cot a)^2}{\mathrm{R}^2};$$

thus (118) becomes
$$\frac{du}{d\theta} = \frac{\cot a}{\mathrm{R}};$$
$$\therefore \quad r = \frac{\mathrm{R}\tan a}{\theta-\gamma}; \tag{123}$$
which is the equation to the reciprocal spiral.

(3) Let $\mu - h^2$, that is, $\mu - \mathrm{R}^2\mathrm{V}^2(\sin a)^2$, be a negative quantity; and let
$$\frac{\mu - h^2}{h^2} = -n^2;$$
therefore $\mathrm{V}^2\mathrm{R}^2 - \mu$ is a positive quantity: and let us therefore suppose
$$\frac{\mathrm{V}^2\mathrm{R}^2 - \mu}{h^2\mathrm{R}^2} = n^2c^2;$$
so that (118) becomes
$$\frac{du^2}{d\theta^2} = n^2(c^2 - u^2);$$
$$\therefore \quad u = \frac{1}{r} = c\cos n(\theta-\gamma). \tag{124}$$

If the central force is repulsive, the sign of μ must be changed; in which case, if $\dfrac{-\mu - h^2}{h^2} = -n^2$, and $\dfrac{\mathrm{V}^2\mathrm{R}^2 + \mu}{h^2\mathrm{R}^2} = n^2c^2$, equation (118) becomes
$$\frac{du^2}{d\theta^2} = n^2(c^2 - u^2);$$
$$\therefore \quad u = \frac{1}{r} = c\cos n(\theta-\gamma). \tag{125}$$

There are therefore generally five different orbits in which a particle may move under the action of a central force which varies inversely as the cube of the distance, and to which the equations are, if the prime radius is judiciously chosen,
$$(1)\ \ r = \frac{2a}{e^{n\theta} - e^{-n\theta}}; \qquad (2)\ \ r = a^{\theta}; \qquad (3)\ \ r = \frac{2a}{e^{n\theta} + e^{-n\theta}};$$
$$(4)\ \ r = \frac{a}{\theta}; \qquad (5)\ \ r = \frac{2a}{\cos n\theta}.$$

One case of the preceding requires notice, viz. (2), wherein $\mu = h^2$; if $\mathrm{V}^2\mathrm{R}^2 = \mu$, then $\cot a = 0$, and the equation of the orbit becomes
$$\frac{du}{d\theta} = 0; \qquad \therefore \quad u = \frac{1}{r} = \frac{1}{\mathrm{R}};$$
$$\therefore \quad r = \mathrm{R}; \tag{126}$$
which is the equation of a circle, the centre of which is the pole. Since in this case
$$\mathrm{V}^2\mathrm{R}^2 = \mu; \qquad \therefore \quad \frac{\mathrm{V}^2}{\mathrm{R}} = \frac{\mu}{\mathrm{R}^3}.$$

From these results it appears that, since $\cot\alpha = 0$, m is projected in a line perpendicular to R; and at the point of projection the centrifugal force, of which $\frac{\text{V}^2}{\text{R}}$ is the representative, is equal to the central force, that is, to $\frac{\mu}{\text{R}^2}$; and as the orbit is circular, this equality of the central and centrifugal forces will hold good at every point of it.

397.] A particle is moving under the action of a central force which varies inversely as the fifth power of the distance, and is attractive: it is required to determine the nature of the orbits.

Let R be the distance from the centre of the point of projection, α the angle between R and the line of projection, V = the velocity of projection: then

$$h = \text{VR}\sin\alpha;$$

and thus h is given in terms of known quantities. And for the central force we have

$$\text{P} = \frac{\mu}{r^5} = \mu u^5;$$

so that (21) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^3}{h^2};$$

$$\therefore\quad h^2\left(\frac{du^2}{d\theta^2} + u^2\right) - \text{V}^2 = \frac{\mu u^4}{2} - \frac{\mu}{2\text{R}^4};$$

$$\therefore\quad h^2\frac{du^2}{d\theta^2} = \frac{\mu u^4}{2} - h^2u^2 + \text{V}^2 - \frac{\mu}{2\text{R}^4}. \tag{127}$$

Now suppose $\text{V}^2 - \frac{\mu}{2\text{R}^4} = 0$, in which case the velocity of projection is equal to that acquired at a distance R in moving towards the centre of force from an infinite distance; then (127) becomes

$$\frac{du^2}{d\theta^2} = \frac{\mu u^4}{2h^2} - u^2.$$

Let $\frac{\mu}{2h^2} = a^2$, and substituting $u = \frac{1}{r}$, we have

$$\frac{-dr}{(a^2-r^2)^{\frac{1}{2}}} = d\theta; \qquad \therefore\quad \cos^{-1}\frac{r}{a} = \theta - \gamma;$$

$$\therefore\quad r = a\cos(\theta-\gamma); \tag{128}$$

the equation of a circle, on the circumference of which the pole is, whose diameter is inclined at an angle γ to the prime radius, and of which the diameter is a.

Again (127) may be integrated when the right-hand member is a complete square; in which case, we have

$$\mathrm{V}^2 - \frac{\mu}{2\,\mathrm{R}^4} = \frac{h^4}{2\mu};$$

and therefore (127) becomes

$$\begin{aligned} h\frac{du}{d\theta} &= u^2\left(\frac{\mu}{2}\right)^{\frac{1}{2}} - \frac{h^2}{(2\mu)^{\frac{1}{2}}} \\ &= \left(\frac{\mu}{2}\right)^{\frac{1}{2}}\left\{u^2 - \frac{h^2}{\mu}\right\} \\ &= \left(\frac{\mu}{2}\right)^{\frac{1}{2}}\{u^2 - a^2\}, \end{aligned}$$

if $a^2 = \dfrac{h^2}{\mu}$;

$$\therefore\quad \log\frac{u-a}{u+a} = 2^{\frac{1}{2}}(\theta-\gamma);$$

$$\therefore\quad r = \frac{\mu^{\frac{1}{2}}}{h}\,\frac{1-e^{2^{\frac{1}{2}}(\theta-\gamma)}}{1+e^{2^{\frac{1}{2}}(\theta-\gamma)}}; \qquad (129)$$

which is the equation to the orbit.

398.] In problems such as those of the preceding Articles, where the species of curve depends on the velocity with which the particle is projected, and consequently, by the principle of conservation of work, on the velocity in the orbit, it is convenient to have an absolute standard with which the velocity may be compared, so that the species of orbit may be discriminated with reference to such an absolute condition. Two such standards of comparison are suggested by general considerations, and are indeed supplied by the equations of motion.

The first arises out of the theory of centrifugal force: for if a particle is projected from an apse with a velocity such that the centrifugal force due to it is exactly equal to the central force, the effects of these two, both being normal, neutralize each other, and the particle remains always at the same distance from the centre of force, and thus describes a circle uniformly with the velocity of projection. Thus whatever the law of force is, the orbit may under certain initial circumstances be a circle; and the corresponding velocity gives a standard for other velocities to be compared with.

Thus if P is the central force, V is the velocity at an apse whose distance from the centre of force is R, and if the central and the centrifugal forces are equal,

$$\mathrm{P} = \frac{\mathrm{V}^2}{\mathrm{R}};$$

$$\therefore\quad \mathrm{V}^2 = \mathrm{PR};$$

which assigns the relation between the velocity, P, and R, when the orbit is a circle, and gives a standard with which the actual velocity at any point of a particle's orbit may be compared. This velocity is commonly called *the velocity in a circle at the same distance*.

As this relation between V, R, and P holds good, whatever R is, we may replace R and V by their general values r and v; and we have

$$\mathrm{P} = \frac{v^2}{r} = \frac{h^2}{r^3} = h^2 u^3. \tag{130}$$

But in the general equation of motion, viz. (21), Art. 380, if this condition is true, $\frac{d^2 u}{d\theta^2} = 0$; and we have the singular solution of this differential equation; because this solution arises not from any particular value of the arbitrary constants of integration, but because the variable P in the original equation is equated to a constant quantity. If however $\mathrm{P} = h^2 u^3$, and $\frac{d^2 u}{d\theta^2} = 0$, we have the orbit (123) of Art. 396, and this is the reciprocal spiral. But the particular form of it is a circle; for if there is an apse at one point, there are apses at all points, and such a curve is a circle.

399.] The second standard of comparison is given by the velocity which a particle would acquire in moving from rest at an infinite distance under the action of the central force to its place on the curve at a distance r from the centre. Thus, as in Art. 286, if the force varies inversely as the nth power of the distance and is attractive, we have

$$\frac{d^2 x}{dt^2} = -\frac{\mu}{x^n}; \tag{131}$$

and therefore

$$\frac{dx^2}{dt^2} = \frac{2\mu}{(n-1)\,x^{n-1}}, \tag{132}$$

since the particle is at rest when $x = \infty$; and consequently if V is the velocity at the distance R, and is the velocity acquired in moving from rest at an infinite distance,

$$\mathrm{V}^2 = \frac{2\mu}{(n-1)\,\mathrm{R}^{n-1}}; \tag{133}$$

and this is a velocity with which the velocity in the curve may be compared; and on its relation to which the species of orbit will depend.

Now this value of the velocity is that for which the values of the two sides of the first integral of the equation of motion at the lower limit are equal and cancel each other. For if $\mathrm{P} = \mu u^n$,

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu u^{n-2}}{h^2};$$

$$\therefore \quad h^2\left\{\frac{du^2}{d\theta^2} + u^2\right\} - \mathrm{V}^2 = \frac{2\mu}{n-1}\left\{u^{n-1} - \frac{1}{\mathrm{R}^{n-1}}\right\}; \qquad (134)$$

and consequently by reason of (133)

$$\frac{du^2}{d\theta^2} + u^2 = \frac{2\mu}{h^2(n-1)}\, u^{n-1}. \qquad (135)$$

$$\text{Let} \quad \frac{2\mu}{h^2(n-1)} = a^{n-3}; \qquad \therefore \quad \frac{du}{(a^{n-3}u^{n-1} - u^2)^{\frac{1}{2}}} = d\theta;$$

$$\frac{du}{u^{\frac{n-1}{2}}(a^{n-3} - u^{-(n-3)})^{\frac{1}{2}}} = d\theta;$$

$$\frac{-d.u^{-\frac{n-3}{2}}}{(a^{n-3} - u^{-(n-3)})^{\frac{1}{2}}} = \frac{n-3}{2}\, d\theta;$$

$$\therefore \quad \cos^{-1}\frac{u^{-\frac{n-3}{2}}}{a^{\frac{n-3}{2}}} = \frac{n-3}{2}(\theta - \gamma);$$

$$\therefore \quad r^{\frac{n-3}{2}} = a^{\frac{n-3}{2}} \cos\frac{n-3}{2}(\theta - \gamma); \qquad (136)$$

which is the equation to the orbit: and thus the equation to the orbit can be found whatever is the value of n. The following are particular cases:

$n = 2$, $r = \dfrac{2a}{1 + \cos\theta}$, the equation of a Parabola.

$n = 3$, then by means of (135), $r = a^{\theta}$, the Logarithmic Spiral.

$n = 4$, $r = \dfrac{a}{2}(1 + \cos\theta)$, the equation of a Cardioid.

$n = 5$, $r = a\cos\theta$, the equation of a Circle.

$n = 6$, $r^3 = \dfrac{a^3}{2}(1 + \cos 3\theta)$.

$n = 7$, $r^2 = a^2\cos 2\theta$, the equation of a Lemniscata.

And so for other values of n.

400.] If the form of P is

$$\mathrm{P} = \frac{\mu}{r^3} + \frac{k^2}{r^3} f(\theta) \qquad (137)$$

$$= \mu u^3 + k^2 u^3 f(\theta),$$

where $f(\theta)$ represents any function of θ, then equation (21) becomes

$$\frac{d^2u}{d\theta^2}+\left(1-\frac{\mu}{h^2}\right)u=\frac{k^2}{h^2}f(\theta); \tag{138}$$

or

$$\frac{d^2u}{d\theta^2}+n^2u=a^2f(\theta); \tag{139}$$

which is a linear differential equation of the second order, and of which the integral is

$$u=\frac{1}{r}=\frac{a^2}{n}\left\{\sin n\theta\int f(\theta)\cos n\theta\,d\theta-\cos n\theta\int f(\theta)\sin n\theta\,d\theta\right\}$$
$$+\,\text{C}\sin(n\theta-\gamma); \tag{140}$$

where C and γ are constants undetermined, and dependent on the initial circumstances of the motion.

Suppose $f(\theta)=1$; so that the central force, see (137), varies partly inversely as the square, and partly inversely as the cube, of the distance, then (140) becomes

$$\frac{1}{r}=\frac{a^2}{n^2}+\text{C}\sin(n\theta-\gamma);$$

which represents a revolving conic: see Art. 385.

And if $f(\theta)=1$, and $\mu=0$, so that $n=1$, then

$$\frac{1}{r}=a^2+\text{C}\sin(\theta-\gamma);$$

which is the equation of a conic.

And the differential equation can always be reduced to simple quadrature, if the central force is a homogeneous function of the second order in terms of x and y, because in that case $\text{P}=\frac{f(\theta)}{r^2}$. Also if

$$\text{P}=\frac{\mu}{r^3}+\mu' r=\mu u^3+\frac{\mu'}{u};$$

then

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu u}{h^2}+\frac{\mu'}{h^2u^3};$$

$$\therefore\quad \frac{d^2u}{d\theta^2}+\left(1-\frac{\mu}{h^2}\right)u=\frac{\mu'}{h^2u^3};$$

therefore multiplying by $2du$, integrating, and introducing an arbitrary constant,

$$\frac{du^2}{d\theta^2}+\left(1-\frac{\mu}{h^2}\right)u^2=-\frac{\mu'}{h^2u^2}+\text{C};$$

the integral of which is of the form

$$u^2=\frac{1}{r^2}=a+k\cos 2n(\theta-\gamma). \tag{141}$$

If $n=1$, that is, if $\mu=0$, the central force varies directly as the distance, and (141) is the equation of an ellipse, the centre of which is at the centre of force.

401.] Some few problems are added, so that the principles of the preceding Articles may be exhibited in forms somewhat different to those which have been already discussed.

Ex. 1. A particle m moves under the action of a central force; and its velocity varies inversely as the nth power of the distance from the centre of force: it is required to determine the law of force, and the equation of the orbit.

Let v and V be the velocities at the distances r and R respectively; and let $v = \frac{k}{r^n}$; then by (16), Art. 381,

$$\frac{k^2}{r^{2n}} - \mathrm{V}^2 = -2\int_{\mathrm{R}}^{r} \mathrm{P}\, dr;$$

whence differentiating, $-\frac{2nk^2}{r^{2n+1}} = -2\mathrm{P}$;

$$\therefore \quad \mathrm{P} = \frac{nk^2}{r^{2n+1}};$$

which gives the law of attracting force.

Also since $\quad$ (the velocity)$^2 = h^2\left(u^2 + \frac{du^2}{d\theta^2}\right)$;

$$\therefore \quad k^2 u^{2n} = h^2\left(u^2 + \frac{du^2}{d\theta^2}\right);$$

of which the integral is

$$r^{n-1} = \frac{k}{h}\cos(n-1)(\theta-\gamma).$$

Ex. 2. A particle m is moving in an ellipse, at the focus of which is the centre of force, and as m passes successively through the apse which is nearer to the centre of force, the absolute force is increased in the ratio of n to 1 : it is required to determine the nature of the orbit after p passages through the apse.

Since m moves in an ellipse, the velocity at every point of the orbit, and therefore at the apse, is less than that due to an infinite distance; see Art. 391. Now the velocity at the apse is always the same, see Art. 380, in the successive passages through it, whatever is the value of the absolute force: and therefore if the absolute force is increased, the velocity at it becomes proportionately less and less than that acquired from an infinite distance; the orbit therefore is still an ellipse.

And as the velocity at the nearer apse is unchanged, by the increase of μ, h is the same in the orbit after the pth increase as it was in the original ellipse. Let $2a$ and e be the major axis and the eccentricity of the original ellipse, $2a_p$ and e_p those of the final ellipse: then equating the values of h, see (93), Art. 392,

$$\mu a(1-e^2) = n^p \mu a_p (1-e_p^2);$$

also as the distance of the apse from the focus is the same in both orbits,

$$a(1-e) = a_p(1-e_p); \tag{142}$$

$$\therefore \quad 1+e = n^p(1+e_p),$$

which gives the eccentricity of the final ellipse.

Thus as n is greater than unity, the eccentricity becomes less and less, and $= 0$, when

$$p = \frac{\log(1+e)}{\log n};$$

in which case the orbit is a circle: and as the revolutions continue, the orbit again becomes an ellipse, but the apse which was the nearer becomes that farther from the focus: and ultimately, when $p = \infty$, that is, after an infinite number of revolutions, $e_p = -1$; in which case from (142), $2a_p$ becomes equal to the distance between the focus and the apse, and thus the particle passes through the centre of force.

Ex. 3. A particle under the action of a central force which varies partly inversely as the cube, and partly inversely as the fifth power, of the distance, is projected from a given point with the velocity which would be acquired in motion from an infinite distance, at $\tan^{-1} 2^{\frac{1}{2}}$ with the distance, and the forces are equal at the point of projection. It is required to determine the orbit.

Let R be the distance of the point of projection from the centre of force: and let

$$\text{P} = \frac{\mu}{r^3} + \frac{\mu'}{r^5} = \mu u^3 + \mu' u^5;$$

and since the forces are equal at the point of projection,

$$\frac{\mu}{\text{R}^3} = \frac{\mu'}{\text{R}^5}; \qquad \therefore \quad \mu' = \mu \text{R}^2.$$

Let V = the velocity of projection: therefore

$$\text{V}^2 = \frac{\mu}{\text{R}^2} + \frac{\mu'}{2\text{R}^4} = \frac{3\mu}{2\text{R}^2};$$

$$\therefore \quad h^2 = \text{V}^2 \text{R}^2 (\sin \tan^{-1} 2^{\frac{1}{2}})^2 = \mu.$$

The equation of motion is

$$h^2\left(\frac{d^2u}{d\theta^2} + u\right) = \mu u + \mu' u^3;$$

$$\therefore \quad h^2\left(\frac{du^2}{d\theta^2}+u^2\right)-\mathrm{V}^2 = \mu u^2+\frac{\mu' u^4}{2}-\frac{\mu}{\mathrm{R}^2}-\frac{\mu'}{2\mathrm{R}^4};$$

$$\therefore \quad h^2\left(\frac{du^2}{d\theta^2}+u^2\right) = \mu u^2+\frac{\mu'}{2}u^4$$

$$= \mu u^2+\frac{\mu \mathrm{R}^2 u^4}{2};$$

$$\therefore \quad \frac{du^2}{d\theta^2} = \frac{\mathrm{R}^2}{2}u^4; \qquad \therefore \quad r = \frac{\mathrm{R}}{2^{\frac{1}{2}}}(\theta+2^{\frac{1}{2}}),$$

if R coincides with the prime radius.

Ex. 4. A particle is projected from an apse, and moves under the action of an attractive central force which varies inversely as the seventh power of the distance: it is required to find the orbit, the velocity of projection being equal to $3^{-\frac{1}{2}}\mathrm{V}_c$, where V_c is the velocity with which a particle projected from an apse would describe a circle about the centre of force in its centre.

In this case,

$$\mathrm{P} = \frac{\mu}{r^7} = \mu u^7;$$

therefore if $\mathrm{V} =$ the velocity of projection, $\mathrm{R} =$ the distance of projection, by Art. 398,

$$\mathrm{V}_c^2 = \frac{\mu}{\mathrm{R}^6}; \qquad \therefore \quad \mathrm{V}^2 = \frac{\mu}{3\mathrm{R}^6};$$

$$\therefore \quad h^2 = \mathrm{V}^2\mathrm{R}^2 = \frac{\mu}{3\mathrm{R}^4}.$$

The equation of motion is $h^2\left(\frac{d^2u}{d\theta^2}+u\right) = \mu u^5$;

$$\therefore \quad h^2\left(\frac{du^2}{d\theta^2}+u^2\right)-\frac{\mu}{3\mathrm{R}^6} = \frac{\mu}{3}u^6-\frac{\mu}{3\mathrm{R}^6};$$

$$\therefore \quad \frac{du^2}{d\theta^2} = \mathrm{R}^4u^6-u^2;$$

$$\therefore \quad r^2 = \mathrm{R}^2\cos 2\theta,$$

if the prime radius coincides with R: and this is the equation of the lemniscata.

Ex. 5. A particle describing a circle under the action of a central force which varies as any power of the distance is slightly displaced from its position in the circular orbit. Under what circumstances will its orbit be stable, and what will be the angle between two successive apsidal distances?

Let the law of force be represented by $u^2 f(u)$, so that $\mathrm{P} = u^2 f(u)$; let a be the value of u at the point where the particle moves with the velocity in the circle, and where the displacement takes place; so that if v is the velocity at the point, and the velocity is not disturbed,

$$v^2 = h^2 a^2 = af(a); \qquad \therefore \; h^2 = \frac{f(a)}{a}.$$

Let $u = a + z$, z being a small quantity of which the squares and higher powers may be neglected, the displacement being supposed to be slight: hence the equation of motion is

$$\frac{d^2 z}{d\theta^2} + a + z = \frac{a}{f(a)} f(a+z)$$

$$= \frac{a}{f(a)} \{f(a) + z f'(a)\};$$

$$\therefore \; \frac{d^2 z}{d\theta^2} + \left\{1 - \frac{af'(a)}{f(a)}\right\} z = 0, \tag{143}$$

which is an equation of harmonic motion when $f(a)$ is greater than $af'(a)$; consequently the orbit is stable or unstable according as $f(a)$ is greater or less than $af'(a)$. If $f(a) = af'(a)$, z is constant and the orbit is circular.

If the orbit is stable, the angle between two successive apsidal distances

$$= \frac{\pi}{\left\{1 - \frac{af'(a)}{f(a)}\right\}^{\frac{1}{2}}}. \tag{144}$$

If the force varies inversely as the nth power of the distance, $f(u) = \mu u^{n-2}$; and $f'(u) = \mu (n-2) u^{n-3}$: consequently

$$1 - \frac{af'(a)}{f(a)} = 3 - n; \tag{145}$$

and the orbit is stable or unstable according as n is less than or greater than 3. If $n = 3$, the orbit is still circular.

Also the angle between two successive apsidal distances $= \frac{\pi}{(3-n)^{\frac{1}{2}}}$. If $n = 2$, that is, if the force varies inversely as the square of the distance, the angle between two successive apses $= \pi$. If the force varies directly as the distance, that is, if $n = -1$, the angle between two successive apses $= \frac{\pi}{2}$.

This example is Prop. XLV, Sect. IX, Bk. I, Newton's Principia.

Ex. 6. Two heavy particles m and m' are connected by a thin inextensible string of given length a: the string passes through a small hole in a smooth horizontal plane; m' hangs vertically downwards, and m moves in a curve on the plane: determine the motion of m and of m', and the tension of the string.

Let r and r' be the respective distances of m and of m' from the hole at the time t, the string being assumed to be always stretched; so that $r+r'=a$. Let T be the tension of the string; then evidently

$$\begin{aligned} \text{T} &= m'g - m'\frac{d^2r'}{dt^2} \\ &= m'g + m'\frac{d^2r}{dt^2}. \end{aligned} \qquad (146)$$

Let the motion of m be referred to a system of polar coordinates of which the hole is the pole. Then as no transversal force but only a central force, viz. T, acts on m, the equations of motion are, see (62), Art. 324,

$$\begin{aligned} m\left\{\frac{d^2r}{dt^2} - r\frac{d\theta^2}{dt^2}\right\} &= -\text{T} \\ &= -m'g - m'\frac{d^2r}{dt^2}; \end{aligned}$$

$$\frac{r^2 d\theta}{dt} = \text{a constant} = h;$$

$$\therefore \quad (m+m')\frac{d^2r}{dt^2} - \frac{mh^2}{r^3} + m'g = 0;$$

and integrating, if $\frac{dr}{dt} = 0$, when $r = a$,

$$(m+m')\frac{dr^2}{dt^2} + \frac{mh^2}{r^2} - \frac{mh^2}{a^2} + 2m'g(r-a) = 0,$$

$$\frac{(m+m')h^2}{r^4}\frac{dr^2}{d\theta^2} + \frac{mh^2}{r^2} - \frac{mh^2}{a^2} + 2m'g(r-a) = 0; \qquad (147)$$

whence the orbit of m may be derived.

If the orbit of m is a circle, $\frac{d^2r}{dt^2}=0$; and we have $m'g=\frac{mh^2}{r^3}$; that is, the weight of m' is equal to the centrifugal force of m.

Also
$$\text{T} = \frac{mm'}{m+m'}\left(g + \frac{h^2}{r^3}\right). \qquad (148)$$

SECTION 3.—*The Elements of Physical Astronomy.*

402.] The science of physical astronomy consists in the application of mechanical principles and equations to the motion, figure, and other circumstances of the celestial bodies; and, as such, it has been called Celestial Mechanics, or, as Laplace has named it in his great work, *Mécanique Céleste*. It is the most important application of the two preceding Sections;

and I propose therefore to investigate the process by which our equations, deduced as they are from the abstract principles of dynamics, may be so applied; and also to prove some few elementary astronomical theorems which are immediately consequent upon them. I must observe that the explanations will be of motions as they are, and not as they appear to us on the surface of the earth: we are upon one of the planets which circulate about the sun, as a secondary about its primary; and we are therefore subject to displacement; and to displacements of two kinds: one of which is due to the absolute motion of the earth in space, by reason of which its centre is carried forward about 19 miles every second of time, and the distance between its places at an interval of six months is about 184 millions of miles: and the other is owing to the diurnal rotation of the earth about its own axis, so that as the diameter of the earth is nearly 8000 miles, a person at the equator suffers a displacement to this extent in the course of every 12 hours, and of an observer at a place the latitude of which is λ, the displacement is 8000 miles $\times \cos\lambda$; these effects are called *parallax*: the former is the *annual*, the latter the *diurnal*, parallax: so that the motions and displacements of the heavenly bodies do not appear to us to be what they actually are. Thus another planet appears to us to have a motion sometimes in one direction, sometimes in another, and at other times appears to be stationary. In the following account, to anticipate the sequel, the sun, which is the centre of our system, will be supposed to be at rest, and the planets will be supposed to circulate around it; and the system of reference is said to be heliocentric.

Now the equations and results which have been proved in the preceding pages are, as I have said, founded on principles of the abstract science of motion: thus the cinematical principles have been deduced from the relation of moving matter to time and space: inertia has been assumed to be a property of matter as the subject of motion, and the principle of equalizing impressed and expressed momenta has been founded on it. Instances also have been quoted wherein cosmical matter fulfils these conditions, and, as the laws of inductive philosophy authorize, a property which is proved to be true of some is extended to all cosmical matter. Other members of the solar system however do not admit of direct experiment, and therefore the proof is, in a great measure, *a posteriori*. In the preceding pages these properties have been applied to various kinds of laws of impressed momenta: some have had their source at an infinite distance, so that the lines of action of the impressed momenta on a particle in its different positions are parallel: of some the source has been in a point or centre at a finite distance, which can be conveniently taken as the origin of reference: and thus we have shewn that the pure science is comprehensive enough for all kinds and for all laws of force. But as our object now is the investigation of the motion of a system of bodies which exist in cosmos, the questions which first meet us are, What is the nature of the force which acts on these bodies? is it a central force? or is the centre, if there is one, at a distance so great that the lines of action of it must be considered parallel? And again, if the acting force is a central force, with its centre at a finite distance, what is the law of force? is it attractive or repulsive? is it periodic; that is, does the law of it change from time to time? and at regular intervals? and is it a central force varying as some power of the distance only? and what is the law according to which it varies? does it vary inversely as the square of the distance? or according to what, if any, power of the distance? And what is its absolute force? is that constant for each one body in its orbit? is it the same for all bodies in their orbits? or is it periodical? Surely to all these questions the reply must be sought in observation. *Hypotheses non fingamus.* Although when the nature and the law of force, which the Almighty Creator has impressed on cosmical matter, is known, we do perceive wisdom and fitness in it; yet it is not our prerogative to assert that such or such *must* be the law: it may be that the law of the inverse square of the distance is simple, because then spherical and spheroidal shells attract an internal particle equally in all directions, see Arts. 198, 205; because then a sphere composed of a series of concentric homogeneous shells attracts an external particle with an effect the same as if it were condensed into a particle at its centre, Art. 195: because it is the law of attraction of matter, when none of the influence is lost in the process of propagation, see Art. 215: because it is that law for which the angle between two successive apsidal distances is 180°, see Art. 401: yet of no one of these reasons, nor of all collectively, is the cogency such as to necessitate the existence of that law. We are therefore compelled to have recourse in the first place to observation, by which we may detect such facts as will enable us to discover the mechanical laws of which they are the effects. The necessary observation was made by John Kepler with marvellous industry and ingenuity, and the three laws which he discovered, and which have since been known by the name of Kepler's Laws, constitute the basis on which mechanical principles are applied to the solution of the problem of planetary and lunar motion. John Kepler was born in 1571: Galileo, to whom we owe the law of inertia, and therein the first correct principles of mechanics, was born in 1564: and Newton in 1642. Thus when Galileo had first stated in a true form the fundamental axioms of mechanics, and when John Kepler had enunciated those laws, which, when translated into mathematical language, as we shall just now do, are pregnant (approximately) with all the results of physical astronomy, that great philosopher Sir Isaac Newton was born; for whom it was reserved to construct the general science of which Galileo had given the axioms, and to apply it to the explanation of the solar system, the observed laws of which had been discovered by Kepler. This is one, and perhaps the most remarkable, instance of a great genius being brought into the world exactly at the time when, so to speak, the materials of his work were ready. The Greek geometers had sufficiently examined the properties of conics in addition to the ordinary elements of geometry: Galileo had laid the foundation of dynamics; Kepler had enunciated the planetary laws: it remained for Newton to assimilate these elements, and out of them to form that structure of physical astronomy which the Principia contain: D'Alembert, Lagrange, and Laplace subsequently elaborated, and wellnigh completed, the work which he began. Adams and Le Verrier drew inferences which observation confirmed, and thereby the severest test of truth was satisfied.

403.] Kepler's observations were at first chiefly confined to the planet Mars; and after long and assiduous study of its motion he enunciated the first two of the three following laws: and subsequently his observations were extended, and he enunciated the third law. The first two laws were extended by him analogically to all the planets: the third law he discovered from a comparison of the numbers which a table of the quantities corresponding to the several planets exhibited. And however unsatisfactory his proofs of these laws may seem to us, yet modern astronomy has completely verified them.

The three laws are—

I. *The sectorial areas described by the radius-vector of a planet in its motion about the sun are proportional to the times of describing them.*

II. *The orbits of the planets are ellipses, in one of the foci of which is the sun.*

III. *The squares of the periodic times of the planets are proportional to the cubes of the semi axis major* (*or mean distance**).

Let us translate these into their mathematical equivalents, and deduce from them such mechanical laws as they contain.

(1) By law I. equal areas are described in equal times: therefore when the areas and times are infinitesimal, the law still is true; in which case we have

$$r^2\,d\theta = h\,dt: \tag{149}$$

and changing to rectangular coordinates,

$$r^2 = x^2+y^2, \qquad \theta = \tan^{-1}\frac{y}{x};$$

$$\therefore\quad d\theta = \frac{x\,dy-y\,dx}{x^2+y^2};$$

so that (149) becomes $\quad x\,dy-y\,dx = h\,dt;$
therefore differentiating,

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2} = 0;$$

$$\therefore\quad \frac{\dfrac{d^2x}{dt^2}}{x} = \frac{\dfrac{d^2y}{dt^2}}{y} = \frac{-\mathrm{P}}{r}, \tag{150}$$

if P = the resultant expressed velocity-increment, and tends to diminish x and y;

$$\therefore\quad \frac{d^2x}{dt^2} = -\mathrm{P}\frac{x}{r}, \qquad \frac{d^2y}{dt^2} = -\mathrm{P}\frac{y}{r}; \tag{151}$$

that is, the axial-components of the velocity-increment are the resolved parts of a central force P; the force therefore, under the action of which a planet moves, is a central force, of which the place of the sun is the centre.

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* In astronomy if a quantity contains periodic and non-periodic terms, the value of it, when the periodic terms are omitted, is called its *mean* value. Thus the focal radius-vector of an ellipse is, (100), Art. 393, $r = a(1-e\cos u)$, where $\cos u$ is a periodical quantity: and thus a is the mean value of r: a is for this reason called the *mean* distance.

(2) Since the orbit is an ellipse, having the sun's centre, which is the centre of force, in one of the foci, let the equation to it be

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$

If therefore P = the central force, by Art. 380, equation (21),

$$\begin{aligned} \text{P} &= h^2u^2\left\{\frac{d^2u}{d\theta^2}+u\right\} \\ &= \frac{h^2u^2}{a(1-e^2)} = \frac{h^2}{a(1-e^2)}\frac{1}{r^2}; \end{aligned} \qquad (152)$$

and thus the central force varies inversely as the square of the distance. And therefore from Kepler's second law we infer that the planets move about the sun under the action of a force in the sun's centre which varies inversely as the square of the distance: this law is called *the law of gravitation*, and the planets are said to gravitate towards the sun.

(3) Let T = the periodic time in the planet's orbit: then the third law asserts that $\text{T}^2 = ca^3$; where c is a constant. Now if μ = the absolute central force, from (152),

$$\mu = \frac{h^2}{a(1-e^2)}; \qquad \therefore \quad h^2 = \mu a(1-e^2).$$

Also from (13), Art. 379,

$$\text{T}^2 = \frac{4\pi^2a^4(1-e^2)}{h^2} = \frac{4\pi^2a^3}{\mu};$$

and equating this to the value of T^2 given by Kepler's third law, we have

$$\mu = \frac{4\pi^2}{c}; \qquad (153)$$

and since c is constant for all the planets, so is μ: that is, the same absolute force in the centre acts on all the planets; and modified by the distance of the planets retains them all in their orbits about the sun. And here I cannot but quote the eloquent words of Sir John Herschel: "Of all the laws to which induction from pure observation has ever conducted man, this third law of Kepler may justly be regarded as the most remarkable, and the most pregnant with important consequences. When we contemplate the constituents of the planetary system from the point of view which this relation affords us, it is no longer mere analogy which strikes us, no longer a general resemblance among them as individuals independent of each other, and circulating about the sun, each according to its own peculiar nature, and connected with it by its own peculiar tie. The resemblance is now perceived to be a true *family* likeness; they are bound up in one chain; interwoven in one web of mutual relation and harmonious agreement; subjected to one pervading influence, which extends from the centre to the farthest limit of that great system, of which all of them, the earth included, must henceforth be regarded as members *."

Thus from these three laws of planetary motion we infer (1) the nature of the force acting on the secondaries of the solar system; viz. it is a central force, of which the source is in the sun's centre: (2) the law of the central force: viz. it varies inversely as the square of the distance: (3) the fact that the absolute central force is the same for all the planets: and the proper work of physical astronomy is to deduce from these dynamical facts the results which they contain: the nature of the problem therefore is the same as that of those which have been investigated in the present Chapter.

404.] The preceding laws however are only approximately true; it is assumed in them that the sun is fixed, and the motion of only one body about the sun is supposed: whereas the sun is attracted by, and moves towards, the other bodies of its system: and the motion is relative, not absolute: thus as the absolute central force in Art. 365 is the sum of the masses of the attracting and the attracted bodies, so if a is the mean distance of a planet's orbit, and T is the periodic time, by (28), Art. 381,

$$\mathrm{T} = \frac{2\pi a^{\frac{3}{2}}}{(m+\mathrm{M})^{\frac{1}{2}}}; \tag{154}$$

and thus (the periodic time)^2 varies directly as the cube of the mean distance, and inversely as the sum of the masses of the sun and the planet. In the solar system however the numerical correction of Kepler's third law thus introduced is too small to be of any importance, the mass of Jupiter, the largest of the planets, being much less than a thousandth part of that of the sun. Again, in the first two laws we have assumed one body to be moving about the sun, and attraction to exist between these two bodies only; whereas all the planets attract each other; and thus the elliptic orbit of each becomes disturbed. The method of investigation pursued in these and similar problems is, firstly to seek a solution in the simple, although inaccurate, form; and subsequently to correct it by the introduction of other elements which enter into it. Thus the elliptic motion of a planet is a first approximation to the actual orbit: and subsequently we prove that the size and position of the ellipse undergo certain changes, the magnitudes of which can be expressed in terms of the time.

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* Outlines of Astronomy, Art. 488, 4th edition, London, 1851.

Hereafter it will be shewn that when a body of finite dimensions moves in space, and its particles are under the action of many forces, so that it has motions both of translation and of rotation; so far as the motion of translation is concerned, we may consider all the impressed forces to be applied, each in its own intensity and in its own line of action, at the centre of gravity of the body, and the motion of translation of the centre will be the same as it is in the actual motion of the body. By virtue of this theorem therefore we may consider the planets to be condensed into their centres of gravity, and thus to move as material particles about the sun.

405] The motion of the centre of gravity of a planet is, by reason of Art. 378, in one plane, which also contains the centre of the sun. Thus if lines are drawn from the centre of the sun to the centre of the earth in all its positions, and if these lines are produced to the heavenly vault, it is found that all of them are in one and the same plane; and the plane in which the centre of the earth moves is called *the plane of the ecliptic:* to it, for the present, the motions of the other planets are referred. Also the fixed stars which are in the plane of the ecliptic are called *ecliptic stars.* Of the ellipse in which the centre of the earth moves, the mean distance is about 92 millions of miles, and the eccentricity is 0.0167836*; so that the distance of the centre of the ellipse from the sun is about 1,544,091 miles; the eccentricity however is of such a small amount, that if the orbit is drawn with a major axis of 10 feet, and a circle is also drawn on the major axis as a diameter, the difference between the two will not be sensible at any part of the curves. Suppose now in fig. 132 the plane of the paper to be the plane of the ecliptic; and let us suppose the path which the centre of the earth describes about the sun to be (approximately) a circle, of which the centre is s, the place of the sun. The position of the plane of the ecliptic having been determined by observation, it is found that the earth's axis is always inclined at the same angle to that plane:

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* See the synoptical table in Herschel's Outlines of Astronomy.

that is, as the earth moves in its path about the sun, the axis about which it rotates always points (approximately) to the same star in the heavens: and this star receives the name of *the polar star:* the stars being at distances so great that, neglecting small variations, the earth's axis during the whole of the earth's orbit about the sun being produced passes through the same star: hence we infer that the earth's axis is parallel to itself through the whole of the orbit: but as the polar star is not in the normal to the ecliptic, it appears that the axis of the earth's rotation is not perpendicular to the ecliptic: the mean value of the angle between it and the normal to the ecliptic is found to be 23° 27′ 30″. Now this fact gives rise to the seasons, and to the varying lengths of the day and the night on the earth: thus, in fig. 132, let S be the sun, and let A, B, C, D be four positions of the earth in its orbit: PQ being in each position the polar axis of the earth. It is evident that as the polar axis always retains its line parallel to itself, in two positions of the earth's orbit it is at right angles to the line drawn from the centre of the sun to the centre of the earth: let these positions be A and C; and at another position, say B, the angle between the polar axis towards the north, say P, and the sun's radius-vector is the least acute angle, being 66° 32′ 30″; and at another position D, the angle is the greatest obtuse angle, being equal to 113°27′30″. It is evident also that the positions A and C are at the extremities of a diameter of the earth's orbit, and that B and D are at the extremities of another diameter, which is perpendicular to the former. As the sun illumines only that part of the earth which is turned towards it, so is one half of the earth always enlightened, and the other is darkened: now in the position A, the diametral plane, which divides the enlightened and the darkened parts, passes through the two poles: and therefore as the earth during 24 hours revolves uniformly on its axis, so does every place on its surface describe half of its path in the light of the sun, and half of it out of it: thus the length of day and night is equal at every place on the surface of the earth; and hence the term *equinox;* and the earth in such a position is said to be at the equinox. As C is the other place in the orbit where the sun's radius-vector is perpendicular to the earth's polar axis, so is C also the other equinox: the preceding circumstances give, it is to be observed, the physical definition of equinox; viz. that position of the earth in its orbit, at which the earth's polar axis is perpendicular to the heliocentric radius-vector of the earth's centre. Let A be the position of the earth at the vernal equinox; C its position at the autumnal equinox.

Thus B is the position of the earth in the summer, when POS is 66° 32′ 30″, and which is called *the summer solstice*: in this case, as the earth revolves about its axis, the north pole P, and all places within the circle described by B as the earth revolves, are within the enlightened part of the sphere during the whole of a revolution; so that the north pole and all places within the circle (the geographical arctic circle) which is distant 23° 27′ 30″ from the pole have continual light for 24 hours: and the south pole Q and all places within 23° 27′ 30″ of the south pole (within the geographical antarctic circle) are in the darkened part of the sphere, and so have night during the 24 hours: and for all other places, the day decreases as we pass from the arctic to the antarctic circle, at the equator the length of the day being evidently 12 hours. It appears therefore that the north pole has during the summer continual light for six months, and that the south pole has continual darkness. In the fourth position, viz. D, of the earth, all the above phaenomena are exactly inverted, the north pole and all places within the arctic circle are immersed in total darkness during the whole diurnal revolution, and all places within the antarctic circle have continual light; and we have the phaenomena of *the winter solstice.*

406.] If the earth were a perfect sphere, the equinoxes and the solstices would every year take place at the same points of the path of the earth: but as the earth is (approximately) an oblate spheroid and revolves about its polar axis, and as the axis is not perpendicular to the plane of the ecliptic, the sun unequally attracts the protuberant masses at the earth's equator, by reason of the difference of their distances from the sun, and thus produces a change in the position of the axis of rotation. The resultant effect of this inequality of attraction is to cause the equinoxes and solstices to take place at points in the orbit about 50.4″ in advance of the points of the previous year. A similar effect is also produced by the attraction of the moon; and another of smaller amount, and in an opposite direction, by the planets. Let us confine our attention to the vernal equinox, that viz. when the earth has the position A. Thus the position A takes place each year 50.2″ nearer to D than it did in the preceding year. As the earth's axis has the same mean inclination to the normal of the ecliptic, the effect of this shifting of A, or of the precession of the vernal equinox, as it is called, is a change of the polar star; and so that in the course of as many years as 360° contains 50.2″, that is, in the course of 25,868 years, the pole of the earth will describe a circle in the heavens about the pole of the ecliptic, the angular radius of which is 23° 27′30″; and thus the polar star is continually changing. After a lapse of 12,000 years, says Sir John Herschel, the bright star α Lyræ will become the polar star of the earth. In addition to this regular precession of the equinoxes, there are also in the mathematical expression certain periodic terms, by reason of which the axis is sometimes in advance of, and sometimes behind, its mean place: and sometimes nearer to, and at other times farther from, the pole of the ecliptic. These variations are called *Nutation*. A complete investigation is given in the following volume of the present work.

Now a certain line passing through S is required from which angles may be measured: it might at first sight be thought that the major axis of the earth's orbit would serve the purpose: but by reason of the disturbing action of other bodies which circulate about the sun, the position of the major axis changes so much that it is ill adapted to such an use: and the line which is best suited is that which passes through one of the equinoxes or one of the solstices: and it is usual to take that which passes through the vernal equinox: that is, the line drawn from S to the centre of the earth at the position A. The fixed stars in the plane of the ecliptic have been divided into twelve equal parts, each consisting of 30°, and which are known by the names of "the signs of the Zodiac:" and the constellation *Aries* began at the position of the vernal equinox at the time when the names of the zodiacal constellations were given: but by reason of the precessional motion of the equinox, the vernal equinox has now passed out of the constellation Aries and is nearly at the beginning of Pisces; the line however drawn from the centre of the sun to the centre of the earth at the vernal equinox still bears the name of "the line drawn through the first point of Aries," and the earth at the vernal equinox is said to be in "the first point of Aries." Angles measured on the plane of the ecliptic from the line drawn through the first point of Aries are called *Longitudes*, and those perpendicular to the plane of the ecliptic are called *Latitudes*; these terms being used heliocentrically, and thus affording a complete system of spherical coordinates. The effect of the precession of the equinox is to increase the longitudes of all the fixed stars annually by 50.2″; and thus the values of these quantities given in the astronomical tables become yearly farther and farther removed from the true values.

407.] Although the orbit of each planet is in a plane containing the sun's centre, yet all the orbits are not in one and the same plane: neither do all the planes of the planets intersect the ecliptic in one and the same line; and therefore for the determination of the plane in which a planet moves in reference to the ecliptic it is necessary to know (1) the angle of inclination of the plane of the planet's orbit to the plane of the ecliptic: (2) the longitude of the line of intersection of the two planes; this line of intersection is called *the line of nodes*; the point at which the planet passes through the plane of the ecliptic in its passage from the south to the north of the ecliptic being called *the ascending node*, and the other being called *the descending node*. For the determination of the position of the orbit in its own plane, it is necessary to know the position of the major axis of the ellipse with reference to the line of nodes: the points in the orbit which are at the greatest and least distances from the sun, and which are of course the extremities of the major axis, are called *the aphelion* and *the perihelion* of the planet; and the position of the major axis is usually determined by means of the angle between the line of nodes and the perihelion distance; but as the inclination of the planes of all the larger planets to the ecliptic is very small, the projection of this angle on the plane of the ecliptic is very nearly equal to the angle itself in its own plane: and as we are obliged to have recourse to approximations in all problems of this kind, the longitude of perihelion is commonly given; which is the angle between the line of Aries and the perihelion distance, and which is measured on the plane of the ecliptic as far as the line of nodes, and on the plane of the planet's orbit from that line to the perihelion distance. Again, the magnitude of the orbit is determined by means of the semi major axis or mean distance and the eccentricity. These five quantities, which are called *elements of the orbit*, viz. the inclination, the longitude of the ascending node, the longitude of the perihelion distance, the mean distance, and the eccentricity, completely determine the plane, the position, and the magnitude, of the ellipse in which a planet moves. For the determination of its position in its orbit at a given time, two more elements are required: (1) its position at a certain given time: (2) the mean motion, which assigns mean velocity; or, and this is equivalent, the periodic time: this last element, see equation (154), Art. 404, requires a knowledge of the mass of the planet. The time at which the position of a planet is given is called *the epoch*, and is in the ordinary tables of planetary elements taken to be Jan. 1, 1800. The other element is the mass. Thus these two together with the preceding five are sufficient for the complete determination of a planet in its orbit: the first five and the seventh must be found by means of observation: the sixth is evidently arbitrary, and must therefore be conventionally assigned.

408.] Now the problem of the planetary theory in physical astronomy is the determination of the position of a planet at any time, when its position at some previous time is known by observation or otherwise: and the simplest form of the problem is when the planet is assumed to be moving in its own plane about the sun, and to be undisturbed by any other body: and this is the only form that I shall venture to enter upon in this treatise; so that the inclination, the longitude of the line of nodes, and the latitude of the planet will not enter into consideration; we have therefore to determine (1) the angular distance of the planet in its own plane from a certain fixed line in that plane, which I shall assume to be the perihelion distance, the epoch being the time at which the planet is in its perihelion; and (2) the planet's radius-vector, in terms of the time which has elapsed since the epoch. And from Art. 393, the equations which connect these quantities are

$$nt = u - e\sin u, \tag{155}$$

$$\tan\frac{\theta}{2} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan\frac{u}{2}, \tag{156}$$

$$r = a(1 - e\cos u); \tag{157}$$

which are three transcendental equations; in the first u and t are connected; and in the second and third θ and r are given in terms of u; our object therefore is so to express u in terms of t from (155), that we may be able to express θ and r in terms of t also. The problem is known by the name of Kepler's Problem.

We know by observation that the eccentricity of the orbits of all the larger planets is very small; so that the functions which require expansion may be expressed in ascending powers of e, and thereby terms involving powers of e higher than, say, the third may be omitted; n, it will be observed, is given in equation (102), Art. 393; and as the motion is relative, if M = the mass of the sun, m = the mass of the planet, a = the mean distance,

$$n = \frac{(M+m)^{\frac{1}{2}}}{a^{\frac{3}{2}}}. \tag{158}$$

We must in the first place expand θ in terms of u. In (156) let

$$\frac{1+e}{1-e} = m^2;$$

and since
$$(-1)^{\frac{1}{2}} \tan x = \frac{e^{2x\sqrt{-1}}-1}{e^{2x\sqrt{-1}}+1};$$

therefore from (156),

$$\frac{e^{\theta\sqrt{-1}}-1}{e^{\theta\sqrt{-1}}+1} = m\frac{e^{u\sqrt{-1}}-1}{e^{u\sqrt{-1}}+1};$$

$$\therefore \quad e^{\theta\sqrt{-1}} = \frac{(m+1)e^{u\sqrt{-1}}-(m-1)}{(m+1)-(m-1)e^{u\sqrt{-1}}}. \tag{159}$$

Let $\dfrac{m-1}{m+1} = \lambda$;

$$\therefore \quad e^{\theta\sqrt{-1}} = e^{u\sqrt{-1}}\,\frac{1-\lambda e^{-u\sqrt{-1}}}{1-\lambda e^{u\sqrt{-1}}};$$

and taking the logarithms of both sides of the equation,

$$\begin{aligned}
\theta\sqrt{-1} &= u\sqrt{-1}+\log(1-\lambda e^{-u\sqrt{-1}})-\log(1-\lambda e^{u\sqrt{-1}}) \\
&= u\sqrt{-1}-\{\lambda e^{-u\sqrt{-1}}+\frac{\lambda^2}{2}e^{-2u\sqrt{-1}}+\frac{\lambda^3}{3}e^{-3u\sqrt{-1}}+\ldots\} \\
&\qquad\qquad +\{\lambda e^{u\sqrt{-1}}+\frac{\lambda^2}{2}e^{2u\sqrt{-1}}+\frac{\lambda^3}{2}e^{3u\sqrt{-1}}+\ldots\} \\
&= u\sqrt{-1}+\lambda(e^{u\sqrt{-1}}-e^{-u\sqrt{-1}}) \\
&\qquad +\frac{\lambda^2}{2}(e^{2u\sqrt{-1}}-e^{-2u\sqrt{-1}})+\frac{\lambda^3}{3}(e^{3u\sqrt{-1}}-e^{-3u\sqrt{-1}})+\ldots \\
&= u\sqrt{-1}+2\lambda\sqrt{-1}\sin u \\
&\qquad\qquad +2\frac{\lambda^2}{2}\sqrt{-1}\sin 2u+2\frac{\lambda^3}{3}\sqrt{-1}\sin 3u+\ldots;
\end{aligned}$$

$$\therefore \quad \theta = u+2\,\{\lambda\sin u+\frac{\lambda^2}{2}\sin 2u+\frac{\lambda^3}{3}\sin 3u+\ldots\}. \tag{160}$$

But
$$\lambda = \frac{m-1}{m+1} = \frac{(1+e)^{\frac{1}{2}}-(1-e)^{\frac{1}{2}}}{(1+e)^{\frac{1}{2}}+(1-e)^{\frac{1}{2}}}$$
$$= \frac{1-(1-e^2)^{\frac{1}{2}}}{e} = \frac{e}{2} + \frac{e^3}{8} + \dots. \qquad (161)$$

Thus to express θ and r in terms of t we must find u, $\sin u$, $\sin 2u$, $\sin 3u, \dots$ and $\cos u$ in terms of t by means of (155); and for this purpose the theorem of Lagrange, see Differential Calculus, Art. 90, is to be used, being indeed discovered for the solution of this very problem.

409.] Given $u = nt + e\sin u$, it is required to find u, $\sin u$, $\sin 2u$, $\sin 3u, \dots \cos u$, in terms of t and e.

If
$$y = z + x\phi(y), \qquad (162)$$
then by Lagrange's theorem,
$$f(y) = f(z) + f'(z)\phi(z)\frac{x}{1}$$
$$+ \frac{d}{dz}\{f'(z)\{\phi(z)\}^2\}\frac{x^2}{1.2} + \frac{d^2}{dz^2}\{f'(z)\{\phi(z)\}^3\}\frac{x^3}{1.2.3} + \dots;$$
and comparing (162) with $u = nt + e\sin u$, we have, by means of this series,
$$y = z + \sin z\frac{x}{1} + \frac{d}{dz}(\sin z)^2\frac{x^2}{1.2} + \frac{d^2}{dz^2}(\sin z)^3\frac{x^3}{1.2.3} + \dots$$
$$\sin y = \sin z + \cos z \sin z\frac{x}{1} + \frac{d}{dz}\{\cos z(\sin z)^2\}\frac{x^2}{1.2} + \dots$$
$$\sin 2y = \sin 2z + 2\cos 2z \sin z\frac{x}{1} + \frac{d}{dz}\{2\cos 2z(\sin z)^2\}\frac{x^2}{1.2} + \dots$$
$$\sin 3y = \sin 3z + 3\cos 3z \sin z\frac{x}{1} + \frac{d}{dz}\{3\cos 3z(\sin z)^2\}\frac{x^2}{1.2} + \dots$$

. . . .

$$\cos y = \cos z - (\sin z)^2\frac{x}{1} - \frac{d}{dz}(\sin z)^3\frac{x^2}{1.2} - \frac{d^2}{dz^2}(\sin z)^4\frac{x^3}{1.2.3} + \dots.$$
Therefore substituting
$$u = nt + \sin nt\frac{e}{1} + \sin 2nt\frac{e^2}{1.2} + (3\sin 3nt - \sin nt)\frac{e^3}{8} + \dots$$
$$\sin u = \sin nt + \sin 2nt\frac{e}{2} + (3\sin 3nt - \sin nt)\frac{e^2}{8} + \dots$$
$$\sin 2u = \sin 2nt + (\sin 3nt - \sin nt)e + \dots$$
$$\sin 3u = \sin 3nt + \dots$$

. . . .

$$\cos u = \cos nt - (1 - \cos 2nt)\frac{e}{2} + (3\cos 3nt - 3\cos nt)\frac{e^2}{8} + \dots;$$

therefore substituting these values in (157) and (160), and also replacing λ by its value from (161), and omitting terms involving powers of e higher than the third, we have

$$\theta = nt + 2e \sin nt + \frac{5e^2}{4} \sin 2nt + \frac{e^3}{12}(13 \sin 3nt - 3 \sin nt) + \ldots ; \quad (163)$$

$$r = a\{1 - e \cos nt + \frac{e^2}{2}(1 - \cos 2nt) - \frac{3e^3}{8}(\cos 3nt - \cos nt) + \ldots\}; \quad (164)$$

which give us the true anomaly, and the radius-vector, in terms of the mean anomaly or the time, the approximations being carried correctly as far as terms involving the cube of the eccentricity; and omitting periodic terms, we have for the mean values of θ and r,

$$\theta = nt, \text{ (the mean anomaly)}, \quad (165)$$

$$r = a\left(1 + \frac{e^2}{2}\right). \quad (166)$$

If the line from which the angles are measured is that drawn through the first point of Aries, so that we have longitude instead of anomaly; then, if θ is the longitude of the planet at the time t measured in the manner explained in Art. 407, and if ϖ is the longitude of the perihelion, measured in the same manner, θ in (163) must be replaced by $\theta - \varpi$; and if the time begins, that is, if $t = 0$, when the planet is at a place whose mean longitude is ϵ, then in (163) and (164) nt must be replaced by $nt + \epsilon - \varpi$. It is unnecessary however to write down the preceding equations when these substitutions have been made, as the form of them is of course unaltered; ϵ is called the epoch.

410.] The preceding method is that which is most convenient when the equivalent values of r and θ in terms of t are required to high powers of e, say to e^7 or higher. If however only the first three or four terms of each series are required, the following process of successive approximation may be employed:

Since $r = \dfrac{a(1-e^2)}{1 + e\cos\theta}$, $r^2\, d\theta = h\, dt$, and $h = \{\mu a (1-e^2)\}^{\frac{1}{2}}$;

$$\therefore \quad dt = \frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} (1-e^2)^{\frac{3}{2}} (1 + e\cos\theta)^{-2}\, d\theta;$$

whence substituting as in (102) Art. 393, and expanding as far as e^3, we have

$$n\, dt = \left(1 - \frac{3e^2}{2}\right)\{1 - 2e\cos\theta + 3e^2(\cos\theta)^2 - 4e^3(\cos\theta)^3\}\, d\theta$$

$$= \left\{1 - 2e\cos\theta + \frac{3e^2}{2}\cos 2\theta - e^3 \cos 3\theta\right\} d\theta.$$

And supposing $t=0$ when $\theta=0$; that is, assuming the epoch to be when the planet is in perihelion,

$$nt = \theta - 2e\sin\theta + \frac{3e^2}{4}\sin 2\theta - \frac{e^3}{3}\sin 3\theta; \qquad (167)$$

which series gives the mean anomaly in terms of the true anomaly. But as the true anomaly is required in terms of the mean anomaly, it is necessary to revert the series: this process is effected by means of successive approximation. From (167) we have

$$\theta = nt + 2e\sin\theta - \frac{3e^2}{4}\sin 2\theta + \frac{e^3}{3}\sin 3\theta;$$

whence we have the following successive approximations to the value of θ:

(1) $\theta = nt.$

(2) $\theta = nt + 2e\sin nt.$

(3)
$$\begin{aligned}\theta &= nt + 2e\sin\{nt + 2e\sin nt\} - \frac{3e^2}{4}\sin 2nt \\ &= nt + 2e\{\sin nt + 2e\sin nt\cos nt\} - \frac{3e^2}{4}\sin 2nt \\ &= nt + 2e\sin nt + \frac{5e^2}{4}\sin 2nt.\end{aligned}$$

(4)
$$\begin{aligned}\theta &= nt + 2e\sin\left\{nt + 2e\sin nt + \frac{5e^2}{4}\sin 2nt\right\} \\ &\qquad - \frac{3e^2}{4}\sin 2\{nt + 2e\sin nt\} + \frac{e^3}{3}\sin 3nt \\ &= nt + 2e\sin nt + \frac{5e^2}{4}\sin 2nt + \frac{e^3}{12}(13\sin 3nt - 3\sin nt);\end{aligned}$$

which last is the same value as that given in (163). The labour however required in investigating higher approximations is so great, that the former method is far easier. Again,

$$\begin{aligned}r &= \frac{a(1-e^2)}{1+e\cos\theta} = a(1-e^2)(1+e\cos\theta)^{-1} \\ &= a\{1 - e\cos\theta - e^2(\sin\theta)^2\}, \text{ to terms involving } e^2;\end{aligned}$$

whence we have the following successive approximations:

(1) $r = a.$

(2) $r = a(1 - e\cos nt).$

(3) $r = a\left\{1 - e\cos nt + \frac{e^2}{2}(1 - \cos 2nt)\right\};$

which is the same result as (164). It is unnecessary to calculate higher terms, for the labour of doing so becomes very great.

411.] The excess of the true anomaly at any time over the mean anomaly, that is, $\theta - nt$, is called by astronomers *the equation of the centre;* and is positive or negative according as the planet is in advance of or behind its mean place: the equation of the centre evidently is zero at perihelion and at aphelion. The point in the orbit at which it is a maximum may thus be found.

Since by (10) Art. 379, (93) Art. 392, and (102) Art. 393,

$$\begin{aligned} r^2\,d\theta &= h\,dt \\ &= na^2(1-e^2)^{\frac{1}{2}}\,dt; \end{aligned} \tag{168}$$

therefore

$$d(\theta-nt) = \frac{a^2(1-e^2)^{\frac{1}{2}}-r^2}{a^2(1-e^2)^{\frac{1}{2}}}\,d\theta;$$

and this = 0, if

$$r = a(1-e^2)^{\frac{1}{4}}; \tag{169}$$

that is, when r is a mean proportional between the two principal semi-axes. And to determine the corresponding maximum value of the equation of the centre, let E represent it; then

$$\begin{aligned} \text{E} &= \theta-nt; \\ &= \theta-u+e\sin u. \end{aligned} \tag{170}$$

But from the equation to the ellipse, and from (169),

$$\cos\theta = -\frac{3e}{4}-\frac{3e^3}{32};$$

$$\therefore\quad \theta = \frac{\pi}{2}+\frac{3e}{4}+\frac{21e^3}{128}+\cdots\cdot \tag{171}$$

Also from (157) and (169),

$$\cos u = \frac{e}{4}+\frac{3e^3}{32}+\cdots;$$

$$\therefore\quad u = \frac{\pi}{2}-\frac{e}{4}-\frac{37e^3}{384}, \tag{172}$$

$$e\sin u = e-\frac{e^3}{32};$$

$$\therefore\quad \text{E} = 2e+\frac{11e^3}{48}+\cdots, \tag{173}$$

which is the maximum value of the equation of the centre. And since from (172) it appears that u is at the corresponding point less than 90°, it follows that the equation is a maximum at a point before the planet comes to the extremity of the minor axis. It may also by a similar process be shewn that in the passage of the planet from aphelion to perihelion, the equation of the centre is a maximum at a point between the extremity of the minor axis and the perihelion.

Also since mean time depends on the earth's mean position in its orbit, that is, on nt, the equation of the centre is the difference between true solar time and mean solar time, in so far as the difference is due to the varying velocity of the earth in its orbit: for this reason $\theta-nt$ is also called *the elliptic inequality.*

412.] It is necessary for us still to consider certain properties of parabolic orbits, because there is reason to suppose that some comets move either in such orbits, or in ellipses of which the eccentricity is so nearly equal to unity that the orbits appear to us, so far as our calculations go when the comets are within observation, to be parabolic.

The elements of a parabolic orbit are (see Art. 407) only six in number; the inclination and the longitude of the line of nodes are required as in the elliptic orbit, whereby the position of the plane of the comet's orbit may be determined. The longitude of the perihelion distance, the mean motion, and the epoch are also required; and as the eccentricity is unity, the perihelion distance is sufficient for the determination of the magnitude of the orbit.

Let us investigate the relation between the time and the angle measured from the perihelion distance which is due to the time in an elliptic orbit of which the eccentricity is nearly unity; that is, when the orbit is nearly parabolic.

Let $\theta =$ the angle measured from the perihelion distance, and $r =$ the corresponding radius-vector, so that

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$

Let $p =$ the perihelion distance; viz. $p = a(1-e)$;

$$\therefore \quad r = p(1+e)(1+e\cos\theta)^{-1}: \tag{174}$$

and if $t =$ the time during which θ from perihelion is described, then from Art. 381, (26),

$$h^2 = \mu a(1-e^2) = \mu p(1+e); \tag{175}$$

$$\begin{aligned} \therefore \quad dt &= \frac{r^2}{h}\,d\theta \\ &= \frac{p^{\frac{3}{2}}(1+e)^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \frac{d\theta}{\left(1+e\left\{\left(\cos\frac{\theta}{2}\right)^2-\left(\sin\frac{\theta}{2}\right)^2\right\}\right)^2} \\ &= \frac{p^{\frac{3}{2}}(1+e)^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \frac{d\theta}{\left\{(1+e)\left(\cos\frac{\theta}{2}\right)^2+(1-e)\left(\sin\frac{\theta}{2}\right)^2\right\}^2} \\ &= \frac{2p^{\frac{3}{2}}}{\mu^{\frac{1}{2}}(1+e)^{\frac{1}{2}}} \frac{\left(\sec\frac{\theta}{2}\right)^2 d.\tan\frac{\theta}{2}}{\left\{1+\mathrm{E}\left(\tan\frac{\theta}{2}\right)^2\right\}^2}, \end{aligned} \tag{176}$$

if $\mathrm{E} = \frac{1-e}{1+e}$, and which is therefore a small quantity, when e is nearly equal to unity. Therefore expanding (176), and omitting terms involving powers of E higher than E^3, and integrating between the limits $\theta = \theta$ and $\theta = 0$, we have

$$t = \frac{2p^{\frac{3}{2}}}{\mu^{\frac{1}{2}}(1+e)^{\frac{3}{2}}}\left\{\tan\frac{\theta}{2} + \frac{1-2E}{3}\left(\tan\frac{\theta}{2}\right)^3 - \frac{E(2-3E)}{5}\left(\tan\frac{\theta}{2}\right)^5 \right.$$
$$\left. + \frac{E^2(3-4E)}{7}\left(\tan\frac{\theta}{2}\right)^7 - \frac{4E^3}{9}\left(\tan\frac{\theta}{2}\right)^9\right\}. \quad (177)$$

If the orbit becomes a parabola, then $e = 1$, $E = 0$; and we have

$$t = \frac{2^{\frac{1}{2}}p^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}\left\{\tan\frac{\theta}{2} + \frac{1}{3}\left(\tan\frac{\theta}{2}\right)^3\right\}; \quad (178)$$

which result is the same as that before investigated in Art. 381.

The preceding expressions are those commonly used for the calculation of the true anomaly, and also of the true longitude in the case of comets which move in ellipses of eccentricities which are nearly equal to unity, or in parabolas. In the actual use of (178), the determination of θ in terms of t requires the solution of a cubic equation which has only one real root. And to avoid the difficulty of the solution, a table is formed containing the values of t corresponding to the values of θ in the parabola whose least focal distance is unity; and this being multiplied by $p^{\frac{3}{2}}$ gives the time corresponding to the anomaly θ in the given parabola.

As it is very uncertain whether any celestial bodies move in hyperbolic orbits, it is unnecessary for us to develope the equations which express the relation between the time and the longitude in such orbits.

413.] It remains for us still to indicate means by which the masses of the planets may be deduced, at least approximately, from the laws of elliptic motion.

If $T =$ the periodic time of a planet about the sun, $a =$ the mean distance of the planet's orbit, if m and M are the masses of the planet and of the sun respectively, then by (154), Art. 404,

$$T = \frac{2\pi a^{\frac{3}{2}}}{(m+M)^{\frac{1}{2}}}. \quad (179)$$

Let T' be the periodic time of another planet of which the mass is m' and of whose orbit the mean distance is a'; then

$$T' = \frac{2\pi a'^{\frac{3}{2}}}{(m'+M)^{\frac{1}{2}}};$$

$$\therefore \quad \frac{T^2}{T'^2} = \left(\frac{a}{a'}\right)^3 \frac{m'+M}{m+M}. \quad (180)$$

But by Kepler's third law, the squares of the periodic times vary as the cubes of the mean distances; and it is found by observation that this law is true with very slight variations; it follows therefore that the second factor in the right-hand member of (180) is nearly constant: and thus the masses of the planets must in comparison of that of the sun be so small, that the preceding process does not enable us to determine the ratio between them.

Suppose however the planet to have a satellite; and m' to be the mass of the satellite, a' being the mean distance of its orbit about its primary, and T' being its periodic time: then if m is the mass of the planet,

$$\mathrm{T}' = \frac{2\pi a'^{\frac{3}{2}}}{(m+m')^{\frac{1}{2}}};$$

therefore from (179),

$$\frac{\mathrm{T}^2}{\mathrm{T}'^2} = \left(\frac{a}{a'}\right)^3 \frac{m+m'}{\mathrm{M}+m}. \tag{181}$$

Now if m is so small in comparison of M (the mass of the sun) that it may be neglected without very great error; and also if the mass of the satellite be similarly small in comparison of the mass of the planet; then (181) becomes

$$\frac{\mathrm{T}^2}{\mathrm{T}'^2} = \left(\frac{a}{a'}\right)^3 \frac{m}{\mathrm{M}}; \tag{182}$$

and thus T, T', a, a' having been determined by observation, the ratio of m to M is known. When this method is applied to the determination of the mass of Jupiter by means of its fourth satellite, it is found that

$$m = \frac{\mathrm{M}}{1066.09}; \tag{183}$$

and thus may the masses of all the planets which have satellites be compared with the mass of the sun.

The masses however are only *compared* by this process: no one is absolutely determined: and to accomplish this object it is necessary to find the mass of at least one: and the one which naturally offers itself for the purpose is the earth: hence arises the necessity of direct observation of the figure, magnitude, and density of the earth. Many processes have been devised; but the most reliable is that of direct geodesic measurement: and arcs of meridian have been measured in England, France, Russia, India, and the south of Africa: and from them the form and the magnitude of the earth [illegible]termined. The density has been directly investigated by means of Dr. Maskelyne's observations with the pendulum near to the Schehallien mountain in Scotland*. And also by experiments with leaden balls, which were conducted by Cavendish, and have been subsequently repeated with great care by the late Mr. Baily, in the time intervening between Oct. 1838 and May 1842; and from these experiments it appears that the mean density of the earth is equal to 5.66 times that of distilled water. But this is not the place for entering into a detailed explanation of such experiments and observations. The article on the Figure of the Earth in the Encyclopedia Metropolitana, by Mr. Airy, the astronomer royal, contains all necessary information on the former subject: and for that on the latter I must refer the reader to the Philosophical Transactions, and to Vol. XIV of the Memoirs of the Royal Astronomical Society.

SECTION 4.—*The polar equation of motion of a disturbed planet.*

414.] In Article 367 we have calculated the equations of motion of a particle m relatively to M, when both of the particles are acted on by another particle m', the law of attraction of all three being that of the mass directly and of the square of the distance inversely. As the problem is evidently that of the moon moving about the earth, both being acted on by the sun, which is a disturbing body, it deserves further consideration; and I therefore propose to indicate the process by which equations (162), Art. 367, are transformed into their equivalents in terms of polar coordinates.

Let us suppose the earth to be the central body m, relatively to which the motion of M, the moon, is calculated: let m' be the mass of the sun: let the plane of the ecliptic be that of (x, y), so that the sun is always in that plane, and therefore its z-ordinate is zero: let the positions of M and m' at the time t respectively be (x, y, z) and (x', y'): let r and r' be the radii vectores of M and of m': firstly, let the acting forces be resolved in and perpendicular to the ecliptic, which is the plane of (x, y): and, to fix the thoughts, let the supposed system be represented in fig. 133; P being the place of the moon, and P′ that of the sun at the time t. Let the velocity-increments acting on M in the line perpendicular to the plane of (x, y) be called the *orthogonal* velocity-increments. Let ON $= \rho$, the moon's curtate radius-vector; NOM $= \theta$, the moon's longitude; P′OM′ $= \theta'$, the sun's longitude; also let $s =$ the tangent of the angle PON, which is the moon's latitude; let P = the impressed velocity-increment on M in the plane of the ecliptic along ON, and acting to diminish ON; similarly, let T = that which is in the same plane, and is perpendicular to ON, and tends to increase θ; S = that which is orthogonal, and tends to diminish the angle PON, and thus to bring the moon nearer to the plane of the ecliptic. Then using the notation of Art. 367, if $\mu = m + \text{M}$,

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* See Philosophical Transactions, 1811.

$$\left.\begin{aligned}
\text{P} &= \frac{\mu\rho}{r^3} + \cos\theta\left(\frac{d\text{R}}{dx}\right) + \sin\theta\left(\frac{d\text{R}}{dy}\right),\\
\text{T} &= \sin\theta\left(\frac{d\text{R}}{dx}\right) - \cos\theta\left(\frac{d\text{R}}{dy}\right),\\
\text{S} &= \frac{\mu z}{r^3} + \left(\frac{d\text{R}}{dz}\right).
\end{aligned}\right\} \tag{184}$$

And since
$$\left.\begin{aligned} x &= \rho\cos\theta,\\ y &= \rho\sin\theta; \end{aligned}\right\} \qquad \left.\begin{aligned} x' &= r'\cos\theta',\\ y' &= r'\sin\theta'; \end{aligned}\right\} \tag{185}$$

and since $s = \tan \text{PON}$, so that

$$z = \rho s; \tag{186}$$

we have

$$\left.\begin{aligned}
\frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta + \text{P} &= 0;\\
\frac{d^2x}{dt^2}\sin\theta - \frac{d^2y}{dt^2}\cos\theta + \text{T} &= 0;\\
\frac{d^2z}{dt^2} + \text{S} &= 0.
\end{aligned}\right\} \tag{187}$$

Let us first consider the equations as they refer to the projection of the moon's path on the plane of the ecliptic; for which purpose, from the first two of (187), we have

$$\left.\begin{aligned}
\frac{d^2\rho}{dt^2} - \rho\frac{d\theta^2}{dt^2} + \text{P} &= 0,\\
\rho\frac{d^2\theta}{dt^2} + 2\frac{d\rho}{dt}\frac{d\theta}{dt} - \text{T} &= 0;
\end{aligned}\right\} \tag{188}$$

which equations are also given directly in (62), Art. 324.

Multiplying the last of these equations by ρ, we have

$$\begin{aligned}
\rho\text{T} &= \rho^2\frac{d^2\theta}{dt^2} + 2\rho\frac{d\rho}{dt}\frac{d\theta}{dt}\\
&= \frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right);
\end{aligned} \tag{189}$$

$$\therefore \quad 2\rho^3 \mathrm{T}\, d\theta = 2\rho^2\frac{d\theta}{dt}\, d\left(\rho^2\frac{d\theta}{dt}\right);$$

therefore integrating, and introducing an arbitrary constant h^2, the meaning of which will have to be determined hereafter, we have

$$\left(\rho^2\frac{d\theta}{dt}\right)^2 = h^2+2\int\rho^3 \mathrm{T}\, d\theta; \tag{190}$$

$$\therefore \quad \frac{d\theta^2}{dt^2} = \frac{h^2}{\rho^4}+\frac{2}{\rho^4}\int\rho^3 \mathrm{T}\, d\theta. \tag{191}$$

Let $\rho = \frac{1}{u}$; then

$$\frac{d\theta^2}{dt^2} = h^2u^4+2u^4\int\frac{\mathrm{T}}{u^3}\, d\theta. \tag{192}$$

By means of this equation we can eliminate t from the first of (188); and let us assume θ to be equicrescent in the new equations. Since

$$\rho = \frac{1}{u};$$

$$\therefore \quad \frac{d\rho}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\rho^2\frac{d\theta}{dt}\frac{du}{d\theta};$$

$$\therefore \quad \frac{d^2\rho}{dt^2} = -\rho^2\frac{d\theta}{dt}\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}-\frac{du}{d\theta}\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right)$$

$$= -\frac{1}{u^2}\frac{d^2u}{d\theta^2}\frac{d\theta^2}{dt^2}-\frac{du}{d\theta}\frac{\mathrm{T}}{u};$$

therefore from (188),

$$\left(\frac{d^2u}{d\theta^2}+u\right)\frac{d\theta^2}{dt^2}+\mathrm{T}u\frac{du}{d\theta}-\mathrm{P}u^2 = 0; \tag{193}$$

and substituting for $\frac{d\theta^2}{dt^2}$ from (191), we have

$$\frac{d^2u}{d\theta^2}+u-\frac{\dfrac{\mathrm{P}}{u^2}-\dfrac{\mathrm{T}}{u^3}\dfrac{du}{d\theta}}{h^2+2\displaystyle\int\frac{\mathrm{T}}{u^3}\, d\theta} = 0, \tag{194}$$

which is the differential equation of the moon's curtate radius-vector in terms of the longitude.

Again, taking the third equation of (187), since

$$z = \rho s = \frac{s}{u},$$

$$\frac{dz}{dt} = \left(u\frac{ds}{d\theta}-s\frac{du}{d\theta}\right)\rho^2\frac{d\theta}{dt};$$

$$\therefore \quad \frac{d^2z}{dt^2} = \left(u\frac{d^2s}{d\theta^2}-s\frac{d^2u}{d\theta^2}\right)\rho^2\frac{d\theta^2}{dt^2}+\left(u\frac{ds}{d\theta}-s\frac{du}{d\theta}\right)\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right);$$

and substituting from (189), (192) and (194), we have

$$\frac{d^2 s}{d\theta^2} + s + \frac{\dfrac{\text{S} - \text{P}s}{u^3} + \dfrac{\text{T}}{u^3}\dfrac{ds}{d\theta}}{h^2 + 2\displaystyle\int \frac{\text{T}\,d\theta}{u^3}} = 0; \tag{195}$$

which is the differential equation to the moon's motion in latitude.

415.] In respect of the three equations (191), (194), (195) it is to be noticed that the values of P, T, S, as given in (184) and taken from Art. 367, involve both the mass and the coordinates of the place of m', the disturbing body: with this difference however, that T is a function of these quantities only, and that P and S contain also quantities which arise immediately from the action of M on m; and thus, if the disturbing forces are omitted, the equations (188) become identical with (10) and (21) of the present Chapter. Now when substitutions are made for P, T and S in terms of the polar coordinates of M and m', the preceding equations are not integrable; and we are obliged to have recourse to a method of approximation; and for this end, the disturbing function R is first expressed in terms of the polar coordinates of M and m', and its partial derived-functions are expanded in ascending powers of small quantities, the values of which must be given by the observed data of the problem; thus, for instance, the mean distance of the sun from the earth is nearly 400 times as great as the mean distance of the moon; if therefore the distance of the earth from the moon is assumed as the standard of comparison, the distance of the sun from the earth will be such that higher powers of it may be neglected. And when these expansions have been performed, the integration-process takes place. But it is beyond the object of the present work to enter on these investigations.

Equation (195) however, when the disturbing forces are omitted, deserves consideration: for we have only

$$\frac{d^2 s}{d\theta^2} + s = 0;$$

whence integrating, $\qquad s = k \sin(\theta - \gamma), \qquad$ (196)

where k and γ are two constants introduced in integration. Now s is the tangent of the latitude, and θ is the longitude of the moon measured from the first point of Aries; and thus

γ is the longitude of the ascending node. Suppose then, in fig. 134, NMP to be a right-angled spherical triangle, P being the place of the moon, N the ascending node, and NM the plane of the ecliptic, then $NM = \theta - \gamma$, $MP = \tan^{-1} s$; and by Napier's rules,

$$\sin MN = \tan MP \times \cot PNM,$$

or,

$$\sin(\theta - \gamma) = s \times \cot PNM; \tag{197}$$

on comparing which with (196), $\tan PNM = k$, that is, is constant; and therefore the moon moves in a plane making a constant angle with the ecliptic; or, in other words, the moon would move in a plane, if it were undisturbed.

CHAPTER XII.

THE CONSTRAINED MOTION OF PARTICLES, UNDER THE ACTION OF GIVEN FORCES.

SECTION 1.—*The motion of a particle constrained to move on a given curved line.*

416.] The subjects of motion thus far have been particles not constrained by any geometrical conditions; they have therefore been free to take in space such paths as are due to the action of the impressed forces: it remains now to investigate the motion of particles which are constrained; that is, in which the motion is subject to certain geometrical relations: such is that of a particle in a small tube, either smooth or rough, and the bore of which is supposed to be of the same size as the particle: of a small ring sliding on a curved wire, with or without friction: of a particle fastened to a string, or moving on a given surface. The principles of the science which were investigated in Chapter IX are of breadth sufficient for this inquiry. For in addition to the impressed momentum-increments on the moving particle, there will in general be a normal pressure or reaction of the curve or surface, arising from the deflexion of the particle from the path which it would have under the action of the impressed forces alone: and this pressure together with the expressed momentum-increments will be equal to the impressed momentum-increments.

417.] In this section I propose to consider the motion of a particle constrained to move on a given curve; such as that of a particle in a tube, or of a small ring on a wire, or of a particle fastened to a string. And, to take the general case, I shall suppose firstly the curve to be in space, and the motion to be free from friction.

Let the equations to the curve on which the particle, of mass m, is constrained to move, be

$$\mathrm{F}_1(x, y, z) = 0, \qquad \mathrm{F}_2(x, y, z) = 0; \qquad (1)$$

and let (x, y, z) be the place of m at the time t: let X, Y, Z be the coordinate components of the impressed velocity-increments; let R be the normal pressure, and let α, β, γ be the direction-angles of its line of action; so that the equations of motion are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= m\text{X}-\text{R}\cos\alpha, \\ m\frac{d^2y}{dt^2} &= m\text{Y}-\text{R}\cos\beta, \\ m\frac{d^2z}{dt^2} &= m\text{Z}-\text{R}\cos\gamma; \end{aligned}\right\} \qquad (2)$$

and dividing by m, whereby the equations become expressed in terms of velocity-increments,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \text{X}-\frac{\text{R}}{m}\cos\alpha, \\ \frac{d^2y}{dt^2} &= \text{Y}-\frac{\text{R}}{m}\cos\beta, \\ \frac{d^2z}{dt^2} &= \text{Z}-\frac{\text{R}}{m}\cos\gamma. \end{aligned}\right\} \qquad (3)$$

The line of action of R cannot be definitely determined; we know no more of it, at present, than that it is perpendicular to the tangent of the curvilinear path at the place of the particle; so that we have

$$dx\cos\alpha + dy\cos\beta + dz\cos\gamma = 0: \qquad (4)$$

neither do we know the *direction* in which R acts.

Now multiplying the equations of (3) severally by dx, dy, dz, and adding, we have

$$\frac{dx\,d^2x + dy\,d^2y + dz\,d^2z}{dt^2} = \text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz. \qquad (5)$$

Let the circumstances of motion be considered at the times t and t_0; and let us suppose that $\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz$ is such a function of x, y, z as to admit of integration; and let v and v_0 be the corresponding velocities; then integrating (5), we have

$$\frac{mv^2 - mv_0^2}{2} = \int_{t_0}^{t} m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz); \qquad (6)$$

whereby the velocity is found by simple integration at any point of the path.

This is the equation of vis viva and of work: the normal pressure does not appear in it; because the particle moves perpendicularly to its action-line, and it does no work.

Also since

$$\frac{d^2 s}{dt^2} = \frac{d^2 x}{dt^2}\frac{dx}{ds} + \frac{d^2 y}{dt^2}\frac{dy}{ds} + \frac{d^2 z}{dt^2}\frac{dz}{ds};$$

$$\therefore \quad \frac{d^2 s}{dt^2} = \frac{\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz}{ds}; \tag{7}$$

and is therefore independent of the resistance of the curve; the velocity-increment therefore of the particle along the curvilinear path is the same as if the particle were moving freely along that path.

Since $v^2 = \frac{ds^2}{dt^2}$, we have

$$dt = \frac{\pm ds}{\{v_0^2 + 2\int_{t_0}^{t}(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz)\}^{\frac{1}{2}}}; \tag{8}$$

whence the time may be found in terms of the coordinates of the particle at t and t_0.

The normal pressure on the curve may thus be found: multiplying (3) severally by $\cos\alpha$, $\cos\beta$, $\cos\gamma$,

$$\frac{\text{R}}{m} = \Big(\text{X} - \frac{d^2 x}{dt^2}\Big)\cos\alpha + \Big(\text{Y} - \frac{d^2 y}{dt^2}\Big)\cos\beta + \Big(\text{Z} - \frac{d^2 z}{dt^2}\Big)\cos\gamma. \tag{9}$$

Now suppose the line of action of the normal pressure to be the principal normal: then if $\rho =$ the radius of absolute curvature,

$$\cos\alpha = \pm\rho\frac{d}{ds}\frac{dx}{ds}, \qquad \cos\beta = \pm\rho\frac{d}{ds}\frac{dy}{ds}, \qquad \cos\gamma = \pm\rho\frac{d}{ds}\frac{dz}{ds},$$

the double sign referring to the value of ρ, and which is to be positive or negative, according to the *direction* in which it is measured, so that it may finally bear a positive sign, because it is an absolute length. And replacing in (9) $\cos\alpha$, $\cos\beta$, $\cos\gamma$ by these values, we have

$$\frac{\text{R}}{m} = \text{X}\cos\alpha + \text{Y}\cos\beta + \text{Z}\cos\gamma$$
$$\pm\rho\Big\{\frac{d^2 x}{dt^2}\frac{d}{ds}\frac{dx}{ds} + \frac{d^2 y}{dt^2}\frac{d}{ds}\frac{dy}{ds} + \frac{d^2 z}{dt^2}\frac{d}{ds}\frac{dz}{ds}\Big\};$$

$$\therefore \quad \text{R} = m\Big\{\text{X}\cos\alpha + \text{Y}\cos\beta + \text{Z}\cos\gamma \pm \frac{v^2}{\rho}\Big\}. \tag{10}$$

$\frac{v^2}{\rho}$ is the centrifugal force; see Art. 326: thus if the normal pressure on the curve acts along the radius of absolute curvature, it is equal to the algebraical sum of the resolved parts of the impressed momentum-increments along that radius of absolute curvature and of the centrifugal force.

And the normal pressure is along the principal normal when the curve is described by the moving particle without the action of any continually impressed forces: for in this case,

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{\text{R}}{m}\cos\alpha,\\ \frac{d^2y}{dt^2} &= -\frac{\text{R}}{m}\cos\beta,\\ \frac{d^2z}{dt^2} &= -\frac{\text{R}}{m}\cos\gamma;\end{aligned}\right\}$$

therefore

$$\frac{dx\,d^2x+dy\,d^2y+dz\,d^2z}{dt^2} = -\frac{\text{R}}{m}\{dx\cos\alpha+dy\cos\beta+dz\cos\gamma\}$$
$$= 0;$$

$$\therefore\ \frac{ds^2}{dt^2} = (\text{velocity})^2 = \text{a constant} = \frac{1}{k^2},\ \text{say}.$$

Therefore $dt = k\,ds$, and therefore s and t are simultaneously equicrescent: and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are severally proportional to $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$, and are therefore the direction-cosines of the principal normal.

R is, it will be observed, a pressure or a momentum; and is therefore compounded of two factors, of which one is mass and the other is velocity; and as R varies directly as m, $\frac{\text{R}}{m}$ is velocity. Thus equation (9) is homogeneous, and is formed in terms of velocity: equation (10), on the other hand, is formed in terms of momentum or pressure. It is convenient to have a distinctive name for the velocity, which is lost or gained by reason of the motion being constrained, and I shall call it the reaction of the curve: so that reaction is the pressure on the curve of an unit-mass; and the pressure of m on the curve is the product of the mass and the reaction.

Equations (7) and (9) or (10) are the tangential and normal components of the velocity-increments, and the result is therefore equivalent to a tangential and normal resolution.

418.] If however the motion takes place wholly in one plane, we may take that to be the plane of reference, and the general formulae are simplified accordingly.

If the motion is referred to rectangular axes of x and y, and X and Y are the axial-components of the impressed velocity-increments, the equations of motion are

$$\frac{d^2x}{dt^2} = X \mp \frac{R}{m}\frac{dy}{ds}; \qquad \frac{d^2y}{dt^2} = Y \pm \frac{R}{m}\frac{dx}{ds}. \tag{11}$$

If T and N are the tangential and normal components of the impressed velocity-increments,

$$\frac{d^2s}{dt^2} = T; \qquad \frac{v^2}{\rho} = N \pm \frac{R}{m}. \tag{12}$$

If the system of radial and transversal resolution is taken, and P and Q are the radial and transversal components of the impressed velocity-increment, then

$$\left.\begin{aligned} \frac{d^2r}{dt^2} - r\frac{d\theta^2}{dt^2} &= P \pm \frac{R}{m}\frac{r\,d\theta}{ds}; \\ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) &= Q \mp \frac{R}{m}\frac{dr}{ds}. \end{aligned}\right\} \tag{13}$$

The ambiguity of sign of R in all these cases arises from the *a priori* uncertainty of the side of the tube against which the particle presses.

419.] The equation of vis viva, or of work, may be deduced from the preceding expressions in the following manner :

Multiplying the equations (11) respectively by $m\,dx$ and $m\,dy$, and adding; we have

$$m\frac{dx\,d^2x + dy\,d^2y}{dt^2} = m\,(X\,dx + Y\,dy);$$

the left-hand member of which is equal to $\frac{m\,ds\,d^2s}{dt^2}$; consequently integrating with limits corresponding to t and t_0, we have

$$\frac{mv^2 - mv_0^2}{2} = \int_{t_0}^{t} m\,(X\,dx + Y\,dy). \tag{14}$$

Also from (12) we have

$$\frac{mv^2 - mv_0^2}{2} = \int_{t_0}^{t} m\,T\,ds. \tag{15}$$

And from (13),

$$\frac{mv^2 - mv_0^2}{2} = \int_{t_0}^{t} m\,(P\,dr + Q\,r\,d\theta). \tag{16}$$

All these values of vis viva are in strict agreement with the remarks made in Arts. 259 and 325; and the reaction of the curve, viz. R, does not appear in them, because the motion is always perpendicular to its action-line, and consequently it does no work.

420.] The pressure of the particle on the curve may be determined as follows from the preceding equations:

From (11) we have

$$R = \pm m\left\{X\frac{dy}{ds} - Y\frac{dx}{ds} - \frac{dy\,d^2x - dx\,d^2y}{ds\,dt^2}\right\}$$

$$= \pm m\left\{X\frac{dy}{ds} - Y\frac{dx}{ds} \pm \frac{v^2}{\rho}\right\}. \tag{17}$$

From (12),

$$R = \pm m\left\{N - \frac{v^2}{\rho}\right\}. \tag{18}$$

From (13),

$$R = \pm m\left\{Q\frac{dr}{ds} - P\frac{r\,d\theta}{ds} \pm \frac{v^2}{\rho}\right\}. \tag{19}$$

We proceed now to give examples in which these principles and equations are applied.

421.] Ex. 1. I will first take the case of rectilinear motion; and the most simple is that of a heavy particle moving on a smooth inclined plane. The motion of it along the plane has been fully considered in Section 2, Chapter X, by means of the equation of the tangential component, viz. $\frac{d^2s}{dt^2} = g\sin a$. For the normal component we have $R = mg\cos a$; that is, the pressure on the plane is the same at all points of it, and is independent of the velocity of the particle.

Ex. 2. A particle moves within a smooth rectilinear tube under the action of a force varying directly as the distance whose source is in a given position outside the tube. Determine the motion.

Let the line of the tube be the axis of y, and let the centre of force be on the axis of x at a distance a from the origin; let b and y be the distances of m from the origin when $t = 0$ and $t = t$ respectively; and let r be the distance of m from the centre of force when $t = t$; then

$$\frac{d^2y}{dt^2} = -\mu r\frac{y}{r} = -\mu y;$$

and as this is the equation of harmonic motion, see Arts. 279 and 283, the particle oscillates through a distance $2b$; viz. through b on each side of the origin; and the time of an oscillation is $\frac{\pi}{\mu^{\frac{1}{2}}}$.

Ex. 3. The circumstances of constraint and reference being the same as those of the preceding example, the particle moves under the action of a force which is any function of the distance from the source outside the tube, and makes vibrations of small amplitude. Determine the motion.

In this case the equation of motion is

$$\frac{d^2y}{dt^2} = -\frac{y}{r} f(r),$$

if $f(r)$ denotes the impressed velocity-increment along r. Now $r^2 = a^2+y^2$;

$$\therefore \quad r = a\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}$$

$$= a\left(1+\frac{y^2}{2a^2}\right) = a+\frac{y^2}{2a};$$

$$\therefore \quad f(r) = f\left(a+\frac{y^2}{2a}\right)$$

$$= f(a)+\frac{y^2}{2a}f'(a),$$

if we omit powers of y higher than the second. Also

$$\frac{y}{r} = \frac{y}{(a^2+y^2)^{\frac{1}{2}}} = \frac{y}{a}\left(1+\frac{y^2}{a^2}\right)^{-\frac{1}{2}}$$

$$= \frac{y}{a};$$

$$\therefore \quad \frac{d^2y}{dt^2} = -\frac{y}{a}\left\{f(a)+\frac{y^2}{2a}f'(a)\right\}$$

$$= -\frac{f(a)}{a}y,$$

which is the equation of harmonic motion; and consequently the particle oscillates through a distance $2b$; viz. through b on each side of the origin; and the time of an oscillation

$$= \pi\left\{\frac{a}{f(a)}\right\}^{\frac{1}{2}}.$$

Ex. 4. Two equal particles move in smooth tubes which intersect each other at right angles, and attract each other with a force varying directly as the masses and inversely as the square of the distance. Determine the motion of each.

Let the lines of the tubes be the axes; let a and b be the initial distances of the particles from the origin; and x and y the distances at the time t: let $a^2+b^2=c^2$, $x^2+y^2=r^2$. Then the equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{m}{r^2}\frac{x}{r}; \qquad \frac{d^2y}{dt^2} = -\frac{m}{r^2}\frac{y}{r}:$$

$$\therefore \quad y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2} = 0;$$

$$\therefore \quad y\frac{dx}{dt} - x\frac{dy}{dt} = 0,$$

since the particles are initially at rest. Hence

$$\frac{dx}{x} = \frac{dy}{y};$$

$$\therefore \quad \log\frac{x}{a} = \log\frac{y}{b};$$

$$\therefore \quad \frac{x}{a} = \frac{y}{b} = \frac{r}{c};$$

consequently when $x = 0$, $y = 0$ and $r = 0$, and the particles arrive simultaneously at the junction of the two tubes. Also substituting from this last equation in either of the equations of motion, we have $\frac{d^2r}{dt^2} = -\frac{\mu}{r^2}$; and therefore, as in Art. 384,

$$t = \left(\frac{c}{2\mu}\right)^{\frac{1}{2}} \left\{(cr - r^2)^{\frac{1}{2}} + c\cos^{-1}\left(\frac{r}{c}\right)^{\frac{1}{2}}\right\};$$

and when $r = 0$, $t = \frac{\pi c^{\frac{3}{2}}}{(8\mu)^{\frac{1}{2}}}$.

422.] Let us next consider the motion of a heavy particle moving on a smooth curve, or in a smooth tube, in a vertical plane.

Let the plane be that of (x, y); and let the axes of x and y be horizontal and vertical respectively; let y be positive downwards; then the equations of motion in terms of axial-components are

$$\frac{d^2x}{dt^2} = \pm\frac{\text{R}}{m}\frac{dy}{ds}; \qquad \frac{d^2y}{dt^2} = g \mp \frac{\text{R}}{m}\frac{dx}{ds}; \tag{20}$$

and the equations of tangential and normal components are evidently

$$\frac{d^2s}{dt^2} = g\frac{dy}{ds}; \qquad \text{R} = m\left\{g\frac{dx}{ds} \pm \frac{v^2}{\rho}\right\}. \tag{21}$$

From the former of these we have the following equation of vis viva, viz.

$$\frac{mv^2 - mv_0^2}{2} = mg(y - y_0); \tag{22}$$

so that the vis viva acquired during the fall of m in its constrained path along the curve or through the tube varies directly as the perpendicular distance through which it has fallen; and is thus independent of the curvilinear path.

The latter equation in (21) shews that the pressure on the curve is equal to an algebraical sum made up of the normal component of gravity and of the centrifugal force.

We proceed to apply these equations to the following problems, which are of considerable practical importance.

423.] The motion of a heavy particle on a cycloid in a vertical plane.

Let the base of the cycloid be horizontal, and let the vertex of the cycloid be its lowest point, as in fig. 135; OM $=x$, MP $=y$; so that the equation of the curve is

$$y = a \operatorname{versin}^{-1} \frac{x}{a} + (2ax - x^2)^{\frac{1}{2}}. \tag{23}$$

Let the particle m move on the concave side of the curve, and be at P (x, y) at the time t, and at rest at K (h, k) when $t = 0$. Then the normal PG is the line of action of R, and the equations of motion are

$$\frac{d^2x}{dt^2} = -g + \frac{\mathrm{R}}{m}\frac{dy}{ds}; \qquad \frac{d^2y}{dt^2} = -\frac{\mathrm{R}}{m}\frac{dx}{ds}; \tag{24}$$

whence, observing that $v = 0$ when $x = h$, we have

$$\frac{ds^2}{dt^2} = v^2 = 2g(h-x); \tag{25}$$

$$\frac{\mathrm{R}}{m} = g\frac{dy}{ds} + \frac{v^2}{\rho}, \tag{26}$$

since pressure is due to both gravity and the centrifugal force.

From (23), $$\frac{dy}{\{2a-x\}^{\frac{1}{2}}} = \frac{dx}{x^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}},$$

$$\rho = \{8a(2a-x)\}^{\frac{1}{2}}.$$

If OP $= s$, and m is descending towards the lowest point of the curve; and if t is the time of m descending from K to O, then from (25) we have

$$t = -\frac{1}{(2g)^{\frac{1}{2}}}\int_h^0 \frac{ds}{(h-x)^{\frac{1}{2}}} = \left(\frac{a}{g}\right)^{\frac{1}{2}}\int_0^h \frac{dx}{(hx-x^2)^{\frac{1}{2}}}$$

$$= \left(\frac{a}{g}\right)^{\frac{1}{2}}\left[\operatorname{versin}^{-1}\frac{2x}{h}\right]_0^h = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}; \tag{27}$$

thus the time of descent of a particle to the lowest point is independent of h, and is therefore the same whatever is the position of the point on the cycloid from which the particle starts.

And since the velocity of m at O is $(2gh)^{\frac{1}{2}}$, and has therefore its greatest value, and since by (25) $v = 0$ when $x = h$, it follows that m having passed through O ascends the other branch of the cycloid, and reaches a point K′ at the distance h above the horizontal line through O; and the time of the ascent is $\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$: thus the time of a complete oscillation from K to K′ is $2\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$, and is thus independent of the point on the curve at which the motion commences.

This property of a curve is called *tautochronism*; and the cycloid is accordingly called *the tautochronous curve* of a heavy particle.

The result (27) may be arrived at by the following process: Let us resolve tangentially: then

$$\frac{d^2 s}{dt^2} = -g\frac{dx}{ds}.$$

From the equation to the cycloid, $s^2 = 8ax$, $\frac{dx}{ds} = \frac{s}{4a}$;

$$\therefore \quad \frac{d^2 s}{dt^2} = -\frac{g}{4a}s;$$

which is the equation of harmonic motion; and if $s = s_0$, when $t = 0$ and the particle is at rest, see Art. 279,

$$s = s_0 \cos\left(\frac{g}{4a}\right)^{\frac{1}{2}} t; \tag{28}$$

and the periodic time $= 2\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$.

Also from (26), substituting by means of the equation to the cycloid,

$$\frac{\text{R}}{m} = \frac{g(2a+h-2x)}{\{2a(2a-x)\}^{\frac{1}{2}}};$$

which assigns the pressure on the curve; and if $h = 2a$, that is, if m begins to move from the extremity of the base of the cycloid,

$$\frac{\text{R}}{m} = g\left(\frac{4a-2x}{a}\right)^{\frac{1}{2}};$$

and if $x = 0$, that is, at the lowest point of the curve,

$$\text{R} = 2mg;$$

that is, the pressure is twice the weight of the particle.

If the plane in which the cycloid is, and in which m moves, is inclined to the horizon at an angle a, and is smooth, the preceding results are applicable, if we replace g by $g \sin a$.

424.] A particle m may have the cycloidal motion of the preceding Article, if we suppose it to move in a smooth tube, or an open canal, bent into the form of a cycloid. We may also obtain a path of the required kind by the following arrangement. Let the particle m be fixed at the end of a perfectly flexible and inextensible string, which we shall assume to be without weight, of length $4a$; and let the upper end of this string be fastened at a point C, see fig. 135, in a vertical line through O, where $\text{OC} = 4a$; from C to B and from C to B′ let two cycloidal arcs be drawn, equal to OB or OB′; then BOB′ is the involute of CB and CB′; and therefore the end of a string of the length $4a$ fastened at C and wrapping round CB and CB′ will describe the cycloid BOB′.

Now when a heavy solid body oscillates about a fixed horizontal axis it is called *a pendulum;* and when a heavy particle is attached to the axis by means of a string or a rod, without weight and inextensible, this instrument is called *a simple pendulum;* and although such a system never can be perfectly attained, yet approximations may be made to it which are near enough for practical purposes, and by means of it the incidents of pendulums may be compared.

Suppose then the particle m to be fixed at one end of a string, whose length is $4a$, the other end being fastened at C, fig. 135; and suppose the plane of the curves to be vertical; then if CB, CB′ are such surfaces or cheeks that the string may be wrapped round them, m will move in a cycloidal path, and we shall have a cycloidal pendulum. The time of the oscillations will be the same, whatever is the point whence m begins to move; and if l is the length of the string,

$$\text{the time of an oscillation} = \pi \left(\frac{l}{g}\right)^{\frac{1}{2}}. \tag{29}$$

Also the tension of the string corresponds to the pressure on the curve of the preceding Article, and we have

$$\text{the tension of the string} = mg\,\frac{2a+h-2x}{\{2a(2a-x)\}^{\frac{1}{2}}}; \tag{30}$$

and therefore, if $x = 0$,

$$\text{the tension at the lowest point} = \frac{mg}{2a}(2a+h).$$

If $h = 2a$, which is its largest value on the cycloid, the tension $= 2mg$, that is, is equal to twice the weight of the bob; so that the string must be able to bear twice the weight of the bob.

425.] As another example of cycloidal motion, let us suppose a cycloid to be placed in a vertical plane with its base horizontal, as in fig. 136; and let us suppose m to be projected from the highest point with a given velocity along the convex side of the curve: it is required to determine the subsequent motion of m.

Let P, (x, y), be the position of m at the time t; let $u = (2gh)^{\frac{1}{2}}$ be the velocity of projection. Then

$$\frac{ds^2}{dt^2} = v^2 = 2g(h+x);$$

$$\frac{\text{R}}{m} = g\frac{dy}{ds} - \frac{v^2}{\rho} = \frac{g}{(2a)^{\frac{1}{2}}} \frac{2a-h-2x}{(2a-x)^{\frac{1}{2}}}.$$

So long as R has a positive sign, m is in contact with the cycloid; but when $\text{R} = 0$, the particle leaves the curve; and being heavy and moving freely describes a parabolic path; this takes place when $x = a-\frac{h}{2}$; in which case $v^2 = 2g\left(a+\frac{h}{2}\right)$; and the line of motion of m at that time makes with a horizontal line an angle whose tangent is $\left(\frac{2a-h}{2a+h}\right)^{\frac{1}{2}}$; and the latus rectum of the subsequent parabolic path is $\frac{(2a+h)^2}{2a}$.

426.] The motion of a heavy particle in a circular tube in a vertical plane.

Let the radius of the circle be a, and let the origin be taken at its lowest point, the vertical diameter coinciding with the axis of x: so that the equation to the circle is $y^2 = 2ax - x^2$:

$$\therefore \quad \frac{dy}{a-x} = \frac{dx}{y} = \frac{ds}{a}.$$

Let (h, k) be the initial place of m, and v_0 its velocity at that point; then (x, y) being its place at the time t,

$$\frac{ds^2}{dt^2} = v^2 = 2g(h-x)+v_0^2. \tag{31}$$

Hence the velocity is a maximum and $= (2gh+v_0^2)^{\frac{1}{2}}$, when $x=0$; the velocity is a minimum when x has its greatest value; that is, at the highest point of the circle, if the particle makes a complete circuit, in which case $v^2 = 2g(h-2a)+v_0^2$; and if the particle does not describe a complete circle, it comes to rest when $x = h+\frac{v_0^2}{2g}$. And if the particle moves from rest at the highest point of the tube, then $v_0 = 0$, and $h = 2a$; and (31) becomes

$$v^2 = 2g(2a-x). \tag{32}$$

Also
$$\frac{R}{m} = g\frac{dy}{ds} + \frac{v^2}{\rho}$$
$$= \frac{g}{a}\Big\{a+2h+\frac{v_0^2}{g}-3x\Big\}; \tag{33}$$
which is greatest when $x=0$, and has its least value when x has its greatest value.

If the particle starts from rest at the highest point of the circle, then at the lowest point the pressure $= 5mg$; that is, is five times the weight of the particle.

If the centrifugal force of the particle at the highest point is exactly equal to the weight of the particle, that is, if $v_0^2 = ag$, then the pressure at the lowest point $= 6mg$; that is, is six times the weight of the particle.

To find the time taken by the particle in its descent to the lowest point; if OP $= s$, so that s decreases as t increases, from (31),
$$dt = -\frac{ds}{\{2g(h-x)+v_0^2\}^{\frac{1}{2}}} \tag{34}$$
$$= -\frac{a\,dx}{\{2ax-x^2\}^{\frac{1}{2}}\{2g(h-x)+v_0^2\}^{\frac{1}{2}}}. \tag{35}$$
This element-function does not generally admit of integration. In the case in which
$$h+\frac{v_0^2}{2g} = 2a,$$
that is, in which the particle comes to rest at the highest point of the circle, (35) becomes
$$dt = -\frac{dx}{(2g)^{\frac{1}{2}}x^{\frac{1}{2}}(2a-x)}; \tag{36}$$
whence by integration, and observing that $t=0$ when $x=h$, we have
$$t = -\frac{1}{2}\Big(\frac{a}{g}\Big)^{\frac{1}{2}}\log\frac{(2a)^{\frac{1}{2}}+x^{\frac{1}{2}}}{(2a)^{\frac{1}{2}}-x^{\frac{1}{2}}} + \frac{1}{2}\Big(\frac{a}{g}\Big)^{\frac{1}{2}}\log\frac{(2a)^{\frac{1}{2}}+h^{\frac{1}{2}}}{(2a)^{\frac{1}{2}}-h^{\frac{1}{2}}}.$$

427.] Let us however return to the general expression (35) for dt; and to simplify it, let us suppose the initial velocity to be zero, and h to be the vertical abscissa to the point whence m begins to move; then
$$dt = -\frac{a\,dx}{(2g)^{\frac{1}{2}}(h-x)^{\frac{1}{2}}(2ax-x^2)^{\frac{1}{2}}}; \tag{37}$$
this expression does not admit of integration. If however the radius of the circle is large, and the greatest amplitude to which m moves is small, we may expand (37) in a series of terms in ascending powers of $\frac{x}{a}$, and thus approximately find the integral;

and it is necessary to have recourse to this method, bec the problem is that of a circular pendulum; and in the a cations of it to the purposes of time-measuring it is desir to know the extent of error which arises, if we assume one t of the series to be the measure of the time of descent of m f its highest to its lowest point. Let T be the time of an o lation, that is, from $x = h$, through $x = 0$, to $x = h$; then

$$\begin{aligned} \text{T} &= -\left(\frac{a}{g}\right)^{\frac{1}{2}} \int_h^0 \frac{dx}{(hx-x^2)^{\frac{1}{2}}} \left(1-\frac{x}{2a}\right)^{-\frac{1}{2}} \\ &= \left(\frac{a}{g}\right)^{\frac{1}{2}} \int_0^h \left\{1 + \frac{1}{2}\frac{x}{2a} + \frac{1.3}{2.4}\left(\frac{x}{2a}\right)^2 + \cdots \right. \\ &\qquad \left. \cdots + \frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\left(\frac{x}{2a}\right)^n + \cdots \right\} \frac{dx}{(hx-x^2)^{\frac{1}{2}}}. \end{aligned}$$

This series consists of terms each of which is of the f $\int_0^h \frac{x^m\,dx}{(hx-x^2)^{\frac{1}{2}}}$; and by Art. 51, Integral Calculus,

$$\begin{aligned} \int_0^h \frac{x^m\,dx}{(hx-x^2)^{\frac{1}{2}}} &= \frac{2m-1}{2m} h \int_0^h \frac{x^{m-1}\,dx}{(hx-x^2)^{\frac{1}{2}}} \\ &= \frac{(2m-1)(2m-3)\ldots 5.3.1}{2m(2m-2)\ldots 6.4.2} h^m \int_0^h \frac{dx}{(hx-x^2)^{\frac{1}{2}}} \\ &= \frac{(2m-1)(2m-3)\ldots 5.3.1}{2m(2m-2)\ldots 6.4.2} h^m \pi; \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{T} = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}} &\left\{1 + \left(\frac{1}{2}\right)^2 \frac{h}{2a} + \left(\frac{1.3}{2.4}\right)^2 \left(\frac{h}{2a}\right)^2 + \cdots \right. \\ &\left. \cdots + \left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\right)^2 \left(\frac{h}{2a}\right)^n + \cdots \right\}; \qquad (\end{aligned}$$

which is the complete expression for the time of an oscillatio

If h is very small in comparison of a, and if we neglect powers of $\frac{h}{2a}$,

$$\text{T} = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}; \qquad ($$

which is an expression of the same form as that for the tim an oscillation of a heavy particle on a cycloid; see Art. 4 observing however that a here is equal to $4a$ in that case. A if we include the first two terms of (39), we have

$$\text{T} = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}} \left\{1 + \frac{h}{8a}\right\}. \qquad ($$

If 2α is the angle at the centre of the circle which is s tended by the arc of the oscillation,

$$h = a(1-\cos\alpha) = 2a\left(\sin\frac{\alpha}{2}\right)^2 = \frac{a}{2}(\text{chord }\alpha)^2;$$

so that the difference between the expression for T given in (40) and that given in (41) varies as the square of the chord of half the angle of oscillation.

428.] This motion of a heavy particle in a circular arc is approximately realized by attaching the particle to the end of a very thin and light straight rod, which, turning about a fixed point at its other end, moves in a vertical plane. This instrument in its perfect state, which however can never be attained, is called *a simple circular pendulum.* If l is the length of the rod, the time of an oscillation is approximately given by the formula

$$\mathrm{T} = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}}, \tag{42}$$

when the angle of oscillation is very small.

The pressure on the curve becomes in this case the tension of the rod.

The formula (42) is applied to the determination of gravity at the different places of the earth's surface. Let L be the length of a pendulum which vibrates seconds at the place to which g applies; then

$$1 = \pi\left(\frac{\mathrm{L}}{g}\right)^{\frac{1}{2}}; \qquad \therefore \quad g = \pi^2 \mathrm{L}; \tag{43}$$

from this formula g has been calculated at many places on the earth. The method of determining L accurately will be investigated in the following volume.

Equation (42) is also employed for the determination of (1) the height of a mountain, and (2) the depth of a mine.

(1) Let r be the mean-radius of the earth's surface considered spherical; let h be the altitude of the mountain above the surface, and g and g' the values of gravity on the earth's surface and the top of the mountain respectively: then, by Art. 196,

$$\frac{g}{g'} = \left(\frac{r+h}{r}\right)^2.$$

Let $n =$ the number of oscillations which the seconds' pendulum at the top of the mountain makes in 24 hours;

$$\therefore \quad 1 = \pi\left(\frac{\mathrm{L}}{g}\right)^{\frac{1}{2}},$$

$$\frac{24 \times 60 \times 60}{n} = \pi\left(\frac{\mathrm{L}}{g'}\right)^{\frac{1}{2}} = \pi\,\frac{r+h}{r}\left(\frac{\mathrm{L}}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad \frac{h}{r} = \frac{24 \times 60 \times 60}{n} - 1;$$

whereby h is given in terms of r, the radius of the earth. Let us, for the sake of an example, suppose the pendulum to "lose 10 seconds in a day;" that is, to make 10 oscillations less than it would make on the surface of the earth. Then

$$n = 24 \times 60 \times 60 - 10,$$

and $r = 4000$ miles (approximately);

$$\therefore \quad h = 4000 \left\{ \frac{24 \times 60 \times 60}{24 \times 60 \times 60 - 10} - 1 \right\}$$

$$= 4000 \left\{ \left(1 - \frac{1}{24 \times 6 \times 60}\right)^{-1} - 1 \right\}$$

$$h = \frac{4000}{24 \times 6 \times 60} \text{ approximately}$$

$$= .4626 \text{ of a mile.}$$

(2) Let r be the radius of the earth's surface, as in the last case: and let h be the depth of the mine: let g and g' be the values of gravity on the earth's surface and at the bottom of the mine. Then, by Article 198,

$$\frac{g}{g'} = \frac{r}{r-h}.$$

Let n = the number of oscillations which the seconds' pendulum at the bottom of the mine makes in 24 hours; therefore

$$1 = \pi \left(\frac{\mathrm{L}}{g}\right)^{\frac{1}{2}};$$

$$\frac{24 \times 60 \times 60}{n} = \pi \left(\frac{\mathrm{L}}{g'}\right)^{\frac{1}{2}} = \pi \left(\frac{r}{r-h}\right)^{\frac{1}{2}} \left(\frac{\mathrm{L}}{g}\right)^{\frac{1}{2}}$$

$$= \left(\frac{r}{r-h}\right)^{\frac{1}{2}} = \left(1 - \frac{h}{r}\right)^{-\frac{1}{2}};$$

$$\therefore \quad \frac{h}{2r} = \frac{24 \times 60 \times 60}{n} - 1;$$

whence h may be found in terms of r, the earth's radius.

429.] If the arc through which the *bob* of a circular pendulum vibrates is very small, the time of an oscillation is determined by means of tangential resolution more easily than by the preceding process.

Let $2a$ be the angle which the arc of oscillation subtends at the centre of the circle; let θ be the angular distance of m from the lowest point at the time t; and let s be the arc corresponding to θ, so that, if a is the radius of the circle,

$$s = a\theta; \tag{44}$$

then, if we suppose s to decrease as t increases, the equation of motion along the tangent is

$$\frac{d^2s}{dt^2} = -g\sin\theta; \tag{45}$$

and in terms of θ,
$$\frac{d^2\theta}{dt^2} = -\frac{g}{a}\sin\theta. \tag{46}$$

Multiplying by $2\,d\theta$, integrating, and taking the limits such that $\frac{d\theta}{dt} = 0$ when $\theta = \alpha$,

$$\frac{d\theta^2}{dt^2} = \frac{2g}{a}(\cos\theta - \cos\alpha). \tag{47}$$

This expression cannot be again integrated in the form in which it stands: $\cos\theta$ may however be expressed in a series, and the integral may be approximately found. If the oscillations are small, so that θ and α are small, then expanding $\cos\theta$ and $\cos\alpha$ in powers of θ and α respectively, and omitting terms containing the fourth and higher powers of these quantities, we have

$$\frac{-d\theta}{(\alpha^2-\theta^2)^{\frac{1}{2}}} = \left(\frac{g}{a}\right)^{\frac{1}{2}}dt;$$

and since $\theta = \alpha$ when $t = 0$,

$$t = \left(\frac{a}{g}\right)^{\frac{1}{2}}\cos^{-1}\frac{\theta}{\alpha};$$

when $\theta = -\alpha$, $t = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$, which is the time of an oscillation.

Using the same notation, by the equation of the normal components, when the particle moves as the *bob* of a pendulum,

$$\begin{aligned} \text{R} &= \text{the tension of the rod} \\ &= m\left\{g\cos\theta + \frac{v^2}{\rho}\right\} \\ &= mg\,\{3\cos\theta - 2\cos\alpha\}. \end{aligned}$$

If m is projected from the highest point of the circle with a velocity due to the height h, and along the concave side of the circle; then if θ is the angle subtended at the centre of the circle by the arc which m describes in the time t,

$$(\text{the velocity})^2 = 2g\,\{h + a - a\cos\theta\};$$

$$\text{R} = mg\left\{\frac{2h}{a} + 2 - 3\cos\theta\right\}.$$

Hence it appears that the particle will leave the curve if h and θ are such that the following condition is possible; viz.

$$\cos\theta = \frac{2}{3}\,\frac{h+a}{a}.$$

If R $= 0$ when the motion begins, then $h = \frac{a}{2}$; in which at the lowest point of the circle, R $= 6\,mg$.

430.] The following are other and more general problen constrained motion:

Ex. 1. A parabola is placed with its axis vertical, and its ve the highest point; a heavy particle is projected from the ve along the concave side of the curve with a given velocity: required to determine the subsequent circumstances of motic

Let h be the height to which the velocity of projection is and let the equation of the parabola be $y^2 = 4ax$. Then if (is the place of m at the time t,

$$(\text{the velocity})^2 = 2g(x+h);$$

$$\text{R} = m\left\{\frac{v^2}{\rho} - g\frac{dy}{ds}\right\} = mga^{\frac{1}{2}}\frac{h-a}{(a+x)^{\frac{3}{2}}}.$$

Therefore the pressure on the curve $= 0$, if $h = a$: that is, if velocity of projection at the vertex is equal to that acquire falling from the directrix; and in this case the pressure is at all points of the curve: the parabola is therefore the tra tory of m moving freely. This is apparent from the investi tions of Article 350.

Ex. 2. A heavy particle descends down a curve in a vert plane: it is required to determine the equation of the cu when the pressure on it is the same at all its points.

Let R be the constant pressure: h the altitude to which velocity of m at the origin is due; $\alpha =$ the angle which the t gent to the curve at the origin makes with the vertical l which I shall take to be the axis of x: then

$$v^2 = 2g(h+x); \qquad \text{R} = m\left\{\frac{v^2}{\rho} + g\frac{dy}{ds}\right\}.$$

If s is equicrescent, then from equation (18), Art. 285, Dif ential Calculus,

$$\frac{1}{\rho} = \frac{d^2y}{ds^2}\frac{ds}{dx};$$

and substituting this in the preceding equation, we have

$$\frac{\text{R}\,dx}{2(h+x)^{\frac{1}{2}}} = mg\left\{(h+x)^{\frac{1}{2}}\,d.\frac{dy}{ds} + \frac{dx}{2(h+x)^{\frac{1}{2}}}\frac{dy}{ds}\right\};$$

therefore integrating, and taking the assigned limits,

$$\text{R}\,\{(h+x)^{\frac{1}{2}} - h^{\frac{1}{2}}\} = mg\,\{(h+x)^{\frac{1}{2}}\frac{dy}{ds} - h^{\frac{1}{2}}\sin\alpha\};$$

$$\therefore \quad \frac{dy}{ds} = \frac{\mathrm{R}}{mg} + \frac{h^{\frac{1}{2}}(mg \sin \alpha - \mathrm{R})}{mg(h+x)^{\frac{1}{2}}}; \tag{48}$$

whence, if ds is replaced by its equivalent in terms of dy and dx, the equation to the required curve may be found. It is called *the curve of equal pressure.*

Ex. 3. A heavy particle m moves from rest on a curve, and the pressure on the curve varies as the nth power of the vertical distance through which the particle has moved; it is required to determine the nature of the curve.

In this case, $v^2 = 2gx$,

$$kx^n = \frac{2x}{\rho} + \frac{dy}{ds},$$

if k is conveniently assumed: and by a process similar to that of the last example, we have

$$\frac{k}{2n+1} x^{n+\frac{1}{2}} = x^{\frac{1}{2}} \frac{dy}{ds};$$

whence we have

$$dy = x^n \left\{ \left(\frac{2n+1}{k}\right)^2 - x^{2n} \right\}^{-\frac{1}{2}} dx, \tag{49}$$

which is integrable by rationalization, see Art. 44, Integral Calculus, whenever $\frac{n+1}{2n}$ or $\frac{1}{2n}$ is an integer.

Let $n = 1$; and let $\frac{3}{k} = a$: then (49) becomes after integration

$$(y \pm a)^2 + x^2 = a^2.$$

If $n = -1$, the resulting equation is that of the catenary.

Ex. 4. To find the equation of a curve, which is such that a heavy particle m moving on it may describe a given arc in the time in which it would describe the corresponding chord.

Let us take polar coordinates, and let the origin be the point at which the motion begins: let (r, θ) be the place of m at the time t, the prime radius-vector being vertical. Then for the motion on the curve we have

$$\frac{ds^2}{dt^2} = 2gr \cos \theta; \qquad \therefore \quad t = \frac{1}{(2g)^{\frac{1}{2}}} \int_0^r \frac{ds}{(r \cos \theta)^{\frac{1}{2}}};$$

and the time in which m would describe the chord

$$= \left(\frac{2r}{g \cos \theta}\right)^{\frac{1}{2}};$$

$$\therefore \quad \int_0^r \frac{(dr^2 + r^2 d\theta^2)^{\frac{1}{2}}}{(r \cos \theta)^{\frac{1}{2}}} = 2 \left(\frac{r}{\cos \theta}\right)^{\frac{1}{2}};$$

and differentiating,

$$\frac{(dr^2+r^2\,d\theta^2)^{\frac{1}{2}}}{(r\cos\theta)^{\frac{1}{2}}} = \left(\frac{\cos\theta}{r}\right)^{\frac{1}{2}} \frac{dr\cos\theta + r\sin\theta\,d\theta}{(\cos\theta)^2};$$

$$\therefore \quad \frac{\cos 2\theta}{\sin 2\theta}\,d\theta = \frac{dr}{r},$$

$$r^2 = a^2 \sin 2\theta;$$

which is the equation to a lemniscata, the axis of which is inclined at an angle of 45° to the vertical line through the origin.

Ex. 5. From a given point on a parabola a particle is projected with a given velocity u along the concave side of the curve, and is acted on by a force in the focus which is attractive and varies inversely as the square of the distance; it is required to determine the circumstances of motion.

The equation to the parabola in terms of r and p is $p^2 = ar$. Let b be the initial distance of m from the focus; then as $\text{Q} = 0$, and $\text{P} = -\frac{\mu}{r^2}$; from (16),

$$v^2 = u^2 + 2\mu\left\{\frac{1}{r} - \frac{1}{b}\right\},$$

whereby the velocity is known at every point of the curve. And from (19),

$$\frac{\text{R}}{m} = \frac{v^2}{\rho} - \frac{\mu}{r^2}\,\frac{r\,d\theta}{ds} = \frac{a^{\frac{1}{2}}}{2r^{\frac{3}{2}}}\left\{u^2 - \frac{2\mu}{b}\right\};$$

and thus the pressure is known at every point of the curve; and since m is to move on the concave side of the curve, u^2 must be greater than $\frac{2\mu}{b}$; and if u^2 is less than $\frac{2\mu}{b}$, m must move on the convex side; and if $u^2 = \frac{2\mu}{b}$ the pressure on the curve is zero at all its points; and rightly so; because $\left(\frac{2\mu}{b}\right)^{\frac{1}{2}}$ is, see Art. 284, the velocity acquired at the point of projection by m moving from infinity under the action of the central force; and thus under these circumstances the parabola would be the unconstrained orbit.

Ex. 6. A particle moves on the convex side of a parabola, and is tied by an extensible string to a point in the focus; the unstretched length of the string is equal to the focal distance; the particle is placed at rest at the extremity of the latus rectum: it is required to determine the subsequent circumstances of motion.

Let $\text{T} =$ the tension of the string at the time t, so that T is the central force acting on m; then, if r is the distance of m from S at the time t, and if e is the coefficient of elasticity,

$$r = a(1+e\mathrm{T}); \qquad \therefore \quad \mathrm{T} = \frac{r-a}{ea};$$

and therefore from (16),

$$\frac{ds^2}{dt^2} = 2\int_{2a}^{r} \frac{a-r}{ae}\, dr = \frac{2}{ae}\left\{ar - \frac{r^2}{2}\right\}.$$

Let the equation to the parabola be

$$r = \frac{2a}{1+\cos\theta}; \text{ then } ds^2 = \frac{r\, dr^2}{r-a};$$

$$\therefore \quad dt = -\frac{(ae)^{\frac{1}{2}}\, dr}{\{(r-a)\,(2a-r)\}^{\frac{1}{2}}};$$

$$t = (ae)^{\frac{1}{2}} \cos^{-1}\frac{2r-3a}{a};$$

therefore when $r = a$, $t = \pi(ae)^{\frac{1}{2}}$, and this is the time which the particle takes in passing from the extremity of the latus rectum to the vertex; and the velocity is a maximum when $r = a$, so that the particle passes on from the vertex to the other extremity of the latus rectum, at which point it comes to rest; and this oscillatory motion continues, the period of an oscillation being $2\pi(ae)^{\frac{1}{2}}$.

Ex. 7. A particle moves on the convex side of an ellipse under the action of two forces parallel respectively to the axes of x and y, and which vary respectively as the cubes of the corresponding coordinates: the particle is placed at rest at the point (h, k); it is required to determine the pressure on the curve.

In this case, $\quad \mathrm{X} = -\frac{\mu}{x^3}, \qquad \mathrm{Y} = -\frac{\mu'}{y^3};$

therefore from (13),

$$\frac{ds^2}{dt^2} = v^2 = -2\int\left(\frac{\mu\, dx}{x^3} + \frac{\mu' dy}{y^3}\right)$$

$$= \frac{\mu}{x^2} + \frac{\mu'}{y^2} - \frac{\mu}{h^2} - \frac{\mu'}{k^5}.$$

Let the equation to the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then

$$\frac{\mathrm{R}}{m} = \left(\frac{\mu b^2}{x^2} + \frac{\mu' a^2}{y^2}\right)\frac{1}{(a^4 y^2 + b^4 x^2)^{\frac{1}{2}}} - \left(\frac{\mu}{x^2} + \frac{\mu'}{y^2} - \frac{\mu}{h^2} - \frac{\mu'}{k^2}\right)\frac{a^4 b^4}{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}.$$

Ex. 8. A particle moves on the convex side of an ellipse, and is under the action of (1) two central forces varying inversely as the square of the distance and whose sources are at the foci, and (2) a central force varying as the distance and whose source is at the centre of the ellipse; it is required to determine the pressure on the curve.

Let μ_1, μ_2, μ be the absolute forces severally of the foci and of the centre; and let the centre of the ellipse be the origin. Let r_1, r_2, r be the distances of m at the time t from the two foci and from the centre respectively, and let r'_1, r'_2, r' be the initial values of these, that is, when $t = 0$; and let v_0 be the initial velocity. Then

$$\mathrm{X} = -\mu x - \frac{\mu_1 (x+ae)}{r_1^3} - \frac{\mu_2 (x-ae)}{r_2^3},$$

$$\mathrm{Y} = -\mu y - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3};$$

$$v^2 - v_0^2 = -\mu r^2 + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + \mu r'^2 - \frac{2\mu_1}{r'_1} - \frac{2\mu_2}{r'_2}.$$

Also $r^2 = a^2 + b^2 - r_1 r_2$, and $\rho = \dfrac{(r_1 r_2)^{\frac{3}{2}}}{ab}$;

$$\therefore \quad \frac{\mathrm{R}}{m}\rho = \mu r'_1 r'_2 + \frac{\mu_1 r'_2}{a r'_1} + \frac{\mu_2 r'_1}{a r'_2} - v_0^2.$$

If $\mathrm{v}, \mathrm{v}_1, \mathrm{v}_2$ are the velocities with which a particle m being projected from the given point would freely describe the preceding ellipse under the action of the preceding forces acting singly, then

$$\mathrm{v}^2 = \mu r'_1 r'_2, \qquad \mathrm{v}_1^2 = \frac{\mu_1 r'_2}{a r'_1}, \qquad \mathrm{v}_2^2 = \frac{\mu_2 r'_1}{a r'_2}:$$

so that $\quad \dfrac{\mathrm{R}\rho}{m} = \mathrm{v}^2 + \mathrm{v}_1^2 + \mathrm{v}_2^2 - v_0^2;$

if therefore $\quad v_0^2 = \mathrm{v}^2 + \mathrm{v}_1^2 + \mathrm{v}_2^2,$

then $\mathrm{R} = 0$, and the particle would describe the ellipse freely. This result is a particular application of the general Theorem of Article 362.

431.] I proceed now to the investigation of curves which possess certain general properties, and which are suggested by the preceding inquiry. And first of the general equation of tautochronous curves.

It appears from Article 423 that the cycloid, in either a verticle or an inclined plane, with its base horizontal, is such that the time taken by a heavy particle in moving down the curve to its lowest point is the same whatever is the point on the curve from which the particle begins to move. Our object now is to inquire whether any other curve besides the cycloid possesses this property of tautochronism for heavy particles in vacuo; and we shall also extend the inquiry to forces of other kinds.

Let the impressed forces be resolved tangentially; and let T be the tangential component at the time t of the impressed velocity-increment, and tending to diminish s as t increases; let s be the arc measured from a certain point on the curve chosen as the origin, and let a be the initial value of s and correspond to the point at which m is at rest: then

$$\frac{ds^2}{dt^2} = 2\int_s^a \mathrm{T}\,ds; \tag{50}$$

and the time of passage from a to s is given by the following:

$$t = \frac{1}{2^{\frac{1}{2}}}\int_s^a \left\{\int_s^a \mathrm{T}\,ds\right\}^{-\frac{1}{2}} ds. \tag{51}$$

Now this definite integral must be independent of a when $s = a$; and thus the equation must be of the following form:

$$\int_s^a \left\{\int_s^a \mathrm{T}\,ds\right\}^{-\frac{1}{2}} ds = \phi\left(\frac{s}{a}\right);$$

of which taking the s-differential,

$$-\left\{\int_s^a \mathrm{T}\,ds\right\}^{-\frac{1}{2}} = \frac{1}{a}\phi'\left(\frac{s}{a}\right);$$

$$\int_s^a \mathrm{T}\,ds = \left\{\frac{1}{a}\phi'\left(\frac{s}{a}\right)\right\}^{-2};$$

and again taking the s-differential,

$$-\mathrm{T} = \frac{d}{ds}\left\{\frac{1}{a}\phi'\left(\frac{s}{a}\right)\right\}^{-2}; \tag{52}$$

which is the relation between T and the equation to the tautochronous curve. But the equation to the curve must be independent of a; and therefore

$$a^2\left\{\phi'\left(\frac{s}{a}\right)\right\}^{-2}$$

must be independent of a; whence we infer that

$$\left\{\phi'\left(\frac{s}{a}\right)\right\}^2 = -\frac{2}{k}\frac{a^2}{s^2},$$

where k is a constant. Substituting this in (52), we have

$$\mathrm{T} = ks; \tag{53}$$

whence we infer that the tangential force which acts on the particle must vary directly as the length of the path to be described by the particle to the origin; that is, the tangential equation must be the equation of harmonic motion; wherein the periodic time is independent of the amplitude.

From (53) and (50) we have

$$\frac{ds^2}{dt^2} = k(a^2 - s^2); \tag{54}$$

let us suppose that the particle is moving towards the origin, and that τ is the time from $s = a$ to $s = 0$, then

$$k^{\frac{1}{2}}\, dt = \frac{-ds}{(a^2 - s^2)^{\frac{1}{2}}};$$

$$\therefore \quad k^{\frac{1}{2}}\tau = \left[\cos^{-1}\frac{s}{a}\right]_a^0 = \frac{\pi}{2};$$

$$\therefore \quad \tau = \frac{\pi}{2k^{\frac{1}{2}}}; \quad \text{and} \quad k = \frac{\pi^2}{4\tau^2};$$

so that from (53), $$\mathrm{T} = \frac{\pi^2}{4\tau^2}s. \tag{55}$$

432.] Some examples of tautochronism are subjoined.

Ex. 1. Let gravity be the acting force; and let us suppose the axes of coordinates and the origin to be taken so that the axis of x is vertical, and $s = 0$ when $x = 0$; then $\mathrm{T} = g\frac{dx}{ds}$; and thus from (55), $g\frac{dx}{ds} = \frac{\pi^2}{4\tau^2}s$; therefore integrating, and replacing the constants by another constant $8a$, which is chosen conveniently,

$$s^2 = 8ax; \qquad s = 2(2ax)^{\frac{1}{2}}; \qquad ds = \frac{(2a)^{\frac{1}{2}}\, dx}{x^{\frac{1}{2}}};$$

whence $$y = a \operatorname{versin}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}},$$

which is the equation to the cycloid; and which is therefore the only tautochronous curve of a heavy particle in vacuo.

Also since only the length-element of the curve and the *vertical* distance of the initial above the terminal point of the particle's motion are involved in the preceding investigations, the result will be the same if the vertical plane, in which the cycloid is, is wrapped round a vertical cylinder.

Ex. 2. Let the origin and axes be taken as in Ex. 1, and let the force be parallel to the axis of x and $= \mu x$;

$$\therefore \quad \mathrm{T} = \mu x \frac{dx}{ds} = \frac{\pi^2}{4\tau^2}s; \qquad \mu x^2 = \frac{\pi^2}{4\tau^2}s^2;$$

whence we have ultimately $y = \beta x$, where β is an arbitrary constant: this is the equation of a straight line.

Ex. 3. Let $\mathrm{T} = \mu x^n \dfrac{dx}{ds}$; and let the origin and axes be the same as before;

$$\therefore \quad \mu x^n dx = \frac{\pi^2}{4\tau^2} s\, ds; \qquad x^{n+1} = \frac{(n+1)\pi^2}{8\mu\tau^2} s^2;$$

$$\therefore \quad dy = (-1 + k^2 x^{n-1})^{\frac{1}{2}} dx,$$

where k^2 is substituted for other constants. Now if x is small, y will be imaginary, unless $n-1$ is negative. Let $n-1 = -m$; then

$$dy = x^{-\frac{m}{2}} (k^2 - x^m)^{\frac{1}{2}} dx,$$

which is integrable by rationalization, whenever either $\dfrac{2-m}{2m}$ or $\dfrac{1}{m}$ is an integer. Thus let $\dfrac{2-m}{2m} = 1$; or let $m = \dfrac{2}{3}$, then

$$dy = x^{-\frac{1}{3}} (k^2 - x^{\frac{2}{3}})^{\frac{1}{2}} dx; \qquad \therefore \quad y = -(k^2 - x^{\frac{2}{3}})^{\frac{3}{2}};$$

$$y^{\frac{2}{3}} + x^{\frac{2}{3}} = k^2,$$

which is the equation to an hypocycloid.

Ex. 4. Suppose the force to be an attracting central force and to vary inversely as the square of the distance; it is required to find the equation to the tautochronous curve.

Let r and p be the radius-vector and the perpendicular on the tangent of the path of m at the time t: and let the central attracting force be $-\mu r^{-2}$.

$$\therefore \quad \mathrm{T} = \frac{\mu}{r^3} (r^2 - p^2)^{\frac{1}{2}}; \quad \text{also} \quad ds = \frac{r\, dr}{(r^2 - p^2)^{\frac{1}{2}}};$$

therefore differentiating (55), and substituting,

$$d.\frac{\mu}{r^3} (r^2 - p^2)^{\frac{1}{2}} = \frac{\pi^2}{4\tau^2} \frac{r\, dr}{(r^2 - p^2)^{\frac{1}{2}}};$$

$$\therefore \quad \frac{\mu}{r^3} (r^2 - p^2)^{\frac{1}{2}} d.\frac{\mu}{r^3} (r^2 - p^2)^{\frac{1}{2}} = \frac{\pi^2 \mu}{4\tau^2} \frac{dr}{r^2};$$

whence integrating, and introducing the arbitrary constant c,

$$\frac{\mu^2}{r^6} (r^2 - p^2) = \frac{\pi^2 \mu}{2\tau^2} \left\{ \frac{1}{c} - \frac{1}{r} \right\};$$

$$\therefore \quad p^2 = r^2 - \frac{\pi^2 r^5}{2\mu c \tau^2} (r - c);$$

which is the equation to the tautochronous curve.

433.] The simple problem of tautochronism under the action of gravity in vacuo may be solved by the following process:

From (25) we have

$$\frac{ds^2}{dt^2} = 2g(h - x); \qquad \therefore \quad t = \frac{1}{(2g)^{\frac{1}{2}}} \int_0^h \frac{ds}{(h - x)^{\frac{1}{2}}};$$

therefore expanding the denominator,

$$t = \frac{1}{(2g)^{\frac{1}{2}}} \int_0^h \Big\{ h^{-\frac{1}{2}} + \frac{1}{2} x h^{-\frac{3}{2}} + \frac{1.3}{2.4} x^2 h^{-\frac{5}{2}} + \cdots$$
$$\cdots + \frac{1.3 \ldots (2n-1)}{2.4 \ldots 2n} x^n h^{-(n+\frac{1}{2})} + \ldots \Big\} ds.$$

But this series must after integration be homogeneous in terms of x and h, and of no dimensions, because under this circumstance only will h disappear in the definite integral. Therefore taking the general term,

$$\int x^n ds = k x^{n+\frac{1}{2}};$$

$$\therefore \quad x^n ds = \frac{2n+1}{2} k x^{n-\frac{1}{2}} dx;$$

$$\therefore \quad s = (2n+1) k x^{\frac{1}{2}};$$

which is the equation to the cycloid with its base horizontal. The cycloid therefore is the only tautochronous curve for gravity in vacuo.

434.] If there is a family of curves similar and similarly placed, all of which originate at a common point, and if heavy particles move down these curves from the common point, the locus of the points, at which all of them are at the end of a given time, is called the *synchronous curve* of the family of curves. One and the most simple case of this class of curves we have had already in Art. 344, where the circle is shewn to be the synchronous curve of a series of straight lines in either a vertical or an inclined plane originating at a given point.

If the given point at which the curves originate is taken as the origin, and the axis of x is vertical, and that of y horizontal, then (x, y) being the place of m at the time t, and $\frac{ds}{dt}$ being its velocity,

$$\frac{ds^2}{dt^2} = 2gx;$$

$$\therefore \quad t = \int_0^x \frac{ds}{(2gx)^{\frac{1}{2}}}; \tag{56}$$

whereby the time may be found, when ds is expressed in terms of x by means of the equation of the curve. Now the equation of any one of the curves contains an arbitrary parameter, by the variation of which the several individuals of the family are determined; and by the elimination of this variable parameter between the equation of the curves and (56) the equation to the synchronous curve will be found.

For an example let us take a series of cycloids, placed in a vertical plane with their vertices downwards, with a common starting-point, and with their bases along the same horizontal line: then the equation to them is, a being variable,

$$y = a \operatorname{versin}^{-1}\frac{x}{a} - (2ax - x^2)^{\frac{1}{2}}; \tag{57}$$

and if τ is the common time down all to the synchronous curve,

$$\tau = \left(\frac{a}{g}\right)^{\frac{1}{2}} \operatorname{versin}^{-1}\frac{x}{a};$$

between which and (57) if a is eliminated, the resulting equation in terms of x and y will represent the required synchronous curve.

435.] We proceed now to a problem of a more interesting character: viz. the general problem of Brachistochronism: the object of which is to determine the nature of the path which a particle under the action of certain given forces takes, when the time of that passage is a minimum: or if the particle moves in a smooth tube, it is required to determine the form of the tube, when the time along it is less than that along any other tube.

I shall suppose the motion to be along a tube, and not on a surface, and if $\text{X}, \text{Y}, \text{Z}$ are the axial-components of the impressed velocity-increments, I shall suppose $\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz$ to be an exact differential: so that $\text{X}, \text{Y}, \text{Z}$ are functions of x, y, z only. The meaning of this restriction will be explained in the following Chapter. By equation (6), Art. 417, if the velocity at the initial point is zero,

$$v^2 = \frac{ds^2}{dt^2} = 2\int_0^1 (\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz); \tag{58}$$

using the notation of the Integral Calculus to indicate the limits of the definite integral.

The problem evidently requires the Calculus of Variations; and since

$$\frac{ds}{dt} = v; \qquad \therefore \quad dt = \frac{ds}{v};$$

$$\therefore \quad t = \int_0^1 \frac{ds}{v}; \tag{59}$$

and t is the function of x, y, z, according to the assumption above made, which is to be a minimum: and v is given by the equation (58); and no other general condition is given. Some conditions must be given at the limits; for the curve may be drawn either between given points, or from one given curve to another given curve.

Let (x_1, y_1, z_1) (x_0, y_0, z_0) be the terminal and initial positions of m; which, if the points are fixed, do not admit of variation: but if they are on given curves, the variations to which they are subject must be consistent with the equations to the curves. Taking the variation of (59), and equating it to zero, because t is to be a minimum, we have

$$\delta t = 0 = \delta \int_0^1 \frac{ds}{v} = \int_0^1 \left\{ \frac{\delta . ds}{v} - \frac{ds}{v^2} \delta v \right\}$$

$$= \int_0^1 \left\{ \frac{1}{v} \left(\frac{dx}{ds} d.\delta x + \frac{dy}{ds} d.\delta y + \frac{dz}{ds} d.\delta z \right) - \frac{ds}{v^3} (\mathrm{X}\, \delta x + \mathrm{Y}\, \delta y + \mathrm{Z}\, \delta z) \right\}, \quad (60)$$

because from (58),

$$v\, \delta v = \mathrm{X}\, \delta x + \mathrm{Y}\, \delta y + \mathrm{Z}\, \delta z; \quad (61)$$

and integrating by parts the former part of (60), we have

$$0 = \left[\frac{1}{v} \left(\frac{dx}{ds} \delta x + \frac{dy}{ds} \delta y + \frac{dz}{ds} \delta z \right) \right]_0^1$$

$$- \int_0^1 \left\{ \left(d.\frac{dx}{v\,ds} + \frac{\mathrm{X}\, ds}{v^3} \right) \delta x + \left(d.\frac{dy}{v\,ds} + \frac{\mathrm{Y}\, ds}{v^3} \right) \delta y + \left(d.\frac{dz}{v\,ds} + \frac{\mathrm{Z}\, ds}{v^3} \right) \delta z \right\}; \quad (6$$

and of these, the integrated and the unintegrated parts, each must separately be equal to zero. As to the integrated parts, if the limits are fixed points, they admit of no variation, and the expression vanishes identically. If the limits are on given lines, the expression shews that the brachistochron cuts both of them orthogonally.

As to the unintegrated part; since no relation is given between x, y, and z, the coefficients of δx, δy, δz must separately vanish: and therefore

$$\left. \begin{aligned} d.\frac{dx}{v\,ds} + \frac{\mathrm{X}\, ds}{v^3} &= 0, \\ d.\frac{dy}{v\,ds} + \frac{\mathrm{Y}\, ds}{v^3} &= 0, \\ d.\frac{dz}{v\,ds} + \frac{\mathrm{Z}\, ds}{v^3} &= 0; \end{aligned} \right\} \quad (63)$$

and expanding the first terms of these equations, and multiplying by v, we have

$$\left.\begin{aligned} d.\frac{dx}{ds} - \frac{dx}{ds}\frac{dv}{v} + \frac{\text{x}\,ds}{v^2} &= 0, \\ d.\frac{dy}{ds} - \frac{dy}{ds}\frac{dv}{v} + \frac{\text{y}\,ds}{v^2} &= 0, \\ d.\frac{dz}{ds} - \frac{dz}{ds}\frac{dv}{v} + \frac{\text{z}\,ds}{v^2} &= 0. \end{aligned}\right\} \qquad (64)$$

Let ρ be the radius of absolute curvature of the path: then by Art. 377, Differential Calculus,

$$\frac{ds^2}{\rho^2} = \left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2 + \left(d.\frac{dz}{ds}\right)^2; \qquad (65)$$

also, $$\frac{dx}{ds}d.\frac{dx}{ds} + \frac{dy}{ds}d.\frac{dy}{ds} + \frac{dz}{ds}d.\frac{dz}{ds} = 0.$$

Therefore multiplying the equations (64) severally by $d.\frac{dx}{ds}$, $d.\frac{dy}{ds}$, $d.\frac{dz}{ds}$, adding and substituting, we have

$$\frac{ds^2}{\rho^2} + \frac{ds}{v^2}\left\{\text{x}\,d.\frac{dx}{ds} + \text{y}\,d.\frac{dy}{ds} + \text{z}\,d.\frac{dz}{ds}\right\} = 0. \qquad (66)$$

But if λ, μ, ν are the direction-angles of the radius of absolute curvature, or, which is the same thing, of the principal normal, Art. 378, Differential Calculus,

$$\cos\lambda = \frac{\rho}{ds}d.\frac{dx}{ds}, \qquad \cos\mu = \frac{\rho}{ds}d.\frac{dy}{ds}, \qquad \cos\nu = \frac{\rho}{ds}d.\frac{dz}{ds}:$$

so that (66) becomes

$$\frac{v^2}{\rho} + (\text{x}\cos\lambda + \text{y}\cos\mu + \text{z}\cos\nu) = 0; \qquad (67)$$

and therefore, in absolute magnitude, the centrifugal force at every point on the brachistochron is equal to the resolved part of the impressed velocity-increment along the radius of absolute curvature. This is a general property of brachistochronous curves, and is one by which the path may in many cases be found when the laws of the acting forces are given.

If the brachistochron is a plane curve, the centrifugal force is equal to the normal component of the impressed forces.

This general property of the unconstrained brachistochron was discovered by Euler.

Also multiplying the equations (64) severally by

$$dz\,d.\frac{dy}{ds} - dy\,d.\frac{dz}{ds}, \dots;$$

adding, and reducing, we have

$$\text{x}(dz\,d^2y - dy\,d^2z) + \text{y}(dx\,d^2z - dz\,d^2x) + \text{z}(dy\,d^2x - dx\,d^2y) = 0;$$

which shews that the action-line of the resultant of the impressed velocity-increments is perpendicular to the binomial; and consequently that action-line always lies in the osculating plane.

436.] We will now consider some particular cases of brachistochronous curves.

(1) Suppose the velocity of the particle to be constant; so that $\text{X} = \text{Y} = \text{Z} = 0$: then also $dv = 0$; and from (64) we have

$$d.\frac{dx}{ds} = 0, \qquad d.\frac{dy}{ds} = 0, \qquad d.\frac{dz}{ds} = 0;$$

$$\therefore \quad \frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0};$$

which are the equations to a straight line.

(2) Let gravity be the only acting force; and let its line of action be parallel to the axis of z; then $\text{X} = \text{Y} = 0$, $\text{Z} = g$; therefore also $v^2 = 2g(z-z_0)$.

And equations (63) become

$$d.\frac{dx}{v\,ds} = 0; \qquad d.\frac{dy}{v\,ds} = 0; \qquad d.\frac{dz}{v\,ds} + \frac{g\,ds}{v^2} = 0;$$

integrating the first two,

$$\frac{dx}{v\,ds} = a, \qquad \frac{dy}{v\,ds} = \beta;$$

$$\therefore \quad \frac{dx}{a} = \frac{dy}{\beta};$$

and thus the motion takes place in a plane perpendicular to that of (x, y); let the plane of motion be that of (x, z); so that all the terms involving y disappear, and we have

$$d.\frac{dx}{v\,ds} = 0, \qquad d.\frac{dz}{v\,ds} + g\frac{ds}{v^2} = 0;$$

from the first of these

$$\frac{dx}{v\,ds} = \text{a constant} = \frac{1}{(4ga)^{\frac{1}{2}}}, \text{ (say)},$$

where a is an arbitrary constant:

$$\therefore \quad \frac{dx}{ds} = \left(\frac{z-z_0}{2a}\right)^{\frac{1}{2}};$$

so that $\frac{dx}{ds} = 0$, when $z = z_0$; that is, the curve is vertical at the point (x_0, z_0) of departure of m.

And since $ds^2 = dx^2 + dz^2$, we have

$$dx = \frac{(z-z_0)\,dz}{\{2a(z-z_0)-(z-z_0)^2\}^{\frac{1}{2}}};$$

$$x-x_0 = a\,\text{versin}^{-1}\frac{z-z_0}{a}-\{2a(z-z_0)-(z-z_0)^2\}^{\frac{1}{2}};$$

which is the equation to a cycloid, whose base is horizontal, and starting-point is (x_0, z_0). Let (x_0, z_0) be the origin; then the equation becomes

$$x = a\,\text{versin}^{-1}\frac{z}{a}-(2az-z^2)^{\frac{1}{2}};$$

a is still undetermined. It is however to be such that the point (x_n, z_n) may be on the curve, so that for its determination we have

$$x_n = a\,\text{versin}^{-1}\frac{z_n}{a}-(2az_n-z_n^2)^{\frac{1}{2}}.$$

(3) To find the brachistochron when the force is a central attracting force, and varies inversely as the square of the distance.

In this case $\quad v^2 = 2\mu\left(\frac{1}{r}-\frac{1}{r_0}\right); \qquad \rho = \frac{r\,dr}{dp};$

and the normal component of the impressed force $= \frac{\mu}{r^2}\,\frac{p}{r}$.

So that by the general property (67) we have

$$\frac{2\,dp}{p}+\frac{r_0\,dr}{r(r_0-r)} = 0;$$

$$\therefore \quad \log\frac{p^2}{c^2}+\log\frac{r}{r_0-r} = 0;$$

$$\therefore \quad p^2 = c^2\,\frac{r_0-r}{r};$$

where c^2 is an undetermined constant.

437.] Thus far the tubes or curves on which a particle has been constrained to move have been fixed; the tube however may move in the time during which the particle moves in it, so that the actual motion of the particle in space will be compounded of the motion of the tube, and also of its own motion in the tube. This is indeed a case of relative motion, and its solution depends on the principle explained in Arts. 317–319, 332. I will however first solve some examples from first principles, and in the following Article apply the equations of Art. 332. I shall suppose the magnitudes of the particle and the tube to be such that the particle just fills the smooth tube.

The principle of solution is the same in all cases; the reaction of the tube will be along the normal, and the motion along the tube will be the effect of the tangential component of the impressed velocity-increment.

Ex. 1. A tube bent into the form of a plane curve revolves with an uniform angular velocity about a vertical axis in its own plane; it is required to determine the form of the tube, when a heavy particle placed in it remains at rest in all parts of the tube.

Let the vertical line be the axis of z. Let (x, y, z) be the place of m at the time t; let $x^2+y^2 = r^2$, $\theta =$ the angle between r and the plane of (x, z); $\omega =$ the constant angular velocity; so that $\theta = \omega t$, if θ and t simultaneously are equal to zero.

The impressed velocity-increments on m are (1) gravity, (2) the centrifugal force due to the rotation of the tube about the vertical axis: resolving these along the tangent to the tube at the point (r, z), we have

$$\frac{d^2s}{dt^2} = \omega^2 r \frac{dr}{ds} - g\frac{dz}{ds}; \tag{68}$$

and as the particle is to be at rest, $\frac{d^2s}{dt^2} = 0$; therefore

$$\omega^2 r\,dr = g\,dz; \qquad r^2 = \frac{2g}{\omega^2} z,$$

if $r = 0$, when $z = 0$; and this is the equation of a parabola, of which the latus rectum is $\frac{2g}{\omega^2}$.

Since also $\frac{d^2s}{dt^2} = 0$, when the particle moves with a constant velocity, the preceding solution is applicable, when the particle moves along the tube with a constant velocity.

If the equation to the curve of the tube is given we may by means of (68) determine the point at which a particle will rest.

Also if the velocity of the particle is a function of the path which it has passed over, equation (68) may be integrated. Thus if $v^2 = k^2(z^2-a^2)$; then

$$k^2(z^2-a^2) = \omega^2 r^2 - 2gz + \text{C},$$

where C is undetermined; and this is the equation to a conic.

Ex. 2. To determine the motion of a heavy particle placed in a smooth rectilinear tube which is attached to a vertical axis about which it revolves with a given angular velocity.

Let α be the angle at which the tube is inclined to the vertical axis, and let ω be the constant angular velocity: let the vertex of the cone described by the tube be the origin; (x, y, z) the place of m at the time t; let r be the distance of m from the vertex of the cone: so that the centrifugal force at the time t is $\omega^2 r \sin a$; then taking the components of the velocity-increments along the tube, we have

$$\frac{d^2 r}{dt^2} = \omega^2 r (\sin a)^2 - g \cos a.$$

Multiplying these by $2\dfrac{dr}{dt}$, and integrating on the supposition that $\dfrac{dr}{dt} = u$ when $r = 0$, we have

$$\frac{dr^2}{dt^2} - u^2 = \omega^2 r^2 (\sin a)^2 - 2gr \cos a;$$

whence the final integral may easily be found.

If the tube revolves with an uniform velocity ω in a horizontal plane, we have

$$\frac{d^2 r}{dt^2} = \omega^2 r; \qquad \frac{dr^2}{dt^2} = \omega^2 (r^2 - a^2),$$

if $\dfrac{dr}{dt} = 0$, when $r = a$. Hence we have

$$r = \frac{a}{2}\{e^{\omega t} + e^{-\omega t}\}.$$

438.] Many of these problems however require for their complete solution the equations of relative motion which are given in Art. 332.

Ex. 1. A smooth rectilinear tube revolves uniformly in a horizontal plane about a vertical axis, and a particle moves in it under the action of an attracting force which varies as the distance from the point where the axis pierces the tube; determine the motion of the particle.

Let the moving axis of ξ coincide with the axis of the tube; and let ωt be at the time t the angle between the axes of ξ and x: then, since $\eta = 0$, the equations of motion are

$$\frac{d^2 \xi}{dt^2} - \omega^2 \xi = -\mu \xi; \qquad 2\omega \frac{d\xi}{dt} = \text{R}. \tag{69}$$

Let μ be greater than ω^2, and let $\mu - \omega^2 = n^2$; and let a be the initial distance of m from the origin: so that the equation of motion is

$$\frac{d^2 \xi}{dt^2} = -n^2 \xi; \qquad \therefore \quad \frac{d\xi^2}{dt^2} = n^2 (a^2 - \xi^2);$$

$$\therefore \quad \xi = a \cos nt;$$

and thus we have all the incidents of harmonic motion relatively to the tube, the time of a complete vibration being $\frac{\pi}{(\mu-\omega^2)^{\frac{1}{2}}}$. If the absolute path of the particle in the plane is referred to a system of polar coordinates r and θ, the equation to the path is

$$r = a \cos \frac{(\mu-\omega^2)^{\frac{1}{2}}}{\omega} \theta;$$

which is the equation to a revolving circle such as we have explained in Art. 385.

If ω^2 is greater than μ, and if $\omega^2-\mu = n^2$; then the relative motion is given by $\xi = a\,(e^{nt}+e^{-nt})$.

The pressure of the particle against the tube is in all cases given by the second equation of (69).

Ex. 2. A heavy particle moves in a smooth rectilineal tube which revolves uniformly in a vertical plane about a horizontal axis passing through it; it is required to determine the absolute path of the particle.

Let the plane in which the tube revolves be that of (x, z), the axis of y being that about which it revolves with the uniform angular velocity ω: let us suppose the tube to be vertical and the distance of m from the origin to be a, when $t=0$; the axis of ξ coinciding with the axis of the tube; then the equations of motion are

$$\frac{d^2\xi}{dt^2}-\omega^2\xi = -g\cos\omega t; \qquad 2\omega\frac{d\xi}{dt} = \mathrm{R}+g\sin\omega t;$$

from the first of which we have, A and B being arbitrary constants,

$$\xi = -\frac{g}{2\omega^2}\cos\omega t+\mathrm{A}e^{\omega t}+\mathrm{B}e^{-\omega t};$$

but when $t = 0$, $\xi = a$, $\frac{d\xi}{dt} = a\omega$; consequently

$$\xi = -\frac{g}{2\omega^2}\cos\omega t+\left(a+\frac{g}{4\omega^2}\right)e^{\omega t}+\frac{g}{4\omega^2}e^{-\omega t};$$

and this assigns the motion of m in the tube. The mean distance of m from the origin is given by the last two terms, the first term assigning a periodical quantity, of which the maximum value is $\frac{g}{2\omega^2}$, by which m is sometimes nearer to and sometimes farther from the origin than its mean place; the periodic time of this nutatory quantity being $\frac{2\pi}{\omega}$. If θ is the angle between the tube and the z-axis at the time t, the absolute path of m is given by the equation

$$r = -\frac{g}{2\omega^2}\cos\theta + \left(a + \frac{g}{4\omega^2}\right)e^{\theta} + \frac{g}{4\omega^2}e^{-\theta}. \tag{70}$$

Ex. 3. To determine the motion of a particle placed inside a smooth circular tube which revolves uniformly in a horizontal plane about a vertical axis which passes through the tube.

Let ω be the uniform angular velocity of the circular tube; $\text{OC} = \text{CA} = a$; OA being the diameter of the circle which is coincident with the axis of x when $t = 0$; so that $\text{AOX} = \omega t$. Let P be the place of m at the time t; $\text{OP} = r$; $\text{POC} = \text{CPO} = \theta$; therefore $\text{PCA} = 2\theta$. We will moreover suppose the particle to be absolutely at rest at A on OX when $t = 0$. Let OCA be the axis of ξ, and the line through O perpendicular to it the axis of η; then the equations of relative motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2\xi - 2\omega\frac{d\eta}{dt} &= -\frac{\text{R}}{m}\cos 2\theta; \\ \frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega\frac{d\xi}{dt} &= -\frac{\text{R}}{m}\sin 2\theta; \end{aligned}\right\} \tag{71}$$

and $\xi = a(\cos 2\theta + 1)$, $\eta = a\sin 2\theta$;

$$\therefore \quad d\xi = -2a\sin 2\theta\, d\theta, \qquad d\eta = 2a\cos 2\theta\, d\theta;$$

$$\therefore \quad \frac{d\xi\, d^2\xi + d\eta\, d^2\eta}{dt^2} - \omega^2(\xi\, d\xi + \eta\, d\eta) = 0;$$

$$\frac{d\sigma^2}{dt^2} - 4a^2\omega^2 - \omega^2(r^2 - 4a^2) = 0,$$

since $\dfrac{d\sigma}{dt} = 2a\omega$, when $r = 2a$;

$$\therefore \quad \frac{d\sigma}{dt} = \omega r;$$

but $d\sigma = 2a\,d\theta$, and $r = 2a\cos\theta$; consequently

$$\frac{d\theta}{dt} = \omega\cos\theta;$$

$$\therefore \quad \log\frac{1+\sin\theta}{1-\sin\theta} = 2\omega t; \qquad \sin\theta = \frac{e^{\omega t} - e^{-\omega t}}{e^{\omega t} + e^{-\omega t}}; \tag{72}$$

which gives the angle θ in terms of t; and as the relative orbit is of course a circle, the circumstances of relative motion are hereby determined.

In the absolute path of m, if $\text{OP} = r$, $\text{PO}x = \phi$, we have $\theta + \phi = \omega t$, $r = 2a\cos\theta$, and also (72); and if we eliminate θ and ωt from these equations, the resulting equation in terms of r and ϕ will be that of the absolute path of m.

SECTION 2.—*Motion of a particle constrained to move on a given curved surface.*

439.] The particle which has been the subject of motion in the preceding Section has been constrained to move in a given tube: we proceed now to investigate the equations of motion and the results of these equations when the particle is constrained to be in contact with a given surface, but is free to describe on the surface such a path as is compatible with the forces to which it is subject.

Let the equation to the surface be

$$\mathrm{F}(x, y, z) = c;$$

and let (x, y, z) be the place of m at the time t. Let U, V, W express the several partial derived-functions of F, and let

$$\mathrm{Q}^2 = \mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2;$$

let X, Y, Z be the components along the coordinate axes of the impressed velocity-increments; let $\mathrm{R} =$ the normal pressure of the surface on m at the time t: the direction-cosines of the line of action of R are

$$\frac{\mathrm{U}}{\mathrm{Q}}, \quad \frac{\mathrm{V}}{\mathrm{Q}}, \quad \frac{\mathrm{W}}{\mathrm{Q}};$$

so that the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mathrm{X} \pm \frac{\mathrm{R}}{m}\frac{\mathrm{U}}{\mathrm{Q}}, \\ \frac{d^2y}{dt^2} &= \mathrm{Y} \pm \frac{\mathrm{R}}{m}\frac{\mathrm{V}}{\mathrm{Q}}, \\ \frac{d^2z}{dt^2} &= \mathrm{Z} \pm \frac{\mathrm{R}}{m}\frac{\mathrm{W}}{\mathrm{Q}}; \end{aligned}\right\} \qquad (73)$$

if we eliminate R from these equations taken two and two together, three equations will result, which together with the equation to the surface will determine x, y, z in terms of t; and if t is eliminated, two equations will result, which determine surfaces, the line of intersection of which is the line of motion of the particle.

Let us suppose $\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz$ to be an exact differential; and let us multiply equations (73) severally by dx, dy, dz, and add: then, remembering that $\mathrm{U}\,dx + \mathrm{V}\,dy + \mathrm{W}\,dz = 0$, we have

$$\frac{dx\,d^2x + dy\,d^2y + dz\,d^2z}{dt^2} = \mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz;$$

and integrating between limits corresponding to $t = t$ and to $t = t_0$, we have

$$\frac{v^2}{2} - \frac{v_0^2}{2} = \int_{t_0}^{t} \{\text{X}\, dx + \text{Y}\, dy + \text{Z}\, dz\}, \tag{74}$$

if v_0 is the velocity of m when $t = t_0$.

From this equation the time taken by m in passing from one to another point on the surface may be found.

Again, multiplying equations (73) severally by $\frac{\text{U}}{\text{Q}}, \frac{\text{V}}{\text{Q}}, \frac{\text{W}}{\text{Q}}$, and adding, we have

$$\frac{\text{U}\, d^2x + \text{V}\, d^2y + \text{W}\, d^2z}{\text{Q}\, dt^2} = \frac{\text{XU} + \text{YV} + \text{ZW}}{\text{Q}} \pm \frac{\text{R}}{m}, \tag{75}$$

whereby the pressure on the surface is given.

If the path which the particle is taking at (x, y, z) lies in a principal normal section of the surface at that point; and if ρ is the length of the corresponding principal radius of curvature, then by (12), Art. 399, Differential Calculus,

$$\rho = \frac{\text{Q}\, ds^2}{\text{U}\, d^2x + \text{V}\, d^2y + \text{W}\, d^2z}; \tag{76}$$

so that (75) becomes

$$\pm \text{R} = m \left\{ \frac{\text{XU} + \text{YV} + \text{ZW}}{\text{Q}} \pm \frac{v^2}{\rho} \right\}, \tag{77}$$

that is, the pressure on the curve is the algebraic sum of the normal component at that point of the impressed momentum-increment and of the centrifugal force.

If the particle moves on the surface, and is under the action of no force, so that $\text{X} = \text{Y} = \text{Z} = 0$, then equations (73) give

$$\frac{\frac{d^2x}{dt^2}}{\text{U}} = \frac{\frac{d^2y}{dt^2}}{\text{V}} = \frac{\frac{d^2z}{dt^2}}{\text{W}} = \pm \frac{\text{R}}{m\text{Q}}; \tag{78}$$

but the velocity of m under these circumstances, as (74) shews, is constant; and therefore the numerators of (78) are proportional to $\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}, \frac{d^2z}{ds^2}$; and we have, if s is not equicrescent,

$$\frac{d.\frac{dx}{ds}}{\text{U}} = \frac{d.\frac{dy}{ds}}{\text{V}} = \frac{d.\frac{dz}{ds}}{\text{W}}, \tag{79}$$

which are the equations to a geodesic line on the surface; see Art. 336, Integral Calculus; the path therefore of m is a geodesic line.

440.] In application of these equations, let us consider the motion of a heavy particle on a sphere; and, to fix our thoughts, let us suppose the particle to move on the *inside* of the sphere, and take, as in fig. 139, the horizontal plane through the centre of the sphere to be the plane of (x, y), and the axis of z to be vertical downwards through the centre. Then if a is the radius of the sphere,

$$x^2+y^2+z^2 = a^2; \tag{80}$$

so that the equations of motion become

$$\frac{d^2x}{dt^2} = -\frac{\text{R}}{m}\frac{x}{a}, \quad \frac{d^2y}{dt^2} = -\frac{\text{R}}{m}\frac{y}{a}, \quad \frac{d^2z}{dt^2} = g-\frac{\text{R}}{m}\frac{z}{a}. \tag{81}$$

Multiplying these respectively by $2\,dx$, $2\,dy$, $2\,dz$, adding and integrating,

$$\begin{aligned}\frac{ds^2}{dt^2} = v^2 &= v_0{}^2+2g(z-z_0)\\ &= c+2gz, \text{ (say)},\end{aligned} \tag{82}$$

where $c = v_0{}^2-2gz_0$; v_0 and z_0 being the values of v and of z, when $t=0$.

Also from the first two of (81), $x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2} = 0$;

$$\therefore \quad x\,dy-y\,dx = k\,dt, \tag{83}$$

where k is an arbitrary constant. Let the angle $\text{NOM} = \phi$, $\text{ON} = \rho$; so that $x = \rho\cos\phi$, $y = \rho\sin\phi$; and ρ and ϕ are the polar coordinates of the horizontal projection of the path of the particle; therefore, as in Art. 379,

$$\rho^2\,d\phi = k\,dt; \tag{84}$$

and therefore ρ describes equal sectorial areas in equal times.

To find an expression for the time in terms of z; from the equation to the sphere we have

$$x\,dx+y\,dy = -z\,dz;$$

$$\text{also} \qquad x\,dy-y\,dx = k\,dt;$$

$$\therefore \quad (dx^2+dy^2)(x^2+y^2) = z^2\,dz^2+k^2\,dt^2;$$

$$\therefore \quad \frac{ds^2}{dt^2} = c+2gz = \frac{dx^2+dy^2+dz^2}{dt^2} = \frac{a^2\,dz^2+k^2\,dt^2}{(a^2-z^2)\,dt^2};$$

$$\therefore \quad dt = \frac{a\,dz}{\{(a^2-z^2)(c+2gz)-k^2\}^{\frac{1}{2}}}; \tag{85}$$

whence might the time be found in terms of z, if the expression were integrable.

Also, since $\rho^2 = a^2 - z^2$, from (84) we have

$$d\phi = \frac{ka\,dz}{(a^2-z^2)\{(a^2-z^2)(c+2gz)-k^2\}^{\frac{1}{2}}}; \qquad (86)$$

which expression does not admit of integration in a finite form. (85) will give the time taken by the particle in passing from $z = z_0$ to $z = z$; and (86) will give the curve described by m on the spherical surface, which will be a kind of spherical spiral. If we equate to zero $\frac{dz}{d\phi}$, we shall have the values of z, which render z a maximum or a minimum if there is a change of sign, and to which in all cases corresponds a horizontal motion of m. And since $(a^2-z^2)(c+2gz)-k^2 = 0$ is a cubic equation, it has always one real root; and as a factor of the first degree will correspond to this, so will the curve always be such that z will have a maximum or minimum value.

Equations (85) and (86) may also be reduced to elliptic functions, and their properties may be studied in that relation; but it is beside our purpose to proceed further with the inquiry in that direction.

The constant c is known in terms of the initial velocity and of the z-ordinate of the initial position of m. As to k; let ρ_0, v_0 be the initial values of ρ, v; and suppose the line of v_0 to make an angle a with the parallel of latitude at the initial point; then the component of the velocity along that parallel of latitude is $v_0 \cos a$; and ρ_0 is the radius of that parallel of latitude; therefore $\rho_0 v_0 \cos a$ is twice the sectorial area described in one unit of time by ρ_0 on the horizontal plane: and from (84) this quantity is equal to k, therefore

$$k = \rho_0 v_0 \cos a.$$

As to the normal pressure on the surface; multiplying equations (81) severally by x, y, z, and adding, we have

$$\frac{x\,d^2x + y\,d^2y + z\,d^2z}{dt^2} = gz - \frac{\text{R}a}{m};$$

and from the equation to the sphere,

$$x\,dx + y\,dy + z\,dz = 0,$$

$$\begin{aligned} \therefore\quad x\,d^2x + y\,d^2y + z\,d^2z &= -(dx^2 + dy^2 + dz^2) \\ &= -ds^2 \\ &= -v^2\,dt^2; \end{aligned}$$

$$\therefore\quad \text{R} = \frac{m}{a}(v^2 + gz). \qquad (87)$$

$m\dfrac{v^2}{a}$ is the centrifugal force of the path described by m, an as $\dfrac{z}{a}$ is the cosine of the angle between the radius of the spher and the line of action of g, $\dfrac{mgz}{a}$ is the normal component of th weight of m; so that the pressure on the surface is equal to th sum of the centrifugal force and the normal component of th weight of m.

441.] The motion on a sphere can of course be effected b means of a heavy *bob* or mass m attached by a string or thi rod of a given length a to a point about which it can turn i all directions; and thus the preceding investigations become o importance, because they are those of the motion of a spheric pendulum: and although the expressions do not generally admi of integration, yet when the distance of m from the vertical lin does not exceed a small quantity, we can expand in ascendin powers of that small quantity, and obtain results which ar approximately exact.

Let COP$=\theta$, and let α be the initial value of θ; α and θ bein always so small that we shall omit all powers of them above th second; and let us suppose the initial velocity v_0 to be in horizontal line, so that $k = \rho_0 v_0$; therefore

$$z = a\cos\theta \qquad\qquad z_0 = a\cos\alpha$$

$$= a - \frac{a\theta^2}{2}; \qquad\qquad = a - \frac{a\alpha^2}{2};$$

$$c = v_0^2 - 2ga + ga\alpha^2,$$

$$k^2 = \rho_0^2 v_0^2 = (a^2 - z_0^2) v_0^2$$

$$= a^2\alpha^2 v_0^2; \qquad (88$$

so that (85) and (86) become, after simplification by putting

$$v_0^2 = ga\beta^2,$$

$$dt = \left(\frac{a}{g}\right)^{\frac{1}{2}} \frac{-\theta\, d\theta}{\{(\alpha^2-\theta^2)(\theta^2-\beta^2)\}^{\frac{1}{2}}}, \qquad (89$$

$$d\phi = \frac{-\alpha\beta\, d\theta}{\theta\,\{(\alpha^2-\theta^2)(\theta^2-\beta^2)\}^{\frac{1}{2}}}; \qquad (90$$

from these it appears that θ must always be intermediate to and β; and therefore if $\beta = \alpha$, or $v_0^2 = ga\alpha^2$, θ is always equa to α; the pendulum, that is, describes a circular cone, of whic α is the semi-vertical angle, and the bob moves in a circle; an dividing (89) by (90), we have in this case,

$$dt = \left(\frac{a}{g}\right)^{\frac{1}{2}} d\phi;$$

and if $\phi = 0$ when $t = 0$, the time of a revolution is

$$2\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}, \tag{91}$$

which is twice the time, see (40), Art. 427, of an oscillation, when the pendulum vibrates in one vertical plane. If therefore two pendulums of the same length a start simultaneously from the same line OA, which is inclined to the vertical at the angle α, the one from rest, the other with a velocity equal to $\alpha\,(ga)^{\frac{1}{2}}$ in a line perpendicular to the vertical plane containing OA, both will return again simultaneously to the same line OA.

Let however α and β be unequal: then from (89) we have

$$dt = \left(\frac{a}{g}\right)^{\frac{1}{2}} \frac{-\theta\, d\theta}{\left\{\left(\frac{\alpha^2-\beta^2}{2}\right)^2 - \left(\theta^2 - \frac{\alpha^2+\beta^2}{2}\right)^2\right\}^{\frac{1}{2}}};$$

whence integrating, and observing that $\theta = \alpha$ when $t = 0$,

$$t = \frac{1}{2}\left(\frac{a}{g}\right)^{\frac{1}{2}} \cos^{-1}\frac{2\theta^2-\alpha^2-\beta^2}{\alpha^2-\beta^2}; \tag{92}$$

$$\therefore\quad \theta^2 = \frac{\alpha^2+\beta^2}{2} + \frac{\alpha^2-\beta^2}{2}\cos 2t\left(\frac{g}{a}\right)^{\frac{1}{2}} \tag{93}$$

$$= \alpha^2\left\{\cos t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2 + \beta^2\left\{\sin t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2. \tag{94}$$

Hence it appears that θ^2 is periodic, and that its greatest and least values are α^2 and β^2; the time elapsing between $\theta = \alpha$ and $\theta = \beta$ is

$$\frac{\pi}{2}\left(\frac{a}{g}\right)^{\frac{1}{2}}.$$

For the azimuthal motion of the vertical plane which contains the pendulum we have, from (89) and (90),

$$d\phi = \left(\frac{g}{a}\right)^{\frac{1}{2}} \frac{\alpha\beta\, dt}{\theta^2}$$

$$= \left(\frac{g}{a}\right)^{\frac{1}{2}} \frac{\alpha\beta\, dt}{\alpha^2\left\{\cos t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2 + \beta^2\left\{\sin t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2};$$

integrating which, and supposing that $\phi = 0$ when $t = 0$, we have

$$\alpha \tan\phi = \beta\tan t\left(\frac{g}{a}\right)^{\frac{1}{2}}; \tag{95}$$

ϕ therefore does not vary directly as the time, as is the case when $\beta = \alpha$; but the plane revolves through 90°, during a time which is equal to

$$\frac{\pi}{2}\left(\frac{a}{g}\right)^{\frac{1}{2}}.$$

From (95) we have

$$(\cos\phi)^2 = \frac{\alpha^2\left\{\cos t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2}{\alpha^2\left\{\cos t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2+\beta^2\left\{\sin t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2}, \tag{96}$$

$$(\sin\phi)^2 = \frac{\beta^2\left\{\sin t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2}{\alpha^2\left\{\cos t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2+\beta^2\left\{\sin t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2}. \tag{97}$$

Also if x and y refer to the place of m at the time t, we have

$$\left.\begin{aligned} x^2 &= (a^2-z^2)(\cos\phi)^2 = a^2\theta^2(\cos\phi)^2, \\ y^2 &= (a^2-z^2)(\sin\phi)^2 = a^2\theta^2(\sin\phi)^2; \end{aligned}\right\} \tag{98}$$

therefore from (96) (97) and (98),

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2} = a^2; \tag{99}$$

which is the equation of an ellipse, of which the semi-axes are $a\alpha$ and $b\beta$. Hence we infer that the projection on a horizontal plane of the path described by the bob of the pendulum is an ellipse, whose centre is in a vertical line drawn through the centre of suspension, and one of whose principal axes lies in the vertical plane perpendicular to the line along which m is at first projected. Now an ellipse may be described by a particle moving under the attraction of a central force situated in the centre of the ellipse and which varies directly as the distance; we may therefore suppose the bob of the pendulum to move under the action of such a central force. Let μ be the absolute central force; then by Art. 382, the periodic time $= \frac{2\pi}{\mu^{\frac{1}{2}}}$; but from above, the periodic time $= 2\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$; therefore $\mu = \frac{g}{a}$; so that if P is the central force, $\text{P} = \frac{g}{a}\rho$.

And this is the resolved part of the tension of the rod along the direction of ρ referred to an unit-mass; as may thus be shewn: if $\text{R} =$ this tension, then the resolved part of R along ρ

$$= \frac{v^2+gz}{a}\sin\theta, \text{ by reason of (87)},$$

$$= \left(1+\beta^2+\alpha^2-\frac{3\theta^2}{2}\right)g\theta;$$

and omitting cubes of small quantities, if $\text{P} =$ this resolved part, $\text{P} = g\theta$; but from (98), $x^2+y^2 = \rho^2 = a^2\theta^2$;

$$\therefore\ \text{P} = \frac{g}{a}\rho;$$

that is, varies directly as ρ.

In the Mécanique Analytique, second part, Sect. VIII, Ch. II, § I, the approximations are carried on so as to involve powers of a and β higher than those which we have taken account of; and M. Bravais, after correcting some errors of M. Lagrange, shews that the angle between two successive points corresponding to $\theta = a$ and to $\theta = \beta$ is $\frac{\pi}{2}\left(1+\frac{3\,a\beta}{8}\right)$, instead of being $\frac{\pi}{2}$, as it is in the preceding investigations. Hence there is a progressive motion in azimuth of the apse, in the direction in which the pendulum moves.

442.] As another example let us consider the motion of a material particle on a surface of revolution, and acted on by forces in a plane passing through the axis. Let the axis of revolution of the surface be the coordinate axis of z; and let $x^2+y^2 = \rho^2$; so that the equation to the surface is

$$\rho^2 = x^2+y^2 = f(z);$$

$$\therefore \quad \mathrm{F}(x, y, z) = x^2+y^2-f(z) = 0; \tag{100}$$

$$\therefore \quad \mathrm{U} = 2x, \qquad \mathrm{V} = 2y, \qquad \mathrm{W} = -f'(z);$$

$$\therefore \quad \mathrm{Q}^2 = 4\rho^2+\{f'(z)\}^2.$$

Since the impressed forces act in a meridian plane only, let them be resolved parallel to the axis of z, and perpendicular to that axis; of which let the components be severally Z and P; then of P's line of action the x- and y-direction-cosines are $\frac{x}{\rho}$, $\frac{y}{\rho}$; and let R, the pressure on the surface, be resolved in a similar way: so that its components are

$$\mathrm{R}\frac{dz}{ds}\frac{x}{\rho}, \qquad \mathrm{R}\frac{dz}{ds}\frac{y}{\rho}, \qquad \mathrm{R}\frac{d\rho}{ds},$$

where $ds^2 = d\rho^2+dz^2$; and the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mathrm{P}\frac{x}{\rho}+\frac{\mathrm{R}}{m}\frac{dz}{ds}\frac{x}{\rho} = \left(\mathrm{P}+\frac{\mathrm{R}}{m}\frac{dz}{ds}\right)\frac{x}{\rho},\\ \frac{d^2y}{dt^2} &= \mathrm{P}\frac{y}{\rho}+\frac{\mathrm{R}}{m}\frac{dz}{ds}\frac{y}{\rho} = \left(\mathrm{P}+\frac{\mathrm{R}}{m}\frac{dz}{ds}\right)\frac{y}{\rho},\\ \frac{d^2z}{dt^2} &= \mathrm{Z}+\frac{\mathrm{R}}{m}\frac{d\rho}{ds}. \end{aligned}\right\} \tag{101}$$

From the first two we have $x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2} = 0$;

$$\therefore \quad x\,dy-y\,dx = h\,dt; \tag{102}$$

so that the projections on a plane perpendicular to the axis of revolution of the sectorial areas described by ρ vary as the times in which they are described.

Again,

$$\frac{dx\,d^2x+dy\,d^2y+dz\,d^2z}{dt^2} = \mathrm{P}\frac{x\,dx+y\,dy}{\rho}+\mathrm{Z}\,dz$$

$$= \mathrm{P}\,d\rho+\mathrm{Z}\,dz;$$

$$\therefore\quad \frac{v^2}{2}-\frac{v_0^2}{2} = \int_{t_0}^{t}\{\mathrm{P}\,d\rho+\mathrm{Z}\,dz\}. \tag{103}$$

Without carrying these general investigations farther, I shall apply them to a few examples.

443.] To determine the circumstances under which a heavy particle will describe a parallel of latitude when the axis of revolution of the surface is vertical, and the velocity of projection is a function of the coordinates of the point of projection.

In this case $\mathrm{P} = 0$, $\mathrm{Z} = -g$; the particle is projected in a horizontal plane, and along the tangent to the parallel of latitude. Then the equations become

$$\frac{d^2x}{dt^2} = \frac{\mathrm{R}}{m}\frac{dz}{ds}\frac{x}{\rho}, \qquad \frac{d^2y}{dt^2} = \frac{\mathrm{R}}{m}\frac{dz}{ds}\frac{y}{\rho},$$

$$\frac{d^2z}{dt^2} = -g+\frac{\mathrm{R}}{m}\frac{d\rho}{ds};$$

$$\therefore\quad \frac{v^2}{2}-\frac{v_0^2}{2} = g(z_0-z);$$

where v_0 and z_0 are the initial values of v and z; but $z = z_0$, because the particle describes a parallel of latitude; therefore the velocity is constant.

Let us consider the equation to the meridian curve as it is in the plane of (x, z): and let the initial velocity at the point (x, z) be $\{2gf(x,z)\}^{\frac{1}{2}}$; then resolving along the meridian curve the centrifugal force and gravity, we have

$$\frac{2gf(x,z)}{x}\frac{dx}{ds} = g\frac{dz}{ds};$$

$$\therefore\quad f(x,z) = \frac{x}{2}\frac{dz}{dx}. \tag{104}$$

(1) Let $f(x, z) = a$; so that the velocity is constant;

$$\therefore\quad \frac{2\,dx}{x} = \frac{dz}{a}, \qquad x^2 = ce^{\frac{z}{a}}.$$

(2) Let $f(x,z) = \frac{x^2}{a}$;

$$\therefore\quad \frac{2x\,dx}{a} = dz, \qquad x^2 = az.$$

444.] To complete the subject of brachistochronous curves, we must consider the properties of those paths which a particle moving on a curved surface under the action of given forces takes, when the time of passing from one point to another, or from one curve to another, is a minimum.

Now on referring to Article 435, the investigation is the same as far as equation (62); and as to the integrated part, if the initial and terminal points are given, it vanishes identically; if the curves on which they are to be on the surface are given, the equation shews that the brachistochron cuts both curves at right angles: this result is evident from general reasoning.

As to the unintegrated part of (62), δx, δy, δz are no longer independent of each other; but if the equation to the surface is

$$\mathrm{F}(x, y, z) = 0, \tag{105}$$

and if U, V, W are the partial derived-functions of it, we have

$$\mathrm{U}\,\delta x + \mathrm{V}\,\delta y + \mathrm{W}\,\delta z = 0; \tag{106}$$

and as this relation exists at *all* points of the brachistochron, we have from the comparison of it with the unintegrated part of (62),

$$\frac{d.\frac{dx}{v\,ds} + \frac{\mathrm{X}\,ds}{v^2}}{\mathrm{U}} = \frac{d.\frac{dy}{v\,ds} + \frac{\mathrm{Y}\,ds}{v^2}}{\mathrm{V}} = \frac{d.\frac{dz}{v\,ds} + \frac{\mathrm{Z}\,ds}{v^2}}{\mathrm{W}}; \tag{107}$$

which are the general equations to the brachistochron; because the integrals of these equations will give two surfaces, the intersection of which is the brachistochronous curve.

If no forces act, $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = 0$, and the velocity is constant: so that (107) become

$$\frac{d.\frac{dx}{ds}}{\mathrm{U}} = \frac{d.\frac{dy}{ds}}{\mathrm{V}} = \frac{d.\frac{dz}{ds}}{\mathrm{W}}, \tag{108}$$

which are the equations to a geodesic line; in this case therefore the geodesic joining the two given points, or that which is orthogonal to the two given curves, is the brachistochron.

In these investigations I shall suppose $\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz$ to be an exact differential; then since, see Art. 435,

$$d.v^2 = 2\,\{\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz\}; \tag{109}$$

$$\therefore \quad dv = \frac{\mathrm{X}}{v}\,dx + \frac{\mathrm{Y}}{v}\,dy + \frac{\mathrm{Z}}{v}\,dz; \tag{110}$$

$$\therefore \quad \mathrm{X} = v\left(\frac{dv}{dx}\right), \qquad \mathrm{Y} = v\left(\frac{dv}{dy}\right), \qquad \mathrm{Z} = v\left(\frac{dv}{dz}\right). \tag{111}$$

Now v is a function of x, y, z; therefore $v = c$ is the equation of a surface; and since, when $v = 0$, $\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz = 0$, I shall, as in Art. 232, call $v = c$ the equilibrium-surface: it is evidently such that at all points of the brachistochronous path, the resultant line of action of the impressed forces is normal to the equilibrium-surface.

445.] It is also convenient to consider the brachistochronous lines with reference to lines of another property which can be drawn on a surface. Suppose the particle m, which is the subject of motion, to be at rest at a given point on a surface, and to be under the action of given impressed forces; then, if the particle is constrained to move on the surface, the first element of its path will be that length-element on the surface which makes the least angle with the line of action of the resultant of the impressed forces; and at the end of this first length-element another element will have the same property; and so on: thus for the system of forces there will be a series of such elements on the surface, and we shall have a curve, which I propose to call *a line of easy motion* on the surface. Such a line will at all its points be normal to the curve of intersection of the equilibrium-surface and of the given surface. Its equations are found in the following manner:

Let $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ be the direction-cosines of the length-element of the line of easy motion; then if $R^2 = X^2 + Y^2 + Z^2$, $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$ are the direction-cosines of the line of action of R; and if θ is the angle which is to be a minimum,

$$\cos\theta = \frac{1}{R}\left\{X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right\};$$

also
$$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2,$$

$$0 = U\frac{dx}{ds} + V\frac{dy}{ds} + W\frac{dz}{ds};$$

therefore differentiating these, and equating to zero $d.\cos\theta$, we have

$$d.\cos\theta = 0 = \frac{1}{R}\left\{X\,d.\frac{dx}{ds} + Y\,d.\frac{dy}{ds} + Z\,d.\frac{dz}{ds}\right\},$$

$$0 = \frac{dx}{ds}\,d.\frac{dx}{ds} + \frac{dy}{ds}\,d.\frac{dy}{ds} + \frac{dz}{ds}\,d.\frac{dz}{ds},$$

$$0 = U\,d.\frac{dx}{ds} + V\,d.\frac{dy}{ds} + W\,d.\frac{dz}{ds};$$

multiplying the second and third by indeterminate multipliers λ and μ, and adding, we have

$$(\text{X}+\lambda\frac{dx}{ds}+\mu\text{U})d.\frac{dx}{ds}+(\text{Y}+\lambda\frac{dy}{ds}+\mu\text{V})d.\frac{dy}{ds}+(\text{Z}+\lambda\frac{dz}{ds}+\mu\text{W})d.\frac{dz}{ds}=0.$$

Let

$$\left.\begin{aligned}\text{X}+\lambda\frac{dx}{ds}+\mu\text{U}&=0,\\ \text{Y}+\lambda\frac{dy}{ds}+\mu\text{V}&=0,\\ \text{Z}+\lambda\frac{dz}{ds}+\mu\text{W}&=0;\end{aligned}\right\} \tag{112}$$

then multiplying these severally by dx, dy, dz, and adding, we have, by reason of (110),

$$v\,dv+\lambda\,ds=0; \qquad \therefore\ \lambda=-\frac{v\,dv}{ds};$$

so that (112) become

$$\left.\begin{aligned}\text{X}-\frac{v\,dv}{ds}\frac{dx}{ds}+\mu\text{U}&=0,\\ \text{Y}-\frac{v\,dv}{ds}\frac{dy}{ds}+\mu\text{V}&=0,\\ \text{Z}-\frac{v\,dv}{ds}\frac{dz}{ds}+\mu\text{W}&=0;\end{aligned}\right\} \tag{113}$$

therefore replacing X, Y, Z by their values from (111), we have

$$\frac{\left(\frac{dv}{dx}\right)ds-\frac{dx}{ds}dv}{\left(\frac{d\text{F}}{dx}\right)}=\frac{\left(\frac{dv}{dy}\right)ds-\frac{dy}{ds}dv}{\left(\frac{d\text{F}}{dy}\right)}=\frac{\left(\frac{dv}{dz}\right)ds-\frac{dz}{ds}dv}{\left(\frac{d\text{F}}{dz}\right)}; \tag{114}$$

from which, by integration, two functions of x, y, z may (theoretically) be found: the line of intersection of the surfaces represented by which are the lines of easy motion.

The paths taken by water in its descent towards a lower level, by avalanches in their descent, by balls in their descent through bent tubes, are all cases of lines of easy motion of heavy bodies under the action of gravity.

When gravity is the only force which acts on m, the line of easy motion is called *the line of greatest slope.* In this case let the axis of z be vertical, so that

$$\text{X}=v\left(\frac{dv}{dx}\right)=0, \qquad \text{Y}=v\left(\frac{dv}{dy}\right)=0, \qquad \text{Z}=v\left(\frac{dv}{dz}\right)=g;$$

whereby (114) become

$$\frac{dx}{\left(\frac{d\text{F}}{dx}\right)}=\frac{dy}{\left(\frac{d\text{F}}{dy}\right)}=-\frac{dx^2+dy^2}{\left(\frac{d\text{F}}{dz}\right)dz}; \tag{115}$$

which are the equations to the line of greatest slope.

Thus on the ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

we have, from the first two of (115),

$$\frac{a^2\,dx}{x}=\frac{b^2\,dy}{y};$$

$$\therefore\quad a^2\log\frac{x}{x_0}=b^2\log\frac{y}{y_0};\qquad \therefore\quad \left(\frac{x}{x_0}\right)^{a^2}=\left(\frac{y}{y_0}\right)^{b^2};$$

which is the equation to a cylinder perpendicular to the pla
of (x, y); the line of intersection of which with the ellipsoid
the line of greatest slope. In the sphere, $a=b$, and the meridi
line is the line of greatest slope.

Also in surfaces of revolution, whose equations are comprise
in the form, $x^2+y^2=f(z)$,
and where the axis of revolution is vertical, we have

$$\left(\frac{d\text{F}}{dx}\right)=2x,\qquad \left(\frac{d\text{F}}{dy}\right)=2y;$$

and thus the first two of (115) become

$$\frac{dx}{x}=\frac{dy}{y};\qquad \therefore\quad \frac{x}{x_0}=\frac{y}{y_0};$$

which is the equation to a meridian plane: the meridian there
fore which passes through a given point on a surface of revolu
tion is the line of greatest slope at that point.

446.] The general equations of the brachistochron (107) ma
be expressed in the following form by means of (111),

$$\frac{d.\dfrac{dx}{v\,ds}-ds\dfrac{d}{dx}\dfrac{1}{v}}{\left(\dfrac{d\text{F}}{dx}\right)}=\frac{d.\dfrac{dy}{v\,ds}-ds\dfrac{d}{dy}\dfrac{1}{v}}{\left(\dfrac{d\text{F}}{dy}\right)}=\frac{d.\dfrac{dz}{v\,ds}-ds\dfrac{d}{dz}\dfrac{1}{v}}{\left(\dfrac{d\text{F}}{dz}\right)};\qquad (11$$

which may again be expressed in the form

$$\frac{v d.\dfrac{dx}{ds}}{\left(\dfrac{d\text{F}}{dx}\right)}+\frac{\left(\dfrac{dv}{dx}\right)ds-\dfrac{dx}{ds}dv}{\left(\dfrac{d\text{F}}{dx}\right)}=\frac{v d.\dfrac{dy}{ds}}{\left(\dfrac{d\text{F}}{dy}\right)}+\frac{\left(\dfrac{dv}{dy}\right)ds-\dfrac{dy}{ds}dv}{\left(\dfrac{d\text{F}}{dy}\right)}$$

$$=\frac{v d.\dfrac{dz}{ds}}{\left(\dfrac{d\text{F}}{dz}\right)}+\frac{\left(\dfrac{dv}{dz}\right)ds-\dfrac{dz}{ds}dv}{\left(\dfrac{d\text{F}}{dz}\right)};\qquad (11$$

now these equations are satisfied, if we have simultaneously,

$$\frac{d.\frac{dx}{ds}}{\left(\frac{dF}{dx}\right)} = \frac{d.\frac{dy}{ds}}{\left(\frac{dF}{dy}\right)} = \frac{d.\frac{dz}{ds}}{\left(\frac{dF}{dz}\right)}, \tag{118}$$

and

$$\frac{\left(\frac{dv}{dx}\right)ds - \frac{dx}{ds}dv}{\left(\frac{dF}{dx}\right)} = \frac{\left(\frac{dv}{dy}\right)ds - \frac{dy}{ds}dv}{\left(\frac{dF}{dy}\right)} = \frac{\left(\frac{dv}{dz}\right)ds - \frac{dz}{ds}dv}{\left(\frac{dF}{dz}\right)}; \tag{119}$$

but these are severally the equations to the geodesic, and to the line of easy motion; therefore a curve which is at the same time a geodesic and a line of easy motion is also a brachistochron. Hence also we conclude that of curves which have the properties of being geodesic, lines of easy motion, and brachistochronous, any line which possesses two of these properties possesses also the third.

The brachistochron at the point whence m starts, and generally at all points where the velocity of m is zero, touches the line of easy motion.

For if we take equations (117), if $v = 0$, we have only the second members of each; so that at the points where $v = 0$, the brachistochron is identical with the line of easy motion.

447.] I proceed, lastly, to consider the case where gravity is the only acting force, and particularly when m moves on a surface of revolution, the axis of which is vertical.

In this case $X = Y = 0$, $Z = g$;

$$\therefore \quad v^2 = 2g(z-z_0);$$

so that the first two of (107) become

$$\frac{d.\frac{dx}{ds(z-z_0)^{\frac{1}{2}}}}{\left(\frac{dF}{dx}\right)} = \frac{d.\frac{dy}{ds(z-z_0)^{\frac{1}{2}}}}{\left(\frac{dF}{dy}\right)}. \tag{120}$$

Ex. 1. Let the surface be the vertical plane of (x, z): so that

$$\left(\frac{dF}{dx}\right) = 1, \qquad \left(\frac{dF}{dy}\right) = 0;$$

$$\therefore \quad d.\frac{dx}{ds(z-z_0)^{\frac{1}{2}}} = 0; \qquad \frac{dx}{ds(z-z_0)^{\frac{1}{2}}} = \text{a constant},$$

which is the differential equation to a cycloid whose base is horizontal.

Ex. 2. Let the surface be the inclined plane, whose equation

$$\frac{y}{b} = \frac{z}{c};$$

$$\therefore \quad \left(\frac{d\text{F}}{dx}\right) = 0, \qquad \left(\frac{d\text{F}}{dy}\right) = \frac{1}{b}, \qquad \left(\frac{d\text{F}}{dz}\right) = -\frac{1}{c};$$

so that from the first of (120),

$$\frac{dx}{ds\,(z-z_0)^{\frac{1}{2}}} = c \text{ (a constant)},$$

which will be the equation to a cycloid on the inclined plane.

Ex. 3. Let the surface be one of revolution, of which let t
equation be $x^2+y^2 = f(z)$;

$$\therefore \quad \left(\frac{d\text{F}}{dx}\right) = 2x, \qquad \left(\frac{d\text{F}}{dy}\right) = 2y, \qquad \left(\frac{d\text{F}}{dz}\right) = -f'(z);$$

so that from (120),

$$y d.\frac{dx}{ds(z-z_0)^{\frac{1}{2}}} - x d.\frac{dy}{ds\,(z-z_0)^{\frac{1}{2}}} = 0;$$

whence integrating,

$$y\,dx - x\,dy = c\,ds\,(z-z_0)^{\frac{1}{2}}, \qquad (1$$

where c is an arbitrary constant. Let $x^2+y^2 = r^2$, $x = r$ co
$y = r \sin\theta$; then (121) becomes

$$r^2\,d\theta = c\,ds(z-z_0)^{\frac{1}{2}}.$$

If v is the velocity of m at the time t,

$$v^2 = 2g\,(z-z_0) = \frac{ds^2}{dt^2}; \qquad \therefore \quad r^2\,d\theta = \text{A}\,v^2\,dt;$$

so that the sectorial area described by r in a horizontal pla
varies as the square of the velocity of the particle.

If the equation to the generating curve of the surface is

$$r = f(z),$$

$$\text{then} \qquad dr = f'(z)\,dz;$$

$$\begin{aligned} \therefore \quad ds^2 &= dz^2 + dr^2 + r^2\,d\theta^2 \\ &= dz^2\,\{1+(f'(z))^2\} + (f(z))^2\,d\theta^2; \end{aligned}$$

$$\text{so that} \qquad d\theta = \frac{c\,dz}{f(z)} \left\{ \frac{\{1+(f'(z))^2\}(z-z_0)}{(f(z))^2 - c^2(z-z_0)} \right\}^{\frac{1}{2}}; \qquad (12$$

whence the equation to the brachistochron may be found
terms of θ and x.

SECTION 3.—*Constrained motion of material particles in resisting media.*

448.] In the present Section we shall find it convenient to employ the method of tangential resolution; and as the principles which have been investigated in the preceding pages are sufficient for the solution of the problem of a particle moving in a resisting medium, it is necessary for us only to give examples.

Ex. 1. The most simple case is that of a heavy particle m moving down a smooth inclined plane. Let a be the angle of inclination of the plane to the horizon; and let s be measured along the plane and be the distance of m from the origin at the time t; then, if the resistance varies as the velocity, the equation of motion is

$$\frac{d^2 s}{dt^2} = g \cos a - k \frac{ds}{dt},$$

which is of the same form as (122) in Art. 296; and the results therein deduced are also applicable to this case.

Ex. 2. If the resistance varies as the square of the velocity, the equation of motion is

$$\frac{d^2 s}{dt^2} = g \cos a - k \left(\frac{ds}{dt}\right)^2,$$

which is of the same form as (107) in Art. 294; and therefore similar results may be deduced.

We proceed next to the consideration of curvilinear motion, and in the first place that along the cycloid.

Ex. 3. A heavy particle moves on a cycloid in a vertical plane with its base horizontal and vertex downwards, and in a medium of which the resistance varies as the velocity; it is required to determine the circumstances of motion.

Let the origin be taken at the lowest point of the cycloid; let the axis of x be vertical; let a be the radius of the generating circle; s = the arc of the cycloid measured from its lowest point; α = the initial value of s; $2k$ = the resistance of the medium for an unit-mass. Therefore the equation of motion along the curve is, as in Art. 423,

$$\frac{d^2 s}{dt^2} = -\frac{g}{4a} s - 2k \frac{ds}{dt},$$

$$\frac{d^2 s}{dt^2} + 2k \frac{ds}{dt} + \frac{g}{4a} s = 0; \qquad ([illegible])$$

which is a linear differential equation of the second order with constant coefficients; and of which the general integral is, see Art. 471, Integral Calculus,

$$s = e^{-kt}\left\{c_1 e^{\left(k^2-\frac{g}{4a}\right)^{\frac{1}{2}}t} + c_2 e^{-\left(k^2-\frac{g}{4a}\right)^{\frac{1}{2}}t}\right\}, \tag{124}$$

where c_1 and c_2 are undetermined constants.

Here we have three cases; viz. according as the radical in the exponent is imaginary or possible or vanishes.

(1) Let the radical be imaginary, that is, let $k^2 - \frac{g}{4a} = -h^2$.

$$\begin{aligned} \therefore \quad s &= e^{-kt}\{c_1 e^{ht\sqrt{-1}} + c_2 e^{-ht\sqrt{-1}}\} \\ &= e^{-kt}(a_1 \cos ht + a_2 \sin ht), \end{aligned}$$

where a_1 and a_2 are undetermined constants: to determine them we have the following data: when $t = 0$, $s = a$, $\frac{ds}{dt} = 0$; therefore

$$\left.\begin{aligned} a &= a_1, \\ 0 &= ha_2 - ka_1; \end{aligned}\right\} \qquad \therefore \quad \begin{aligned} a_1 &= a, \\ a_2 &= \frac{ka}{h}; \end{aligned}$$

$$\therefore \quad s = e^{-kt}\frac{a}{h}\{h \cos ht + k \sin ht\}. \tag{125}$$

Let T = the time from the point where $s = a$ to the lowest point; therefore

$$\tan h\text{T} = -\frac{h}{k}; \tag{126}$$

which is independent of a, and is therefore the same whatever is the point on the cycloid whence m begins to descend. The cycloid therefore is tautochronous in a medium wherein the resistance varies as the velocity, as well as in vacuo.

On comparing the results of this problem with observed facts, they are found so nearly to coincide, that the resistance of the air seems to vary nearly as the velocity, when the arc through which m moves is small, and when consequently the velocity is small: in this case also k is small, and if a is not very large, h, which is equal to $\left(\frac{g}{4a}\right)^{\frac{1}{2}}$, is not small;

$$\begin{aligned} \therefore \quad h\text{T} &= \tan^{-1}\left(-\frac{h}{k}\right) \\ &= \frac{\pi}{2} + \tan^{-1}\frac{k}{h} \\ &= \frac{\pi}{2} + \frac{k}{h} \text{ (approximately)}; \end{aligned}$$

$$\therefore \quad \text{T} = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}} + \frac{4a}{g}k; \tag{127}$$

on comparing which with (27), Art. 423, it appears that the time of descent to the lowest point is greater than it is in vacuo, by a quantity which varies as the coefficient of resistance.

Again, from (125) we have

$$\frac{ds}{dt} = -\frac{a(h^2+k^2)}{h} e^{-kt} \sin ht; \tag{128}$$

therefore $\frac{ds}{dt} = 0$, when $t = 0$, $t = \frac{\pi}{h}$, $= \cdots$; and the time of a complete oscillation is $\frac{\pi}{h}$. And substituting these values of t successively in (125), the amplitudes of vibration successively become

$$a, \qquad a e^{-\frac{k}{h}\pi}, \qquad a e^{-\frac{2k}{h}\pi}, \cdots\cdots;$$

so that the amplitudes decrease in geometrical progression.

(2) Let $k^2 - \frac{g}{4a} = h^2$; then (124) becomes

$$s = e^{-kt}\{c_1 e^{ht} + c_2 e^{-ht}\}; \tag{129}$$

and since when $t = 0$, $s = a$, and $\frac{ds}{dt} = 0$, we have

$$s = \frac{a}{2h}\{(h+k)e^{-(k-h)t} - (k-h)e^{-(h+k)t}\}; \tag{130}$$

in which expression $s = 0$, only when $t = \infty$; that is, the particle never reaches the lowest point of the cycloid.

And differentiating twice (130), it will be found that $\frac{d^2 s}{dt^2} = 0$, that is, that $\frac{ds}{dt}$ is a maximum, when $t = \frac{1}{h} \log \frac{k+h}{k-h}$.

(3) If $k^2 = \frac{4a}{g}$, that is, if two roots of the auxiliary equation of (123) are equal to each other, then the integral of (123) is

$$s = a e^{-kt}(1 + kt); \tag{131}$$

which formula shews that $s = 0$, only when $t = \infty$.

449.] Let us in the next place consider the motion of a circular pendulum in a resisting medium, when the resistance varies as the velocity, and when the amplitudes of vibration are small.

Let a be the radius of the circle, and thus the length of the pendulum; $\theta =$ the angle between the pendulum and the vertical line at the time t; $\alpha =$ the greatest value of θ; then the equation of motion is

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g}{a}\sin\theta = 0; \tag{132}$$

and as θ is always small, $\sin\theta$ may be replaced by θ, and we have

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g}{a}\theta = 0, \tag{133}$$

which is of the same form as (123). Now k is in this case small, so that k^2 is less than $\frac{g}{a}$; therefore let

$$\frac{g}{a} - k^2 = \lambda^2;$$

thus the integral is the same as (125), and we have

$$\theta = \frac{\alpha}{\lambda} e^{-kt}\{\lambda\cos\lambda t + k\sin\lambda t\}, \tag{134}$$

$$\text{and} \quad \frac{d\theta}{dt} = -\frac{\alpha(\lambda^2+k^2)}{\lambda} e^{-kt}\sin\lambda t; \tag{135}$$

whereby the position and the velocity of the pendulum at any time are known.

$\frac{d\theta}{dt} = 0$, whenever $t = \frac{n\pi}{\lambda}$; so that the time of an oscillation

$$= \frac{\pi}{\lambda} = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}\left(1 - \frac{k^2 a}{g}\right)^{-\frac{1}{2}},$$

and is independent of α. Thus, see (40), Art. 427, the time of an oscillation in vacuo is to that in the resisting medium as 1 to $\left(1 - \frac{k^2 a}{g}\right)^{-\frac{1}{2}}$. The amplitudes of the oscillation, as it has been shewn in the last Article, diminish successively in geometrical progression.

On these results M. Poisson remarks in Art. 187 of his Traité de Mécanique, Vol. I, 2nd edition, that experiments in air shew how the amplitudes of the vibrations (approximately) decrease in a geometrical progression. In an experiment made by Borda, where α was one-third of a degree, the amplitudes were evidently reduced in geometrical progression, and the greatest amplitude was reduced by about two-thirds after 1800 oscillations.

450.] Let us assume that the resistance of the air varies as the square of the velocity; so that if k is the coefficient of resistance for an unit-mass, the equation of motion is

$$\frac{d^2\theta}{dt^2} - ak\left(\frac{d\theta}{dt}\right)^2 + \frac{g}{a}\sin\theta = 0;$$

$$\therefore \quad d\left(\frac{d\theta}{dt}\right)^2 - 2ak\left(\frac{d\theta}{dt}\right)^2 d\theta = -\frac{2g}{a}\sin\theta\, d\theta,$$

which is a linear differential equation of the first order, see Art. 382, Integral Calculus; and of which the general integral is

$$\left(\frac{d\theta}{dt}\right)^2 = ce^{2ak\theta} + \frac{2g}{a}\,\frac{\cos\theta + 2ak\sin\theta}{1+4a^2k^2};$$

but $\theta = \alpha$, when $\frac{d\theta}{dt} = 0$; therefore

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{a(1+4a^2k^2)}\,\{\cos\theta + 2ak\sin\theta - (\cos\alpha + 2ak\sin\alpha)e^{-2ak(\alpha-\theta)}\}. \quad (136)$$

Let $-\alpha_1$ be the value of θ, when the pendulum comes to rest on the other side of the vertical;

$$\therefore \quad (\cos\alpha_1 - 2ak\sin\alpha_1)e^{2ak\alpha_1} = (\cos\alpha + 2ak\sin\alpha)\,e^{-2ak\alpha}. \quad (137)$$

As k is small, let us expand the exponential and omit terms involving the square and higher powers of k; then we have

$$\cos\alpha_1 - 2ak(\sin\alpha_1 - \alpha_1\cos\alpha_1) = \cos\alpha + 2ak(\sin\alpha - \alpha\cos\alpha).$$

Let $\Delta\alpha =$ the decrement of the amplitude; so that

$$\alpha = \alpha_1 + \Delta\alpha;$$

then neglecting the square and higher powers of $\Delta\alpha$, and also the product $k\Delta\alpha$, we have

$$\Delta\alpha\sin\alpha = 4ak(\sin\alpha - \alpha\cos\alpha)$$

$$\Delta\alpha = \frac{4ak(\sin\alpha - \alpha\cos\alpha)}{\sin\alpha};$$

and if we suppose the arcs of oscillation to be small, then neglecting the cubes and higher powers of α, we have

$$\Delta\alpha = \frac{4ak\alpha^2}{3}; \qquad \therefore \quad \alpha_1 = \alpha - \frac{4ak\alpha^2}{3};$$

and m having come to rest will again descend and ascend until $\theta = \alpha_2$ (say); where, as a process similar to the preceding one will shew,

$$\alpha_2 = \alpha_1 - \frac{4ak\alpha_1^2}{3};$$

and so on: until finally the oscillations will cease, and the pendulum will be in a vertical position.

For the determination of the time of an oscillation, and the successive decrements of it, I must refer the reader to M. Poisson's Traité de Mécanique, Vol. I, 2nd edition, p. 356. And for an inquiry into various other circumstances connected with the motion of a pendulum in air to (1) a memoir of M. Poisson entitled "Mémoire sur les mouvements simultanés d'un pendule et de l'air environnant," and contained in the Mém. de l'Académie des Sciences de Paris, tome XI; (2) a paper by Professor Stokes of Cambridge, and contained in the Transactions of the Cambridge Philosophical Society, Vol. IX, Part II, 1851, in the introduction to which will be found a succinct account of all the investigations which had been previously made on the subject. These inquiries however are hydrodynamical, and properly belong to a future part of our treatise.

451.] It remains for us still to investigate the general equation of tautochronous curves in a resisting medium; and with this object I shall inquire into the most general expression for the tangential component of tautochronous curves.

Let O be the common extremity of the arcs, which are to be described in the same time τ; let $s =$ the distance along the curve of m from O at the time t, and let σ be the initial value of s. And let T be the tangential force for which the curve is tautochronous; let v be the velocity at the time t; then

$$\tau = -\int_0^{\sigma} \frac{ds}{v}; \tag{138}$$

also $$\mathrm{T} = \frac{d^2 s}{dt^2} = \frac{d}{dt}\frac{ds}{dt} = v\frac{dv}{ds},$$

which is to be determined.

By reason of the property of tautochronism, τ is to be independent of σ; therefore

$$\frac{d\tau}{d\sigma} = 0. \tag{139}$$

Let us in the first place so transform the right-hand member of (138) that the limits of integration may be constant. For this purpose let $s = \psi(z)$, where z is a new variable, and where $\psi(z) = 0$, when $z = 0$; and let $z = u\mathrm{z}$, where u is another new variable, and where z is the value of z, when $s = \sigma$; so that

$$s = \psi(z) = \psi(u\mathrm{z}); \tag{140}$$

therefore $u = 1$ when $s = \sigma$, or when $z = \mathrm{z}$; also $\sigma = \psi(\mathrm{z})$.

From these we have, z being independent of s,

$$ds = \psi'(u\mathrm{z})\,\mathrm{z}\,du;$$

$$\therefore\quad \tau = -\int_0^1 \frac{\psi'(u\mathrm{z})\,\mathrm{z}\,du}{v} = -\int_0^1 \frac{\psi'(z)\,\mathrm{z}\,du}{v}; \tag{141}$$

therefore

$$\frac{d\tau}{d\sigma} = -\int_0^1 \frac{\left\{\psi''(z)\dfrac{dz}{d\sigma}\mathrm{z} + \psi'(z)\dfrac{d\mathrm{z}}{d\sigma}\right\}v - \mathrm{z}\psi'(z)\left\{\dfrac{dv}{d\sigma} + \dfrac{dv}{ds}\dfrac{ds}{d\sigma}\right\}}{v^2}\,du. \tag{142}$$

Now $$\frac{dz}{d\sigma} = u\frac{d\mathrm{z}}{d\sigma} = u\mathrm{z}' \text{ (say)},$$

$$\frac{ds}{d\sigma} = \psi'(u\mathrm{z})\,u\frac{d\mathrm{z}}{d\sigma} = \psi'(z)\,u\mathrm{z}';$$

so that (142) becomes

$$\frac{d\tau}{d\sigma} = -\frac{1}{\mathrm{z}}\int_0^{\sigma} \frac{\{\psi''(z)z\mathrm{z}'+\psi'(z)\mathrm{z}'\}\,v - \left\{\frac{dv}{d\sigma}\mathrm{z}+\frac{dv}{ds}\psi'(z)z\mathrm{z}'\right\}\psi'(z)}{v^2}\,dz. \quad (143)$$

Let the quantity under the sign of integration be represented by $$d.f(s,\sigma); \quad (144)$$
also from (140),

$$\frac{dz}{ds} = \frac{1}{\psi'(z)}; \qquad \therefore \quad \frac{d^2z}{ds^2} = -\frac{\psi''(z)}{(\psi'(z))^3};$$

and let $\frac{dz}{ds}$, $\frac{d^2z}{ds^2}$ be represented by z', z'', respectively: so that

$$\psi'(z) = \frac{1}{z'}, \qquad \psi''(z) = -\frac{z''}{z'^3}; \quad (145)$$

therefore, since $\frac{d\tau}{d\sigma} = 0$, we have, substituting in (143),

$$\mathrm{z}\frac{dv}{d\sigma} + \frac{z\mathrm{z}'}{z'}\frac{dv}{ds} - \mathrm{z}'v\left(1-\frac{zz''}{z'^2}\right) + \frac{d.f(s,\sigma)}{ds}v^2 = 0; \quad (146)$$

which is a partial differential equation of the first order, where $\frac{dv}{ds}$ and $\frac{dv}{d\sigma}$ are partial derived-functions of v; let it be divided through by z'; then by (84), Art. 384, Integral Calculus, we have

$$\frac{\mathrm{z}'}{\mathrm{z}}d\sigma = \frac{z'}{z}ds = \frac{dv}{v\left(1-\frac{zz''}{z'^2}\right) - \frac{v^2}{\mathrm{z}'}\frac{d.f(s,\sigma)}{ds}}; \quad (147)$$

and our object is to find two integrals of these equations. Let us take the first two of these three equalities; and let

$$\frac{z}{z'} = \phi(s); \qquad \therefore \quad \frac{\mathrm{z}}{\mathrm{z}'} = \phi(\sigma); \quad (148)$$

so that we have $$\frac{d\sigma}{\phi(\sigma)} = \frac{ds}{\phi(s)}. \quad (149)$$

Let $$\int\frac{d\sigma}{\phi(\sigma)} = \omega(\sigma); \qquad \therefore \quad \int\frac{ds}{\phi(s)} = \omega(s); \quad (150)$$

and (149) becomes

$$d.\omega(\sigma) - d.\omega(s) = 0;$$

$$\therefore \quad \omega(\sigma) - \omega(s) = c_1; \quad (151)$$

where c_1 is an arbitrary constant of integration.

Now let us take the last two of (147): from (148) we have

$$1-\frac{zz''}{z'^2} = \phi'(s_1), \qquad \frac{1}{z'} = \phi(\sigma)\, e^{-\int\frac{d\sigma}{\phi(\sigma)}}$$
$$= \phi(\sigma)\, e^{-\omega(\sigma)};$$

so that we have $$\frac{ds}{\phi(s)} = \frac{dv}{v\phi'(s)-v^2\phi(\sigma)\,e^{-\omega(\sigma)}\dfrac{d.f(s,\sigma)}{ds}};$$

$$\therefore\quad \frac{v\phi'(s)\,ds-\phi(s)\,dv}{v^2} = \phi(\sigma)\,e^{-\omega(\sigma)}d.f(s,\sigma); \qquad (152)$$

in the right-hand member of this equation let σ be replaced by its value in terms of s from (151), and let the indefinite integral of the quantity after the substitution be $\chi(s)$: then integrating (152) we have

$$\frac{\phi(s)}{v} = \chi(s)+c_2; \qquad (153)$$

where c_2 is another arbitrary constant; by the general theory of the integration of partial differential equations, $c_1 = \mathrm{F}_1(c_2)$, where F_1 is the symbol of an arbitrary function: therefore in this case we have

$$\omega(\sigma)-\omega(s) = \mathrm{F}_1\Big(\frac{\phi(s)}{v}-\chi(s)\Big); \qquad (154)$$

F_1 being such that $v=0$, when $s=\sigma$; and this is the general integral of the differential equation (146).

Let us take the s-differential of it, and replacing $d.\omega(s)$ from (150), we have

$$-\frac{1}{\phi(s)} = d.\mathrm{F}_1\Big(\frac{\phi(s)}{v}-\chi(s)\Big)\Big\{\frac{v\phi'(s)-\phi(s)\dfrac{dv}{ds}}{v^2}-\chi'(s)\Big\}.$$

Let $$d.\mathrm{F}_1\Big(\frac{\phi(s)}{v}-\chi(s)\Big) = \Big\{\mathrm{F}\Big(\frac{\phi(s)}{v}-\chi(s)\Big)\Big\}^{-1}; \qquad (155)$$

$$\therefore\quad v\frac{dv}{ds} = \mathrm{T} = \frac{v^2}{\{\phi(s)\}^2}\Big\{\mathrm{F}\Big(\frac{\phi(s)}{v}-\chi(s)\Big)+\frac{\phi(s)\phi'(s)}{v}-\phi(s)\chi'(s)\Big\}; \quad (156)$$

which is the most general expression for the tangential velocity-increment, where F, ϕ and χ are symbols of arbitrary functions.

As a particular case of (156), let us suppose $\chi(s)=0$; then

$$\mathrm{T} = \frac{v^2}{\{\phi(s)\}^2}\Big\{\mathrm{F}\Big(\frac{\phi(s)}{v}\Big)+\frac{\phi(s)\phi'(s)}{v}\Big\}. \qquad (157)$$

Let $\phi(s) = \xi;\qquad \therefore\ \phi'(s) = \dfrac{d\xi}{ds};$

let $\dfrac{v}{\phi(s)}\mathrm{F}\Big(\dfrac{\phi(s)}{v}\Big) = \Pi\Big(\dfrac{v}{\phi(s)}\Big) = \Pi\Big(\dfrac{v}{\xi}\Big)$, where Π is the symbol of an arbitrary function: then (157) becomes

$$\text{T} = \frac{v^2}{\xi}\left\{\Pi\left(\frac{v}{\xi}\right) + \frac{d\xi}{ds}\right\}; \tag{158}$$

which formula was given by Lagrange in the Mém. de l'Académie de Berlin, 1765, 1770. The general formula (156) is due to M. Brioschi, and is given by him in "Annali di Scienze Matematiche e Fisiche compilati da B. Tortolini, Roma, 1852," p. 362. For the preceding references I am indebted to M. Jullien, "Problèmes de Mécanique rationelle," Vol. I, page 393. Mallet-Bachelier, Paris, 1855.

452.] As an application of formula (158), let us investigate the cases of tautochronism relative to a heavy particle, which moves on a rough curve in a medium, of which the resistance varies as any function of the velocity.

Let the axis of x be vertical, let $f(v)$ be the resistance of the medium, and let μ be the coefficient of friction; so that the tangential and normal components of the velocity-increments are respectively

$$g\frac{dx}{ds} - f(v), \qquad \text{and} \qquad g\frac{dy}{ds} + \frac{v^2}{\rho};$$

and as the friction varies as the pressure on the curve, we have

$$\text{T} = -g\frac{dx}{ds} + \mu g\frac{dy}{ds} + \mu\frac{v^2}{\rho} + f(v); \tag{159}$$

and this expression is to be identical with the right-hand member of (158).

If in (159) T is differentiated thrice with respect to v, and once with respect to s, the result $= 0$; so that

$$\frac{d^4\text{T}}{dv^3\,ds} = 0; \tag{160}$$

the particular form of (158) must therefore be consistent with this condition.

Let $\quad \dfrac{v}{\xi}\Pi\left(\dfrac{v}{\xi}\right) = \pi\left(\dfrac{v}{\xi}\right), \qquad$ and $\quad \dfrac{1}{\xi}\dfrac{d\xi}{ds} = \chi(\xi);$

so that (158) becomes

$$\text{T} = v\pi\left(\frac{v}{\xi}\right) + v^2\chi(\xi); \tag{161}$$

and as s is a function of ξ, (160) becomes

$$\frac{d^4\text{T}}{dv^3\,d\xi} = 0: \tag{162}$$

applying this to (161), we have

$$\frac{v^2}{\xi^5}\pi''''\left(\frac{v}{\xi}\right) + \frac{6v}{\xi^4}\pi'''\left(\frac{v}{\xi}\right) + \frac{6}{\xi^3}\pi''\left(\frac{v}{\xi}\right) = 0; \tag{163}$$

and if $\frac{v}{\xi} = \zeta$ we have

$$\zeta^2 \pi''''(\zeta) + 6\zeta\pi'''(\zeta) + 6\pi''(\zeta) = 0;$$

whence integrating, we have

$$\zeta^2 \pi'(\zeta) = \mathrm{A}\zeta^2 + \mathrm{B}\zeta + \mathrm{C};$$

$$\therefore \quad \pi(\zeta) = \mathrm{A}\zeta + \mathrm{B}\log\zeta - \frac{\mathrm{C}}{\zeta} + \mathrm{D};$$

where A, B, C, D are arbitrary constants. Substituting this in (161) we have

$$\mathrm{T} = -\mathrm{B}v\log\frac{v}{\xi} + \mathrm{D}v + \frac{v^2}{\xi}\left(\mathrm{A} + \frac{d\xi}{ds}\right) - \mathrm{C}\xi. \qquad (164)$$

That this and (159) may be identical, we must have

$$(1) \quad \mathrm{B} = 0, \qquad (2) \quad \mathrm{D}v = f(v), \qquad (3) \quad \frac{1}{\xi}\left(\mathrm{A} + \frac{d\xi}{ds}\right) = \frac{\mu}{\rho},$$

$$(4) \quad -\mathrm{C}\xi = g\left(\mu\frac{dy}{ds} - \frac{dx}{ds}\right);$$

therefore from (2) the resistance of the medium varies directly as the velocity; and from (3) and (4), after all reductions, we have

$$\frac{d^2x}{ds^2} = -\frac{\mathrm{AC}}{g(1+\mu^2)} = \text{a constant}; \qquad (165)$$

and this is the equation to a cycloid.

Thus the cycloid is tautochronous both in vacuo and in a medium, of which the resistance is proportional to the velocity, and with friction; it is also the only case of tautochronism with friction which is given by (158).

If a is the radius of the generating circle of the cycloid, and α is the distance from the lowest point of the cycloid of the common extremity of all the tautochronous arcs, then from the equation to the curve we have

$$\frac{dx}{ds} = \frac{s+\alpha}{4a}.$$

Since all the arcs are tautochronous, α is evidently the distance along the arc from the lowest point of that point on the rough curve at which m being placed will remain at rest.

CHAPTER XIII.

GENERAL THEOREMS IN THE MOTION OF A PARTICLE.

SECTION 1.—*The principle of vis viva, or of work.*

453.] In the present Chapter I propose to investigate certain theorems which are either generally or under certain circumstances true of the motion of a material particle. And also to explain a method of investigation by which, when direct processes fail, approximate solutions of certain physical problems may be obtained.

Firstly, let us consider more generally than heretofore the principle of vis viva or of work in its application to the motion of a material particle. It will be hereafter considered in relation to a system of moving particles. In previous parts of the work particular forms of it have been frequently met with, and we are consequently now in a condition to appreciate the use of the principle.

Let the equations of motion of m moving freely be

$$m\frac{d^2x}{dt^2} = m\text{X}, \qquad m\frac{d^2y}{dt^2} = m\text{Y}, \qquad m\frac{d^2z}{dt^2} = m\text{Z}; \tag{1}$$

let (x, y, z), (x_0, y_0, z_0) be the places of m, v and v_0 be its velocity, when $t = t$ and $t = t_0$ respectively: then multiplying the several equations of (1) by dx, dy, dz respectively, and integrating, we have

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = \int_{t_0}^{t} m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz); \tag{2}$$

of which the left-hand member is the increase of vis viva during the time $t-t_0$, and the right-hand member is the sum of the works done by the several forces during the same time.

454. Now if the right-hand member of this equation has a meaning and admits of a physical interpretation, the element-function of the right-hand member must be an exact differential so that the integral may be found, and the definite integral may be determined. Let us suppose $\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz$ to be such an

exact differential, and to be the differential of a function of x, y, z, which we will call V; so that

$$\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz = d\text{V}; \tag{3}$$

in which case $\text{X}, \text{Y}, \text{Z}$ must be functions of x, y, z only, and must not involve t *explicitly*; then (2) becomes

$$\frac{mv^2}{2}-\frac{mv_0{}^2}{2} = m(\text{V}-\text{V}_0); \tag{4}$$

where v_0 and V_0 are the values of v and V, when m is at (x_0, y_0, z_0), its place at the time t_0.

Thus the increase of vis viva, which is also the work done by the acting forces on m during the time $t-t_0$, depends only on the positions of m at the times t and t_0, and not on the time occupied in the passage from one point to the other, nor on the path taken by m during that interval. This theorem indeed follows immediately from the principle on which work is estimated. It is called *the principle of vis viva* and *the principle of work.*

Hence whenever the point (x, y, z) comes to (x_0, y_0, z_0), the right-hand member of (4) vanishes, and consequently no work is done: that is, the work spent or lost is exactly equal to the work gained; and in this case there is no change of vis viva. Thus whenever the moving particle passes through the same point, the vis viva of m at that point is always the same.

If $\text{X} = \text{Y} = \text{Z} = 0$, that is, if no force acts on the particle, the vis viva is always the same; that is, no work is done, because no force acts to do work. This theorem is known as *the principle of conservation of vis viva of a particle.*

If the function assumed in (3), viz. $\text{V} = c$, represents a surface, so that $\left(\frac{d\text{V}}{dx}\right) = \text{X}$, $\left(\frac{d\text{V}}{dy}\right) = \text{Y}$, $\left(\frac{d\text{V}}{dz}\right) = \text{Z}$, then at every point on this surface the action-line of the resultant of the impressed forces is normal to the surface, so that in reference to the system of forces the surface is an equilibrium-surface; see Art. 232; and the particle would be at rest under the action of the forces at every point on the outside of the surface supposed to be a rigid shell. Similarly $\text{V}_0 = c_0$ may be another equilibrium-surface: and thus

$$\frac{mv^2}{2}-\frac{mv_0{}^2}{2} = m(c-c_0). \tag{5}$$

Consequently the gain of vis viva is always the same whatever points on the first and second surfaces are taken to be the terminal and initial positions of m; the relative positions of these two places are determined by the forces. Hence also whenever m is on the same equilibrium-surface, whatever is its place on that surface, the vis viva is always the same.

455.] Now since this function $\mathrm{V} = c$ is such that its x-, y-, z-partial derived-functions are the axial-components of the impressed momentum-increments referred to an unit of mass, V is the potential of the resultant of the forces which act on the unit-particle at (x, y, z), and is the work done by the forces in the passage of the particle from a given point to the point (x, y, z): thus the right-hand member of (4) is the work done by the forces in the passage of m from (x_0, y_0, z_0) to (x, y, z). This being so all that has been said generally of the potential in Section 2, Chapter VI, in reference to statical attractions is true of it in reference to dynamical force, and the equilibrium-surface above mentioned is an equipotential surface. Hence also $\frac{d\mathrm{V}}{ds}$ is the component of the impressed velocity-increments along ds: and $\frac{d\mathrm{V}}{dn}$ is the resultant of the impressed velocity-increments, and acts along the line normal to the equilibrium-surface at the point (x, y, z), if dn is an element of the normal line. Thus there is a series of equilibrium-surfaces no two of which intersect each other, and at every point on the surface of every one the action-line of the resultant of the impressed velocity-increments is normal to the surface. And a curve is formed which cuts orthogonally the series of equilibrium-surfaces, and the tangent to this curve at every point of it is coincident in direction with the action-line of the resultant of the impressed forces at that point. Thus this line is identical with the line of force, see Art. 232; and is identical with Sir W. R. Hamilton's Hodograph, see Art. 306.

If the system of forces is such as to admit of derivation from a potential, then $\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz$ is an exact differential. When this is the case, the following conditions must be satisfied; viz.

$$\left(\frac{d\mathrm{Y}}{dz}\right) = \left(\frac{d\mathrm{Z}}{dy}\right); \quad \left(\frac{d\mathrm{Z}}{dx}\right) = \left(\frac{d\mathrm{X}}{dz}\right); \quad \left(\frac{d\mathrm{X}}{dy}\right) = \left(\frac{d\mathrm{Y}}{dx}\right); \tag{6}$$

and X, Y, Z and also the potential must not explicitly contain t.

These conditions are satisfied under the following circumstances:

(1) Whenever the particle moves under the action of one or more central forces, the intensities of which are functions of the distance between the centre and the place of the particle.

Thus if P is a central force and $= f(r)$, where r is the distance between (x, y, z) the place of m at the time t and (a, b, c) the centre of force, so that

$$r^2 = (x-a)^2+(y-b)^2+(z-c)^2,$$

$$r\,dr = (x-a)\,dx+(y-b)\,dy+(z-c)\,dz,$$

then $$\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz = f(r)\left\{\frac{x-a}{r}\,dx+\frac{y-b}{r}\,dy+\frac{z-c}{r}\,dz\right\}$$

$$= f(r)\,dr,$$

which is an exact differential. If V is the potential of this function, and V_0 is the value of V when $r = r_0$,

$$\mathrm{V}-\mathrm{V}_0 = \int_{r_0}^{r} f(r)\,dr. \tag{7}$$

If m is under the action of many similar forces, then

$$\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz = \Sigma . f(r)\,dr,$$

$$\text{and} \quad \mathrm{V}-\mathrm{V}_0 = \Sigma\int_{r_0}^{r} f(r)\,dr; \tag{8}$$

and (2) becomes

$$\frac{mv^2}{2}-\frac{mv_0^2}{2} = m(\mathrm{V}-\mathrm{V}_0) = m\Sigma\int_{r_0}^{r} f(r)\,dr. \tag{9}$$

(2) If m is acted on by a force whose line of action is always perpendicular to a given plane, and which is a function of the perpendicular distance of m from the plane, the condition (3) is also satisfied. Thus let the equation to the plane be

$$x\cos\alpha+y\cos\beta+z\cos\gamma-p = 0; \tag{10}$$

then if u is the perpendicular distance from (x, y, z) on (10),

$$u = x\cos\alpha+y\cos\beta+z\cos\gamma-p. \tag{11}$$

Let U represent the function of u, which expresses the force; then $\mathrm{X} = \mathrm{U}\cos\alpha, \quad \mathrm{Y} = \mathrm{U}\cos\beta, \quad \mathrm{Z} = \mathrm{U}\cos\gamma;$

$$\therefore \quad \mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz = \mathrm{U}\{dx\cos\alpha+dy\cos\beta+dz\cos\gamma\}$$

$$= \mathrm{U}\,du;$$

$$\therefore \quad \frac{mv^2}{2}-\frac{mv_0^2}{2} = m\int_{u_0}^{u} \mathrm{U}\,du. \tag{12}$$

(3) In the case of gravity, $\mathrm{X} = 0$, $\mathrm{Y} = 0$, $\mathrm{Z} = g$;

$$\therefore \quad \frac{mv^2}{2}-\frac{mv_0^2}{2} = mg(z-z_0); \tag{13}$$

and if V is the potential of this system of forces,

$$\mathrm{V}-\mathrm{V}_0 = mg(z-z_0). \tag{14}$$

(4) If X, Y, Z consist severally of many terms, and if any parts of them, say X_1, Y_1, Z_1, are such that

$$X_1\,dx + Y_1\,dy + Z_1\,dz = 0,$$

these terms may be omitted in the right-hand member of (2).

This is the case if m moves on a smooth surface; for if R is the pressure on the surface, and if U, V, W, as in Art. 439, are proportional to the direction-cosines of its line of action,

$$R\,\{U\,dx + V\,dy + W\,dz\} = 0, \tag{15}$$

so that the principle of vis viva is true also for a particle moving on a smooth surface so far as the action of the surface is concerned. It is similarly true also for a particle moving in a smooth tube. But it is not necessarily true when the surface or the tube in which the particle moves is rough; for in these cases X, Y, Z may not be functions of x, y, z.

If however the particle moves on a smooth surface, then all the points in its path are on the surface, and the coordinates to these points satisfy the equation to the surface: and thus if U, V, W are the partial derived-functions of the equation to the surface,

$$U\,dx + V\,dy + W\,dz = 0; \tag{16}$$

where dx, dy, dz are the projections on the axes of the element of the path of the particle. Let this be multiplied by an undetermined quantity μ and added to $X\,dx + Y\,dy + Z\,dz$; then it is sufficient that

$$(X + \mu U)\,dx + (Y + \mu V)\,dy + (Z + \mu W)\,dz \tag{17}$$

should be an exact differential. Whereby we have

$$\left.\begin{aligned} \frac{d}{dz}(Y + \mu V) &= \frac{d}{dy}(Z + \mu W), \\ \frac{d}{dx}(Z + \mu W) &= \frac{d}{dz}(X + \mu U), \\ \frac{d}{dy}(X + \mu U) &= \frac{d}{dx}(Y + \mu V); \end{aligned}\right\} \tag{18}$$

that is,

$$\left.\begin{aligned} * \quad + W\Big(\frac{d\mu}{dy}\Big) - V\Big(\frac{d\mu}{dz}\Big) &= \frac{1}{\mu}\Big\{\Big(\frac{dY}{dz}\Big) - \Big(\frac{dZ}{dy}\Big)\Big\}, \\ -W\Big(\frac{d\mu}{dx}\Big) + \quad * \quad + U\Big(\frac{d\mu}{dz}\Big) &= \frac{1}{\mu}\Big\{\Big(\frac{dZ}{dx}\Big) - \Big(\frac{dX}{dz}\Big)\Big\}, \\ V\Big(\frac{d\mu}{dx}\Big) - U\Big(\frac{d\mu}{dy}\Big) + \quad * &= \frac{1}{\mu}\Big\{\Big(\frac{dX}{dy}\Big) - \Big(\frac{dY}{dx}\Big)\Big\}; \end{aligned}\right\} \tag{19}$$

which three conditions are requisite when (17) is an exact differential. Since however on multiplying these respectively by U, V, W, and adding, we have

$$\mathrm{U}\Big\{\Big(\frac{d\mathrm{Y}}{dz}\Big)-\Big(\frac{d\mathrm{Z}}{dy}\Big)\Big\}+\mathrm{V}\Big\{\Big(\frac{d\mathrm{Z}}{dx}\Big)-\Big(\frac{d\mathrm{X}}{dz}\Big)\Big\}+\mathrm{W}\Big\{\Big(\frac{d\mathrm{X}}{dy}\Big)-\Big(\frac{d\mathrm{Y}}{dx}\Big)\Big\}=0, \quad (20)$$

this is a condition requisite that the principle of vis viva may be true of a particle moving on a smooth surface.

SECTION 2.—*The principle of least action.*

456.] Closely connected with the preceding is another principle known as that of Least Action, and which is applicable when $\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz$ is an exact differential; that is, when the system of forces is potentially derived.

Let us suppose a particle m to be moving either freely or on a smooth surface, under the action of forces X, Y, Z, which are potentially derived; then the vis viva acquired by m in an unit of time, the increase of velocity being constant during that unit, is called the *action* of the particle in that unit; and if $\frac{mv^2}{2}$ is the vis viva acquired in dt, then $\frac{mv^2}{2}dt$ is the *action* acquired in dt; so that

$$\frac{m}{2}\int_{t_0}^{t} v^2\,dt \qquad (21)$$

is the action acquired during the time of motion. The principle of least action consists in this. The definite integral (21) is for the given forces less for the path which the particle actually takes than it would be for any other path in space when m moves freely, or on the surface when the motion of m is constrained.

Equation (21) may be expressed in the following form: since

$$v=\frac{ds}{dt}; \qquad (22)$$

$$\therefore \quad v^2\,dt = v\,ds. \qquad (23)$$

Let $\frac{mu}{2}$ represent the definite integral; and let the limits be expressed as in the Calculus of Variations; then

$$u=\int_0^1 v\,ds; \qquad (24)$$

and taking the variation,

$$\delta u = \delta . \int_0^1 v\,ds$$

$$= \int_0^1 \{\delta v\,ds + v\,\delta . ds\}. \qquad (25)$$

Now $$\delta . ds = \frac{dx}{ds}\,\delta . dx + \frac{dy}{ds}\,\delta . dy + \frac{dz}{ds}\,\delta . dz, \qquad (26)$$

and $$\delta v = \frac{\mathrm{X}}{v}\,\delta x + \frac{\mathrm{Y}}{v}\,\delta y + \frac{\mathrm{Z}}{v}\,\delta z\,; \qquad (27)$$

so that (25) becomes

$$\delta u = \int_0^1 \left\{\frac{ds}{v}(\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z) + \frac{v}{ds}(dx\,d.\delta x + dy\,d.\delta y + dz\,d.\delta z)\right\}$$

$$= \left[v\left(\frac{dx}{ds}\,\delta x + \frac{dy}{ds}\,\delta y + \frac{dz}{ds}\,\delta z\right)\right]_0^1$$

$$+ \int_0^1 \left\{\left(\frac{\mathrm{X}\,ds}{v} - d.\frac{v\,dx}{ds}\right)\delta x + \left(\frac{\mathrm{Y}\,ds}{v} - d.\frac{v\,dy}{ds}\right)\delta y + \left(\frac{\mathrm{Z}\,ds}{v} - d.\frac{v\,dz}{ds}\right)\delta z\right\}.$$

Now as the particle moves from one given point to another given point, there are no variations of the coordinates of these points, and the first part vanishes of itself. Also

$$\mathrm{X}\frac{ds}{v} - d.\frac{v\,dx}{ds} = \mathrm{X}\,dt - d.\frac{dx}{dt} = dt\left(\mathrm{X} - \frac{d^2x}{dt^2}\right) = 0\,; \qquad (28)$$

similarly each of the other parts in the variation vanishes; therefore $\delta u = 0$; and u is either a maximum or a minimum or a constant. And either of these it may be: generally however it will be a minimum; although we shall presently have an example wherein u is a maximum.

Since $\frac{mu}{2}$ is the sum of all the vires vivae which are in successive elements of time generated in the moving particle, the principle may also be called that of the greatest or least vis viva.

457.] Now assuming the truth of the principle of least action, let us apply it to the motion of a particle under the action of given forces X, Y, Z, and moving (1) freely, (2) on a given smooth surface.

(1) Pursuing exactly the same course as in the last Article, and equating δu to zero, we have

$$0 = \left[v\left(\frac{dx}{ds}\,\delta x + \frac{dy}{ds}\,\delta y + \frac{dz}{ds}\,\delta z\right)\right]_0^1$$

$$+ \int_0^1 \left\{\left(\frac{\mathrm{X}\,ds}{v} - d.\frac{v\,dx}{ds}\right)\delta x + \left(\frac{\mathrm{Y}\,ds}{v} - d.\frac{v\,dy}{ds}\right)\delta y + \left(\frac{\mathrm{Z}\,ds}{v} - d.\frac{v\,dz}{ds}\right)\delta z\right\}; \qquad (29)$$

and as no relation is given between δx, δy, δz, we have

$$\frac{\mathrm{X}\,ds}{v} - d.\frac{v\,dx}{ds} = 0, \qquad \frac{\mathrm{Y}\,ds}{v} - d.\frac{v\,dy}{ds} = 0, \qquad \frac{\mathrm{Z}\,ds}{v} - d.\frac{v\,dz}{ds} = 0; \tag{30}$$

$$\therefore \quad \mathrm{X} = \frac{d}{dt}\frac{dx}{dt}, \qquad \mathrm{Y} = \frac{d}{dt}\frac{dy}{dt}, \qquad \mathrm{Z} = \frac{d}{dt}\frac{dz}{dt}; \tag{31}$$

and if t is equicrescent,

$$\mathrm{X} = \frac{d^2x}{dt^2}, \qquad \mathrm{Y} = \frac{d^2y}{dt^2}, \qquad \mathrm{Z} = \frac{d^2z}{dt^2}; \tag{32}$$

which are the three ordinary equations of motion.

(2) Let the motion of m be constrained to a surface whose equation is $\mathrm{F}(x, y, z) = c$, and of which the partial derived-functions are $\mathrm{U}, \mathrm{V}, \mathrm{W}$; so that $\delta x, \delta y, \delta z$ are connected by the equation

$$\mathrm{U}\,\delta x + \mathrm{V}\,\delta y + \mathrm{W}\,\delta z = 0; \tag{33}$$

$$\therefore \quad \frac{\mathrm{X}\,dt - d.\frac{dx}{dt}}{\mathrm{U}} = \frac{\mathrm{Y}\,dt - d.\frac{dy}{dt}}{\mathrm{V}} = \frac{\mathrm{Z}\,dt - d.\frac{dz}{dt}}{\mathrm{W}} = \lambda\,dt \text{ (say)}; \tag{34}$$

$$\therefore \quad \left.\begin{aligned} \mathrm{X} &= \lambda\mathrm{U} + \frac{d}{dt}\frac{dx}{dt}, \\ \mathrm{Y} &= \lambda\mathrm{V} + \frac{d}{dt}\frac{dy}{dt}, \\ \mathrm{Z} &= \lambda\mathrm{W} + \frac{d}{dt}\frac{dz}{dt}; \end{aligned}\right\} \tag{35}$$

let, as heretofore, $\mathrm{Q}^2 = \mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2$; then multiplying these equations severally by $\frac{\mathrm{U}}{\mathrm{Q}}, \frac{\mathrm{V}}{\mathrm{Q}}, \frac{\mathrm{W}}{\mathrm{Q}}$, and adding, we have, if t is equicrescent,

$$\lambda = \frac{\mathrm{U}\mathrm{X} + \mathrm{V}\mathrm{Y} + \mathrm{W}\mathrm{Z}}{\mathrm{Q}^2} - \frac{\mathrm{U}\,d^2x + \mathrm{V}\,d^2y + \mathrm{W}\,d^2z}{\mathrm{Q}^2\,dt^2}; \tag{36}$$

so that λ is determined. And $\lambda\mathrm{Q}$ is evidently the normal reaction of the surface: for if the motion of m were unconstrained, then a comparison of (32) and (35) shews that $\lambda = 0$; so that λ is a force which is introduced by the surface: and as $\mathrm{U}, \mathrm{V}, \mathrm{W}$ are proportional to the direction-cosines of the normal to the surface at the point (x, y, z), where m is at the time t, the line of action of λ is the normal to the surface; so that if R is the pressure of m on the surface,

$$\frac{\mathrm{R}}{m} = \frac{\mathrm{U}\mathrm{X} + \mathrm{V}\mathrm{Y} + \mathrm{W}\mathrm{Z}}{\mathrm{Q}} - \frac{\mathrm{U}\,d^2x + \mathrm{V}\,d^2y + \mathrm{W}\,d^2z}{\mathrm{Q}\,dt^2}; \tag{37}$$

and (35) become the three usual equations of motion of a particle moving on a surface.

458.] In this and the following Articles I propose to apply the principle of least action to two problems: (1) that of a heavy projectile in vacuo; (2) that of the trajectory of a free particle under the action of a central force, which varies inversely as the square of the distance.

In the former example let the axis of y be vertical, and let the axis of x be horizontal; and let the initial and final positions of the particle be given; let the initial position be the origin; at which point let the velocity be $(2gh)^{\frac{1}{2}}$, and let the line of motion of m make an angle a with the horizontal line: then at the point (x, y),

$$v^2 = 2g(h-y); \tag{38}$$

and therefore

$$u = \int_0^1 \{2g(h-y)\}^{\frac{1}{2}} ds; \tag{39}$$

and to simplify the calculation let us assume the motion to be wholly in the plane of (x, y). The variation of u, being equated to zero, gives

$$\delta u = 0 = (2g)^{\frac{1}{2}} \int_0^1 \left\{ (h-y)^{\frac{1}{2}} \delta . ds - \frac{ds\, \delta y}{2(h-y)^{\frac{1}{2}}} \right\}; \tag{40}$$

but $\quad \delta . ds = \dfrac{dx}{ds} \delta . dx + \dfrac{dy}{ds} \delta . dy;$

$$\begin{aligned} \therefore \quad 0 &= \int_0^1 \left\{ (h-y)^{\frac{1}{2}} \left(\frac{dx}{ds} d.\delta x + \frac{dy}{ds} d.\delta y \right) - \frac{ds\, \delta y}{2(h-y)^{\frac{1}{2}}} \right\} \\ &= \left[(h-y)^{\frac{1}{2}} \left(\frac{dx}{ds} \delta x + \frac{dy}{ds} \delta y \right) \right]_0^1 \\ &\quad - \int_0^1 \left\{ d . \frac{dx}{ds} (h-y)^{\frac{1}{2}} \delta x + \left(d . \frac{dy}{ds} (h-y)^{\frac{1}{2}} + \frac{ds}{2(h-y)^{\frac{1}{2}}} \right) \delta y \right\} : \end{aligned} \tag{41}$$

of this expression the first part vanishes, because the points at which the motion begins and ends are given. And, as no relation is given between x and y, we have

$$d . \frac{dx\, (h-y)^{\frac{1}{2}}}{ds} = 0, \tag{42}$$

$$d . \frac{dy\, (h-y)^{\frac{1}{2}}}{ds} + \frac{ds}{2(h-y)^{\frac{1}{2}}} = 0. \tag{43}$$

Taking (42), and integrating,

$$\frac{dx}{ds} (h-y)^{\frac{1}{2}} = h^{\frac{1}{2}} \cos a, \tag{44}$$

if $\dfrac{dx}{ds} = \cos a$, when $y = 0$; whence we have

$$dx = \frac{\cos a\, h^{\frac{1}{2}}\, dy}{\{h (\sin a)^2 - y\}^{\frac{1}{2}}}; \tag{45}$$

integrating which, and supposing $x = 0$, when $y = 0$, we have

$$y = x \tan a - \frac{x^2}{4h(\cos a)^2}; \tag{46}$$

which is the equation to a parabola, and is the same as equation (67), Art. 350, if $u^2 = 2gh$. (43) is also satisfied by (46).

459.] Let us also investigate, by means of the principle of least action, the path of a particle moving freely in space under the action of a central force which varies inversely as the square of the distance.

In this problem we will use polar coordinates; let v_0 be the initial value of v when $r = a$, so that

$$v^2 = v_0^2 + 2\mu\Big(\frac{1}{r} - \frac{1}{a}\Big) = c + \frac{2\mu}{r},$$

$$\text{if} \qquad c = v_0^2 - \frac{2\mu}{a};$$

$$\therefore \quad u = \int_0^1 \Big(c + \frac{2\mu}{r}\Big)^{\frac{1}{2}} ds, \tag{47}$$

$$\text{where} \qquad ds^2 = dr^2 + r^2 d\theta^2; \tag{48}$$

$$\text{so that} \qquad \delta . ds = \frac{dr}{ds}\delta . dr + \frac{r^2 d\theta}{ds}\delta . d\theta + \frac{r d\theta^2}{ds}\delta r;$$

and taking the variation of (47), and equating it to zero, we have

$$\begin{aligned} 0 = \Big[\Big(c + \frac{2\mu}{r}\Big)^{\frac{1}{2}} \Big(\frac{dr}{ds}\delta r + \frac{r^2 d\theta}{ds}\delta\theta\Big)\Big]_0^1 \\ + \int_0^1 \Big\{\frac{r d\theta^2}{ds} - \frac{\mu}{r^2}\Big(c + \frac{2\mu}{r}\Big)^{-\frac{1}{2}} - d.\Big(c + \frac{2\mu}{r}\Big)^{\frac{1}{2}} \frac{dr}{ds}\Big\} \delta r \\ - \int_0^1 d\Big\{\Big(c + \frac{2\mu}{r}\Big)^{\frac{1}{2}} \frac{r^2 d\theta}{ds}\Big\} \delta\theta : \end{aligned} \tag{49}$$

and equating to zero the coefficient of $\delta\theta$ under the sign of integration, we have

$$\Big(c + \frac{2\mu}{r}\Big)^{\frac{1}{2}} \frac{r^2 d\theta}{ds} = \text{a constant} = k \text{ (say)};$$

whence if $u = \frac{1}{r}$, we have

$$u = \frac{\mu}{k^2} + \frac{(ck^2 + \mu^2)^{\frac{1}{2}}}{k^2}\cos(\theta - \gamma); \tag{50}$$

which is the equation to a conic section.

460.] If the velocity v of the moving particle m is constant, and u is twice the action,

$$u = mv\int_0^1 ds = mvs, \tag{51}$$

and thus is proportional to the length of the path. The path therefore of an unconstrained particle is in this case a straight line. If however the motion is constrained to take place on a given surface, the path of least action is the geodesic on the surface which joins the two given positions, viz. the initial and the final places of the particle. If the surface is closed, as that of a sphere, there will be at least two geodesic paths joining the initial and the terminal places of m; one of which will be a maximum, and the other a minimum; thus in one case the *action* also will be a maximum, and in the other a minimum; and the action also may be constant, whatever is the path, provided that in this case $\delta u = 0$. For suppose the two points to be on a sphere: the great circle of the sphere passing through these two points will be the geodesic; one segment of which will be a maximum, and the other will be a minimum; and if the two points are opposite poles of the sphere, there will be an infinite number of great circles passing through them, and the lengths of all the arcs joining the two points will be the semi-circumference of a great circle, and thus the same for all. In this case u is constant, and thus $\delta u = 0$.

If the velocity is constant,

$$\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz = 0: \tag{52}$$

this is satisfied by $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = 0$, that is, when the particle m is acted on by no force. Also when all the impressed forces act along lines, which are perpendicular to the path of m at every point of it. Thus if a particle moves on a smooth surface, and is subject to no force except the normal reaction of the surface, (52) is satisfied, and the path of the particle is a geodesic line.

SECTION 3.—*The method of variation of parameters.*

461.] In some problems which have been investigated in the preceding Articles (and there are many of the kind), the integrations have been performed without difficulty in the more simple forms of the questions; whereas the integrations have been impossible in finite terms, when another term, which has generally expressed an additional impressed momentum-increment, has been introduced. The most salient instance of this circumstance occurs in equations (162) of Art. 367. When the expressions involving the disturbing function R are omitted, the equations become those marked (138), and admit of integration; and the complete integral is a conic, say, an ellipse; but equations (162) cannot be integrated in their complete form. A method of dealing generally with such questions has been discovered by Lagrange, and largely applied in Physical Astronomy. It is called *the method of variation of parameters*, and will be most conveniently explained by means of an example; and for this purpose I shall take the case of Art. 367, and shall for the sake of simplicity assume all the bodies to be in the plane of (x, y); so that the equations of motion are, when obvious substitutions are made, of the form

$$\frac{d^2x}{dt^2} = \mathrm{X}+\mathrm{X}'; \qquad \frac{d^2y}{dt^2} = \mathrm{Y}+\mathrm{Y}'. \tag{53}$$

Let us suppose that these equations admit of complete integration, when X' and Y' are omitted: and that the integrals of the equations

$$\frac{d^2x}{dt^2} = \mathrm{X}, \qquad \frac{d^2y}{dt^2} = \mathrm{Y}, \tag{54}$$

are

$$x = f(a, \alpha, t), \qquad y = \phi(b, \beta, t); \tag{55}$$

where a, α, b, β are four constants, as yet undetermined, introduced in the process of integration. Let us suppose the solutions of (53) to be of the form (55), in which a, α, b, β are no longer constant, but functions of t; and let them be determined, so that not only shall the particle, whose motion is represented by (53), have the same place at the time t as that whose motion is expressed by (54), but also that the axial-component velocities shall be the same in both cases: in which case

$$\frac{dx}{dt} = \frac{df}{dt}, \qquad \frac{dy}{dt} = \frac{d\phi}{dt}. \tag{56}$$

Now from (55) we have

$$\left.\begin{aligned} \frac{dx}{dt} &= \Big(\frac{df}{dt}\Big) + \Big(\frac{df}{da}\Big)\frac{da}{dt} + \Big(\frac{df}{d\alpha}\Big)\frac{d\alpha}{dt}, \\ \frac{dy}{dt} &= \Big(\frac{d\phi}{dt}\Big) + \Big(\frac{d\phi}{db}\Big)\frac{db}{dt} + \Big(\frac{d\phi}{d\beta}\Big)\frac{d\beta}{dt}; \end{aligned}\right\} \tag{57}$$

therefore

$$\left.\begin{aligned} \Big(\frac{df}{da}\Big)\frac{da}{dt} + \Big(\frac{df}{d\alpha}\Big)\frac{d\alpha}{dt} &= 0, \\ \Big(\frac{d\phi}{db}\Big)\frac{db}{dt} + \Big(\frac{d\phi}{d\beta}\Big)\frac{d\beta}{dt} &= 0. \end{aligned}\right\} \tag{58}$$

Also from (56) we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= \left(\frac{d^2f}{dt^2}\right) + \left(\frac{d^2f}{da\,dt}\right)\frac{da}{dt} + \left(\frac{d^2f}{d\alpha\,dt}\right)\frac{d\alpha}{dt},\\ \frac{d^2y}{dt^2} &= \left(\frac{d^2\phi}{dt^2}\right) + \left(\frac{d^2\phi}{db\,dt}\right)\frac{db}{dt} + \left(\frac{d^2\phi}{d\beta\,dt}\right)\frac{d\beta}{dt};\end{aligned}\right\} \qquad (59)$$

and substituting these in (53), we have

$$\left.\begin{aligned}\left(\frac{d^2f}{da\,dt}\right)\frac{da}{dt} + \left(\frac{d^2f}{d\alpha\,dt}\right)\frac{d\alpha}{dt} &= \mathrm{X}',\\ \left(\frac{d^2\phi}{db\,dt}\right)\frac{db}{dt} + \left(\frac{d^2\phi}{d\beta\,dt}\right)\frac{d\beta}{dt} &= \mathrm{Y}';\end{aligned}\right\} \qquad (60)$$

by (58) and (60) the four quantities a, α, b, β are to be determined.

Since the components of the velocity are the same in both curves at their common point at the given instant, it is evident that the curves at that point touch each other, and thus have a common tangent. And as the parameters, a, α, b, β, which determine the orbit, vary with the time, so does the form of the curve continually undergo change. Hence the curve in which the particle may be imagined to move has received the name of *the instantaneous orbit,* and the forces which produce the change of the instantaneous orbit are called *disturbing forces.* The actual orbit therefore is the envelope of all these instantaneous orbits. I propose to illustrate the method by one or two simple examples; but the most important application, viz. the astronomical one, is beside the scope of our present work.

462.] A heavy particle falls from rest in a medium the resistance of which varies as the square of the velocity; it is required to determine the circumstances of motion.

In this case the equation of motion is

$$\frac{d^2x}{dt^2} = g - k\left(\frac{dx}{dt}\right)^2; \qquad (61)$$

and omitting the last term, the equation becomes

$$\frac{d^2x}{dt^2} = g. \qquad (62)$$

The most general solution of which is

$$x = a + \alpha t + \frac{gt^2}{2}; \qquad (63)$$

$$\therefore \quad \frac{dx}{dt} = \alpha + gt + \frac{da}{dt} + t\frac{d\alpha}{dt}: \qquad (64)$$

if the velocity is the same in both the paths represented by (61) and (62), then

$$\frac{da}{dt}+t\frac{da}{dt}=0: \qquad (65)$$

and from the former part of (64) we have

$$\frac{d^2x}{dt^2}=g+\frac{da}{dt}: \qquad (66)$$

substituting which in (61), we have

$$\frac{da}{dt}=-k\left(\frac{dx}{dt}\right)^2=-k(a+gt)^2;$$

and adding g to both sides of the equation, and dividing by $g-k(a+gt)^2$, we have

$$\frac{da+g\,dt}{g-k(a+gt)^2}=dt;$$

integrating, and taking limits such that a and t are simultaneously zero, we have

$$\log\frac{g^{\frac{1}{2}}+k^{\frac{1}{2}}(a+gt)}{g^{\frac{1}{2}}-k^{\frac{1}{2}}(a+gt)}=2(kg)^{\frac{1}{2}}t; \qquad (67)$$

$$\therefore\quad a=-gt+\left(\frac{g}{k}\right)^{\frac{1}{2}}\frac{e^{2(kg)^{\frac{1}{2}}t}-1}{e^{2(kg)^{\frac{1}{2}}t}+1}. \qquad (68)$$

Also from (65), we have, integrating by parts,

$$\begin{aligned} a &= -at+\int a\,dt \\ &= \frac{gt^2}{2}-\left(\frac{g}{k}\right)^{\frac{1}{2}}\frac{e^{2(kg)^{\frac{1}{2}}t}-1}{e^{2(kg)^{\frac{1}{2}}t}+1}\,t+\frac{1}{k}\log\frac{e^{(kg)^{\frac{1}{2}}t}+e^{-(kg)^{\frac{1}{2}}t}}{2}; \end{aligned} \qquad (69)$$

so that a and a are both known in terms of t; and substituting in (63), we have a result the same as (111) in Art. 294.

463.] Another problem, on account of its importance in the theory of gunnery, may be solved by the preceding process.

To determine the path of a particle projected with a given velocity in a line inclined at a given angle to the horizon, and moving in a medium the resistance of which varies as the square of the velocity.

In this case the equations of motion are, as in Art. 374,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -k\frac{ds}{dt}\frac{dx}{dt}, \\ \frac{d^2y}{dt^2} &= -g-k\frac{ds}{dt}\frac{dy}{dt}. \end{aligned}\right\} \qquad (70)$$

If there is no resisting medium, the equations are

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g; \tag{71}$$

of which let the solutions be

$$x = a+\alpha t, \qquad y = b+\beta t - \frac{gt^2}{2}. \tag{72}$$

As the velocity is to be the same in the disturbed and in the undisturbed paths, we have

$$\frac{dx}{dt} = \alpha, \qquad \frac{dy}{dt} = \beta - gt; \tag{73}$$

$$\therefore \quad \frac{da}{dt} + t\frac{d\alpha}{dt} = 0, \qquad \frac{db}{dt} + t\frac{d\beta}{dt} = 0; \tag{74}$$

therefore from (73),

$$\frac{ds}{dt} = \{\alpha^2 + (\beta - gt)^2\}^{\frac{1}{2}}. \tag{75}$$

Also from (73),

$$\frac{d^2x}{dt^2} = \frac{d\alpha}{dt}, \qquad \frac{d^2y}{dt^2} = \frac{d\beta}{dt} - g;$$

and substituting these in (70), we have

$$\frac{d\alpha}{dt} = -k\alpha\,\{\alpha^2 + (\beta - gt)^2\}^{\frac{1}{2}}, \tag{76}$$

$$\frac{d\beta}{dt} = -k(\beta - gt)\,\{\alpha^2 + (\beta - gt)^2\}^{\frac{1}{2}}; \tag{77}$$

and therefore from (74),

$$\frac{da}{dt} = k\alpha t\,\{\alpha^2 + (\beta - gt)^2\}^{\frac{1}{2}}, \tag{78}$$

$$\frac{db}{dt} = kt(\beta - gt)\,\{\alpha^2 + (\beta - gt)^2\}^{\frac{1}{2}}; \tag{79}$$

from which four equations a, α, b, β are to be found in terms of t.

Eliminating t from (72), we have

$$\left(x - a - \frac{\alpha\beta}{g}\right)^2 = \frac{2\alpha^2}{g}\left(\frac{\beta^2}{2g} + b - y\right), \tag{80}$$

which is the equation to the instanstaneous path, and this is a parabola; of which, if the latus rectum is $4p$, and if (h, l) is the focus,

$$2p = \frac{\alpha^2}{g}, \qquad h = a + \frac{\alpha\beta}{g}, \qquad l = \frac{\beta^2 - \alpha^2}{2g} + b; \tag{81}$$

therefore, if the velocity at $(x, y) = v$,

$$\frac{dp}{dt} = \frac{\alpha}{g}\frac{d\alpha}{dt} = -k\,\frac{\alpha^2}{g}v = -2pkv, \tag{82}$$

$$\frac{dh}{dt} = -\frac{2k\alpha}{g}(\beta - gt)v, \tag{83}$$

$$\frac{dl}{dt} = -\frac{kv}{g}(v^2 - 2\alpha^2). \tag{84}$$

From (82) it appears that the latus rectum of the parabola continually diminishes; and since $v = \frac{ds}{dt}$, we have from it by integration,

$$\log \frac{p}{p_0} = 2k(s_0 - s);$$

so that the logarithm of the ratio of any two latera recta varies as the length of the arc between the points to which they correspond.

464.] This method has also been applied by Mr. Airy* to the calculation of the alterations produced in the amplitudes and the time of oscillation of a cycloidal pendulum, when a small disturbing force acts on it.

Let the forces be resolved normally and tangentially; and let the disturbing force be S, and act along the tangent to the cycloid, and diminish the velocity of the pendulum in its descent. Let a be the radius of the generating circle of the cycloid; then the equation of undisturbed motion is, see Art. 423,

$$\frac{d^2s}{dt^2} = -\frac{g}{4a}s.$$

Let $\frac{g}{4a} = n^2$, where $4a$, be it observed, is the length of the pendulum, see Art. 424, and we have

$$\frac{d^2s}{dt^2} + n^2 s = 0; \tag{85}$$

and the equation for the disturbed motion is

$$\frac{d^2s}{dt^2} + n^2 s = \text{S}. \tag{86}$$

The general integral of (85) is

$$s = c \sin(nt + \alpha), \tag{87}$$

where c and α are arbitrary constants; and where c is the amplitude measured along the cycloidal arc; and where $\frac{\pi}{2n} - \frac{\alpha}{n}$ is the time at which $s = c$.

From (87) the expression for the velocity in the undisturbed path is

$$\frac{ds}{dt} = cn \cos(nt + \alpha); \tag{88}$$

and as the velocity is the same in the disturbed path, when c and α vary, we have

$$\frac{dc}{dt} \sin(nt + \alpha) + c \cos(nt + \alpha) \frac{d\alpha}{dt} = 0; \tag{89}$$

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* See the Transactions of the Cambridge Philosophical Society, Vol. III, Part I.

and differentiating again (88) for the disturbed motion, and substituting in (86), we have

$$n\frac{dc}{dt}\cos(nt+a)-nc\sin(nt+a)\frac{da}{dt} = \text{s}; \tag{90}$$

so that
$$\frac{dc}{dt} = \frac{\text{s}}{n}\cos(nt+a), \tag{91}$$

$$\frac{da}{dt} = -\frac{\text{s}}{cn}\sin(nt+a); \tag{92}$$

which give the variations of c and a in terms of t; and if these equations were always susceptible of integration, the problem would be completely solved. In only a few cases is the solution possible.

If it is required to find the alteration of c due to one vibration, it is necessary to integrate

$$\frac{\text{s}}{n}\cos(nt+a)\,dt,$$

through a range of $nt+a$ equal to π: so that the increase in the amplitude of vibration

$$= \int\frac{\text{s}}{n}\cos(nt+a)\,dt \tag{93}$$

for the corresponding limits.

If it is required to find the alteration in the time of vibration during one oscillation, we proceed as follows. Let a_1 and t_1 be the values of a and t when the pendulum comes to rest, that is, when $\cos(nt+a)$, see (88), $= 0$; and let a_2 and t_2 be the values when the pendulum comes to rest the next time; so that, say,

$$nt_1+a_1 = \frac{\pi}{2}, \qquad nt_2+a_2 = \frac{3\pi}{2};$$

$$\therefore\quad n(t_2-t_1)+a_2-a_1 = \pi; \tag{94}$$

$$\therefore\quad t_2-t_1 = \frac{\pi}{n}-\frac{a_2-a_1}{n}$$

$$= \frac{\pi}{n}+\frac{1}{cn^2}\int\text{s}\sin(nt+a)\,dt, \tag{95}$$

by (92); the integral being definite, and taken between limits corresponding to the extreme values of the arc of vibration. If $\text{s} = 0$, the time of vibration $= \frac{\pi}{n}$; so that expressing (95) in the form

$$t_2-t_1 = \frac{\pi}{n}\left\{1+\frac{1}{cn\pi}\int\text{s}\sin(nt+a)\,dt\right\}, \tag{96}$$

the proportionate increase of the time of vibration is

$$\frac{1}{cn\pi}\int\text{s}\sin(nt+a)\,dt. \tag{97}$$

If s is expressed in terms of t, (93) and (97) may be used; but if s is a function of s, then from (87) and (88) we have

$$\sin(nt+a) = \frac{s}{c}, \qquad \cos(nt+a) = \frac{(c^2-s^2)^{\frac{1}{2}}}{c},$$

and $$dt = \frac{ds}{n(c^2-s^2)^{\frac{1}{2}}};$$

$$\therefore \text{ the increase of amplitude } = \frac{1}{cn^2}\int \mathrm{s}\,ds; \tag{98}$$

$$\text{the prop. increase of time of vibration } = \frac{1}{c^2n^2\pi}\int\frac{\mathrm{s}\,s\,ds}{(c^2-s^2)^{\frac{1}{2}}}. \tag{99}$$

465.] Two examples are subjoined:

Ex. 1. Let the pendulum make small vibrations in a circular arc; then the tangential impressed velocity-increment is

$$-g\sin\frac{s}{a},$$

which is equal to $$-g\left\{\frac{s}{a}-\frac{s^3}{6a^3}+\cdots\right\};$$

and omitting powers of $\frac{s}{a}$ above the cube, we have

$$\mathrm{s} = \frac{gs^3}{6a^3}; \quad \text{and} \quad n^2 = \frac{g}{a}.$$

Therefore the proportionate increase in the time of vibration

$$= \frac{g}{3\pi c^2n^2a^3}\int_0^c\frac{s^4\,ds}{(c^2-s^2)^{\frac{1}{2}}} = \frac{c^2}{16a^2};$$

which result is the same as (41), Art. 427. Also

$$\text{the increase of the amplitude } = \frac{1}{cn^2}\int_{-c}^{c}\frac{gs^3}{6a^3}\,ds$$

$$= \frac{g}{24cn^2a^3}\Big[s^4\Big]_{-c}^{c} = 0.$$

Ex. 2. Let the friction at the point of suspension be such as to cause a constant tangential retardation; thus suppose $\mathrm{s} = -f$;

$$\therefore \text{ the increase of the amplitude } = -\frac{f}{cn^2}\int_{-c}^{c}ds = \frac{-2f}{n^2}.$$

The proportionate increase in the time of vibration is

$$-\frac{f}{c^2n^2\pi}\int_{-c}^{+c}\frac{s\,ds}{(c^2-s^2)^{\frac{1}{2}}} = 0.$$

Other examples will be found in the memoir of Mr. Airy, which is referred to in the note of the preceding Article.

CHAPTER XIV.

ON VIRTUAL VELOCITIES.

466.] In Section 8, Chapter III, Arts. 108–110, it has been shewn that when a body or a system of material particles is at rest under the action of forces, these forces satisfy the condition expressed by the equation

$$\Sigma.\mathrm{P}\,\delta p = 0; \tag{1}$$

and the enunciation of the theorem contained in this equation is as follows:

If a system of forces, acting on a rigid body or on a system of particles which are at relative rest, is in equilibrium, and the body receives an infinitesimal displacement of the most general kind, whereby the points of application of the forces are displaced; but the forces act along lines parallel to and infinitesimally distant from their former lines of action; then the sum of the products of each force and the projection on its line of action of the displacement of its point of application is equal to zero.

This theorem is called *the principle of virtual velocities.* In Section 8, Chapter III, it has been deduced from the six equations of statical equilibrium, and consequently the demonstration of it as therein given depends on the composition and resolution of statical pressures; and thus ultimately on the parallelogram of forces, and accordingly whatever undue assumption or faults of reasoning there may be, if any, in the proof of the latter theorem as given in Arts. 17–20, these faults are still inherent in the demonstration of the theorem of virtual velocities; and as the theorem underlies the whole of statics and dynamics, the fundamental equations of these sciences being directly deducible from it, I propose to give of it an independent proof; and one that is inherent in our primary notions of the effects of force on matter.

As the meaning of the terms *virtual velocity* and *virtual moment* of a force, and the mode of estimating the signs of these quantities, have been explained in Art. 108, it is unnecessary to repeat them, for the reader can refer to that Article for all that is requisite.

467.] When a moving force acts at a point and does work, that work is measured by the product of the moving force and the projection on its line of action of the displacement of its point of application; see Art. 259; and the work is to be estimated as positive or negative according as the projected line falls on the line of action of the force in the direction towards which the force acts or in the opposite direction.

Now suppose a rigid body or a system of material particles to be at rest under the action of a system of forces P_1, P_2, ..., of which let P be the type: and imagine the system to receive the most arbitrary infinitesimal displacement possible, so that the points of application of the forces may undergo displacements, and the forces may do work, acting along lines parallel to and infinitesimally distant from their original lines of action. Then the system in its displaced state must be in some one of the three following conditions: the resultant effect of the forces acting on it may be either to remove the system farther from its original state; or to keep it at rest in its displaced state; or to bring it back to its original state. In the first condition the resultant effect of all the forces as shewn by the aggregate of the work done is in the displaced state less than in the original state; in the second condition the work done is the same in both states; in the third it is greater in its displaced state than in the original state: consequently in the original state the work virtually done by the forces was balanced, and must be either a maximum or a minimum or a constant; so that in all cases a small variation of it vanishes. Let H be the amount of work virtually done by the forces in the state of equilibrium, and let δ denote the change of work done by the forces during the displacement of the system; let δp be the projection on the line of action of P of the displacement of the point of application of P: so that $P\delta p$ is the work done by P in the displacement. Hence the above condition is mathematically expressed by the equation

$$\delta H = \Sigma . P\delta p = 0; \qquad (2)$$

which is the equation of virtual velocities, and thus expresses the condition that in all equilibrium-systems the variation of the work done by the forces vanishes.

468.] The preceding condition is true absolutely and irrespectively of any coordinate- or other system to which the points of application and action-lines of the forces may be referred.

Suppose however the system to be referred to coordinate axes in space: and let α, β, γ be the direction-angles of the line of action of P; and let P at its point of application be resolved into three axial-components $\text{P}\cos\alpha$, $\text{P}\cos\beta$, $\text{P}\cos\gamma$: then (2) takes the form

$$\delta\text{H} = \Sigma.\text{P}(\cos\alpha\,\delta x + \cos\beta\,\delta y + \cos\gamma\,\delta z) = 0. \tag{3}$$

Let us moreover suppose the arbitrary general displacement of the system to be compounded of a displacement of translation, of which the axial-projections are ξ, η, ζ; and of a displacement of rotation through a small angle θ about an axis whose direction-angles are f, g, h; so that, as in Art. 108,

$$\left.\begin{aligned} \delta x &= \xi + (z\cos g - y\cos h)\,\theta, \\ \delta y &= \eta + (x\cos h - z\cos f)\,\theta, \\ \delta z &= \zeta + (y\cos f - x\cos g)\,\theta; \end{aligned}\right\} \tag{4}$$

then substituting these in (3), and equating to zero the coefficients of the six quantities ξ, η, ζ, $\cos f$, $\cos g$, $\cos h$, all of which are arbitrary and independent, we have

$$\Sigma.\text{P}\cos\alpha = 0, \qquad \Sigma.\text{P}\cos\beta = 0, \qquad \Sigma.\text{P}\cos\gamma = 0;$$

$$\Sigma.\text{P}(y\cos\gamma - z\cos\beta) = \Sigma.\text{P}(z\cos\alpha - x\cos\gamma) = \Sigma.\text{P}(x\cos\beta - y\cos\alpha) = 0;$$

which are the six conditions of equilibrium, corresponding to the six degrees of freedom which a perfectly free system is capable of.

If P expresses of itself a statical force, the preceding results give statical theorems, and are those which have been demonstrated in the early part of this work. The principle is applied as follows when a particle m is subject to dynamical action.

Let $m\text{X}$, $m\text{Y}$, $m\text{Z}$ be the axial-components of the impressed momentum-increment, and let $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$, $m\frac{d^2z}{dt^2}$ be the axial-components of the expressed momentum-increment. The difference of these respectively, viz. the excess of the impressed momentum-increment over the expressed momentum-increment, is that which P represents in the preceding theorem; so that in this case (2) becomes

$$m\Big(\text{X} - \frac{d^2x}{dt^2}\Big)\delta x + m\Big(\text{Y} - \frac{d^2y}{dt^2}\Big)\delta y + m\Big(\text{Z} - \frac{d^2z}{dt^2}\Big)\delta z = 0; \tag{5}$$

and as δx, δy, δz are all arbitrary and independent, this equation is equivalent to

$$\text{X}-\frac{d^2x}{dt^2}=0; \qquad \text{Y}-\frac{d^2y}{dt^2}=0; \qquad \text{Z}-\frac{d^2z}{dt^2}=0; \tag{6}$$

which are the equations already established.

If $\delta x, \delta y, \delta z$ are the axial-projections of the space actually described by m in the time dt, as they may be, because $\delta x, \delta y$, and δz are entirely arbitrary; these latter may be replaced by dx, dy, dz; and (5) becomes

$$m\left\{\frac{dx\,d^2x+dy\,d^2y+dz\,d^2z}{dt^2}\right\}=m\,(\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz); \tag{7}$$

$$\therefore \quad \frac{mv^2}{2}-\frac{mv_0{}^2}{2}=m\int_{t_0}^{t}(\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz); \tag{8}$$

and we have the equation of vis viva and of work. This subject will be resumed in the following volume when we shall treat of the dynamics of material systems.

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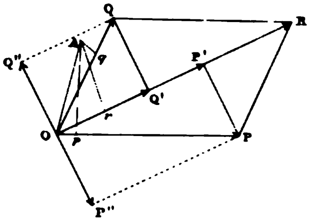

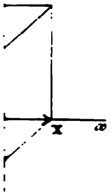

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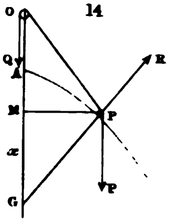

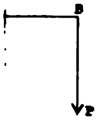

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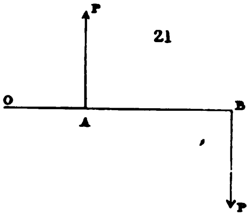

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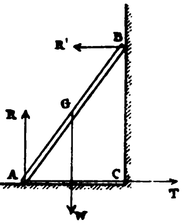

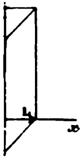

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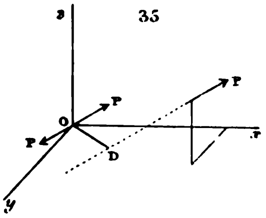

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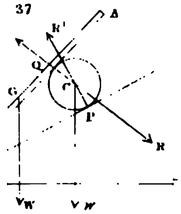

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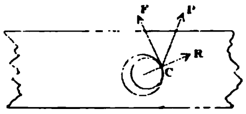

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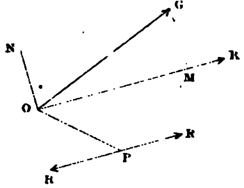

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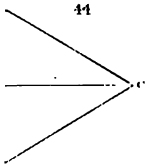

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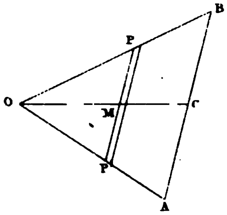

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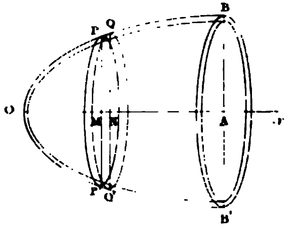

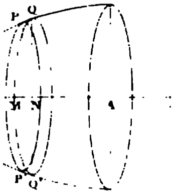

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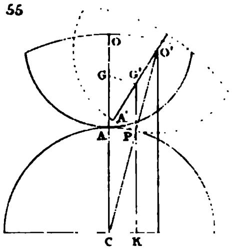

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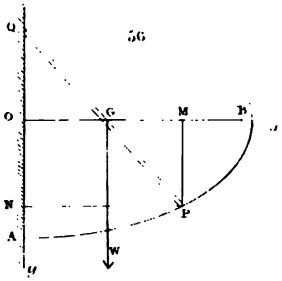

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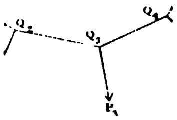

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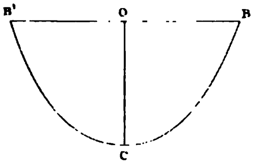

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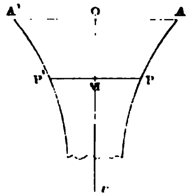

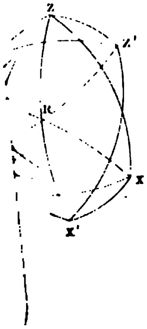

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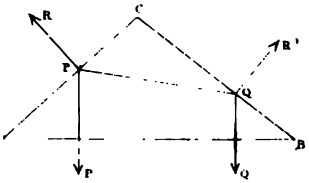

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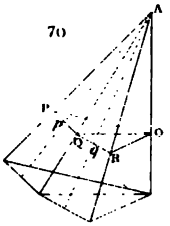

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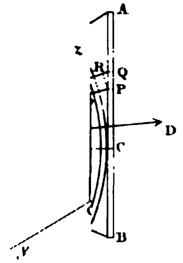

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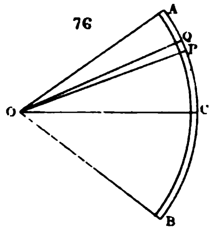

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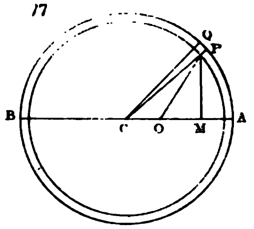

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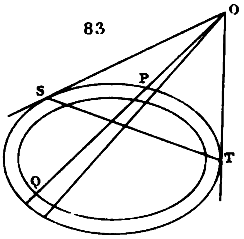

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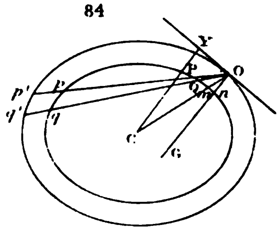

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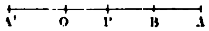

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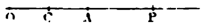

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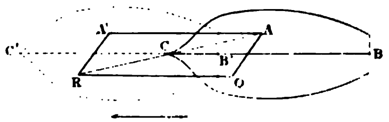

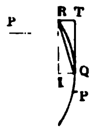

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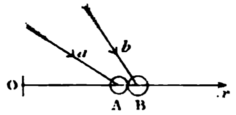

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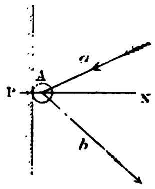

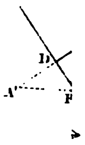

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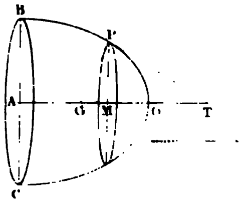

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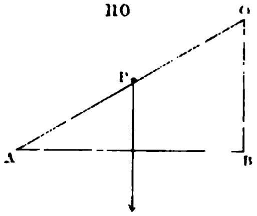

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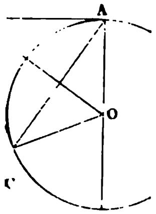

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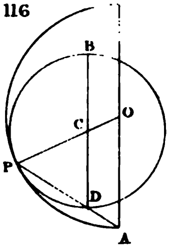

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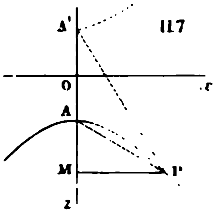

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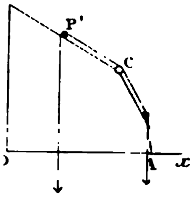

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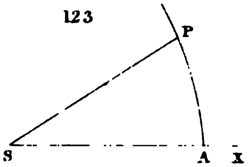

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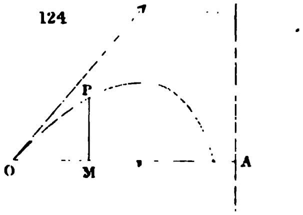

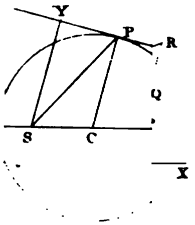

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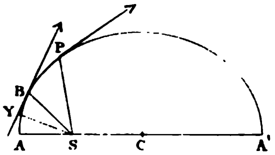

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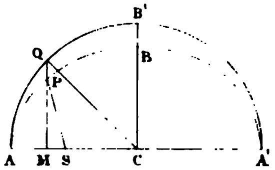

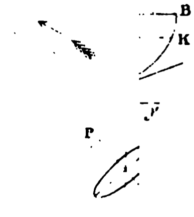

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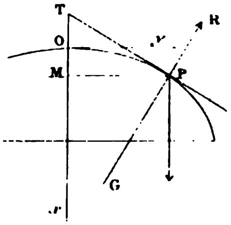

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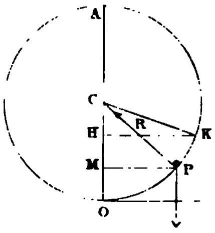

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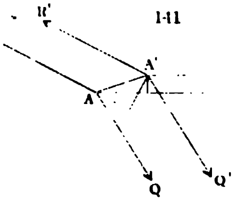

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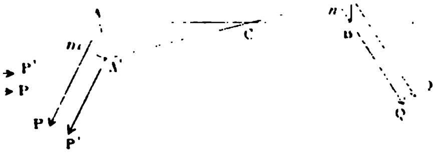

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